LISSAJOUS AND HALO ORBITS IN THE RESTRICTED THREE-BODY PROBLEM

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Abstract. We study the dynamics near the collinear Lagrangian points of the spatial, circular, restricted three–body problem. Following a standard procedure, we reduce the system to the center manifold and we analyze the Lissajous orbits as well the halo orbits, the latter ones arising from bifurcations of the planar Lyapunov family of periodic orbits. To obtain the Lissajous orbits, we perform a classical perturbation theory and we provide a formal approximate solution under suitable non–degeneracy and non–resonance conditions. As for the halo orbits, we construct a normal form adapted to the synchronous resonance: introducing a detuning, measuring the displacement from the resonance, and expanding the energy in series of the detuning, we are able to evaluate the energy level at which the bifurcation takes place. Except for a particular case, the analytical values obtained after a second order resonant perturbation theory are in very good agreement (in some cases up to the fourth decimal digit) with the numerical values found in the literature.

Keywords. Three–body problem, Lagrangian points, Collinear points, Halo orbits, Lissajous orbits.

1. Introduction

The dynamics of small bodies of the solar system can be conveniently described through the celebrated *restricted three–body problem*. This model provides the motion of a small body under the gravitational influence of two primaries. The term *restricted* means that the mass of the minor body is considered to be negligible with respect to that of the primaries, which are therefore assumed to move on Keplerian trajectories around their common barycenter. We will consider a special case of the restricted three–body problem, where the primaries are assumed to move on circular orbits. This model is known as the *spatial, circular, restricted three–body problem* (hereafter SCR3BP). Within the framework of the SCR3BP, Lagrange and Euler showed that the equations of motion in a synodic reference frame (namely, a frame rotating with the angular velocity of the primaries) admit five equilibrium positions ([2, 32]): in two of them the three bodies are located at the vertices of an equilateral triangle, while in the remaining three equilibrium positions the bodies are collinear. The equilateral positions, usually denoted as L_4 and L_5, are linearly stable for most of the mass ratios of the primaries; the collinear positions, denoted as L_1 (located within the primaries), L_2, L_3 (outside the interval joining the primaries) are linearly unstable.

Following the seminal work by C. Conley ([10]) on the existence of transit orbits through L_1, much attention has been devoted to the exploitation of the collinear points for space missions (for more details see the pioneering works in [31, 14, 15, 17, 21, 30], and references therein). Indeed, such locations are viewed for example as privileged positions to observe the Sun (L_1) or to observe the Universe shielding the Sun through the Earth (L_2). Numerical solutions provide high accuracy and a fast way to follow the evolution of a given initial state. However, only an analytical theory can give a thorough insight into the nature of the global behavior of these solutions. Along these lines, we describe some features of the dynamics around the collinear points by means of analytical techniques, most notably Lindstedt series and a perturbation theory based on Lie series ([17]). Since the collinear points are shown to be of the type saddle×center×center, a center manifold reduction is usually performed (see [17]) to separate the hyperbolic and elliptic directions. After the restriction of the dynamics to the center manifold, we obtain a nearly–integrable system with two degrees of freedom, whose integrable part provides a useful approximation of the system ([25]). Within the center manifold one can find quasi–periodic orbits, which form the so–called *Lissajous* family, and periodic orbits, most notably the *planar and vertical Lyapunov* family. Varying the energy, a bifurcation from the planar Lyapunov family gives place to the so–called *halo* periodic orbits when the frequencies in the center manifold are equal.

The aim of this paper is to implement suitable normal forms to get a description of the dynamics in the neighborhood of the collinear points. Under a suitable non-degeneracy condition, a non-resonant normal form is used to provide a parametric representation of the Lissajous tori, whose existence can be established by the Kolmogorov-Arnold-Moser (hereafter KAM) theorem ([20, 1, 26], see also [3, 22]).

Such parametrisation can be explicitly constructed using Lindstedt series (see also [9] and [19] for a numerical approach based on Poincaré sections). A major result of this work is that a 1:1 resonant perturbation theory allows us to investigate the halo family and to determine analytic expressions of the value of the energy at which the bifurcation takes place (compare with [29]).

All computations are performed using the algebraic manipulator `Mathematica`. The results are satisfactory for the collinear points L_1 and L_2, since the analytical prediction of the energy threshold for the bifurcation is very accurate (up to the fourth decimal digit), when compared with numerical data available in the literature (see [14, 15, 16]). Less accurate results are obtained for L_3 when considering small mass-ratios, probably due to the fact that in this case the optimal order of normalization is very low. Indeed, the dynamics around L_3 is rather different with respect to that of the other two collinear points and will be the subject of further investigation.

We stress that the techniques adopted in this work allows us to improve previous analytical approaches based on Lindstedt series ([30], [28]).

This work is organized as follows. In Section 2 we present the equations for the collinear points of the SCR3BP; the corresponding Hamiltonian is simplified in Section 3, while the center manifold reduction is computed in Section 4. The non–resonant perturbation theory and the parametrisation of the Lissajous tori are presented in Section 5. Resonant perturbation theory is implemented in Section 6 to study halo orbits and to derive analytical estimates on the bifurcation values.

2. Collinear points in the three-body problem

We study the motion of a celestial body A with mass m_A, subject to the gravitational attraction of two bodies, to which we refer as the primaries P and S - say, a planet and the Sun - with masses m_P and m_S, respectively. We assume that m_A is much smaller than the masses of the primaries, so that we can neglect the gravitational effect of A on the primaries (*restricted* three–body problem). In particular we assume that the primaries move on circular orbits around their common barycenter (i.e., we consider the SCR3BP).

We consider a *synodic* reference frame centered in the primaries' barycenter and rotating with the angular velocity of the primaries. The X axis is set along the line joining P and S, the Z axis along the angular momentum and the Y axis in such a way to have a positively oriented frame. We normalize the units of measure so that the gravitational constant as well as the sum of the masses of the primaries are unity, and that the period of rotation of the primaries is equal to 2π. Let us rename μ the mass of the smaller primary; then, with the previous normalization it results that the larger primary is located at $(\mu, 0, 0)$, while the smaller is at $(\mu - 1, 0, 0)$. A classical result due to Euler and Lagrange states that the equations of motion of the small body in the synodic reference frame admit five equilibrium points: the so-called *triangular* and *collinear Lagrangian points* ([5, 27]). These equilibrium positions correspond to stationary solutions of the pseudo-potential defined by equation (2.2) below. The triangular points, usually denoted as L_4 and L_5, are linearly stable for μ less than a threshold called *Routh's value*, while one can show that the collinear points, denoted as L_1, L_2, L_3, are always linearly unstable.

Let $(X, Y, Z) \in \mathbb{R}^3$ be the coordinates of the minor body in the synodic reference frame and let $(\dot{X}, \dot{Y}, \dot{Z}) \in \mathbb{R}^3$ be the corresponding velocity vector.

Let us define the symplectic momenta P_X, P_Y, P_Z as

$$P_X = \dot{X} - Y \ , \qquad P_Y = \dot{Y} + X \ , \qquad P_Z = \dot{Z} \ .$$

The 3 degrees-of-freedom Hamiltonian function describing the motion of the minor body is given by

$$H_0^{(in)}(P_X, P_Y, P_Z, X, Y, Z) = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + Y P_X - X P_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \ , \tag{2.1}$$

where r_1, r_2 denote the distances from the primaries:

$$r_1 = r_1(X, Y, Z) \equiv \sqrt{(X-\mu)^2 + Y^2 + Z^2} \ , \qquad r_2 = r_2(X, Y, Z) \equiv \sqrt{(X-\mu+1)^2 + Y^2 + Z^2} \ .$$

The phase space is the *collisionless* domain $\mathcal{P}_s$ of $\mathbb{R}^3 \times \mathbb{R}^3$ defined as

$$\mathcal{P}_s \equiv \{(P_X, P_Y, P_Z), (X, Y, Z) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ r_1(X, Y, Z) \neq 0, \ r_2(X, Y, Z) \neq 0\} \ ,$$

endowed with the standard symplectic form

$$\omega = dP_X \wedge dX + dP_Y \wedge dY + dP_Z \wedge dZ \ .$$

We assume that on $\mathcal{P}_s$ the Hamiltonian (2.1) is integrable and describes the two–body Newtonian interaction. Let us introduce the scalar function, sometimes called *pseudo-potential* (compare with [27]), defined as

$$\Omega(X, Y, Z) \equiv \frac{1}{2}(X^2 + Y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \ . \tag{2.2}$$

The collinear points are defined as the solutions of the system of equations

$$\frac{\partial \Omega}{\partial X} = 0 \ , \qquad \frac{\partial \Omega}{\partial Y} = 0 \ , \qquad \frac{\partial \Omega}{\partial Z} = 0$$

with the constraint $Y = Z = 0$. The literature on the Lagrangian (collinear and triangular) points is very wide and we refer the reader to the classical textbooks of Celestial Mechanics (see, e.g., [2, 32]).

Next task is to translate the origin of the synodic reference frame, actually located at the barycenter of the primaries, so that the new origin coincides with a collinear point;

to this end, we determine the distance γ_j, $j = 1, 2, 3$, of the collinear equilibria from the closest primary as the solution of the fifth order Euler's equations (see, e.g., [17]):

$$\begin{aligned}
\gamma_1^5 - (3-\mu)\gamma_1^4 + (3-2\mu)\gamma_1^3 - \mu\gamma_1^2 + 2\mu\gamma_1 - \mu = 0 \qquad & \text{for } L_1 \\
\gamma_2^5 + (3-\mu)\gamma_2^4 + (3-2\mu)\gamma_2^3 - \mu\gamma_2^2 - 2\mu\gamma_2 - \mu = 0 \qquad & \text{for } L_2 \\
\gamma_3^5 + (2+\mu)\gamma_3^4 + (1+2\mu)\gamma_3^3 - (1-\mu)\gamma_3^2 - 2(1-\mu)\gamma_3 - (1-\mu) = 0 \qquad & \text{for } L_3 \ .
\end{aligned}$$

Afterwards, we introduce new coordinates (x, y, z) through the following transformation, which also takes into account a rescaling of the distances, without altering the symmetry properties of the Hamiltonian:

$$\begin{aligned}
X &= \mp\gamma_j x + \mu + a \ , \qquad Y = \mp\gamma_j y \ , \qquad Z = \gamma_j z \ , \\
p_x &= P_X \ , \qquad\qquad\quad\ \ p_y = P_Y \ , \qquad\ \ p_z = P_Z \ ,
\end{aligned}$$

where the upper signs hold for L_1, L_2, while the lower signs are referred to L_3; moreover, we set $a = -1+\gamma_1$ for L_1, $a = -1-\gamma_2$ for L_2, $a = \gamma_3$ for L_3. Denoting by $\mathcal{P}_n = \mathcal{P}_n(\chi)$ the Legendre polynomial of order n and argument χ, the equations of motion associated to the Hamiltonian (2.1) expressed in the new variables can be written in the following form, where the pseudo-potential Ω has been expanded in terms of the Legendre polynomials:

$$\begin{aligned}
\ddot{x} - 2\dot{y} - (1+2c_2)x &= \frac{\partial}{\partial x}\sum_{n\geq 3} c_n(\mu)\rho^n \mathcal{P}_n\left(\frac{x}{\rho}\right) \\
\ddot{y} + 2\dot{x} + (c_2-1)y &= \frac{\partial}{\partial y}\sum_{n\geq 3} c_n(\mu)\rho^n \mathcal{P}_n\left(\frac{x}{\rho}\right) \\
\ddot{z} + c_2 z &= \frac{\partial}{\partial z}\sum_{n\geq 3} c_n(\mu)\rho^n \mathcal{P}_n\left(\frac{x}{\rho}\right) \ ,
\end{aligned} \tag{2.3}$$

where $\rho = \sqrt{x^2+y^2+z^2}$ and where the coefficients c_n, $n \geq 2$, coincide with one of the following expressions, according to which collinear point we are studying:

$$\begin{aligned}
c_n(\mu) &= \frac{1}{\gamma_1^3}\left(\mu + (-1)^n \frac{(1-\mu)\gamma_1^{n+1}}{(1-\gamma_1)^{n+1}}\right) \qquad \text{for } L_1 \\
c_n(\mu) &= \frac{(-1)^n}{\gamma_2^3}\left(\mu + \frac{(1-\mu)\gamma_2^{n+1}}{(1+\gamma_2)^{n+1}}\right) \qquad \text{for } L_2 \\
c_n(\mu) &= \frac{(-1)^n}{\gamma_3^3}\left(1 - \mu + \frac{\mu\gamma_3^{n+1}}{(1+\gamma_3)^{n+1}}\right) \qquad \text{for } L_3 \ .
\end{aligned} \tag{2.4}$$

Introducing the symplectic momenta $p_x = \dot{x} - y$, $p_y = \dot{y} + x$, $p_z = \dot{z}$, associated to x, y, z, we write the Hamiltonian corresponding to (2.3) as

$$H_1^{(in)}(p_x, p_y, p_z, x, y, z) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + yp_x - xp_y - \sum_{n\geq 2} c_n(\mu)\rho^n \mathcal{P}_n\left(\frac{x}{\rho}\right) . \qquad (2.5)$$

For later use we remark that the relation between $H_0^{(in)}$ in (2.1) and $H_1^{(in)}$ in (2.5) is given by (see [14])

$$H_0^{(in)} = H_1^{(in)}\gamma_1^2 - \frac{1}{2}(1 - \gamma_1 - \mu)^2 - \frac{\mu}{\gamma_1} - \frac{1-\mu}{1-\gamma_1} \qquad (2.6)$$

for L_1, by

$$H_0^{(in)} = H_1^{(in)}\gamma_2^2 - \frac{1}{2}(1 + \gamma_2 - \mu)^2 - \frac{\mu}{\gamma_2} - \frac{1-\mu}{1+\gamma_2} \qquad (2.7)$$

for L_2, and by

$$H_0^{(in)} = H_1^{(in)}\gamma_3^2 - \frac{1}{2}(\gamma_3 + \mu)^2 - \frac{1-\mu}{\gamma_3} - \frac{\mu}{1+\gamma_3} \qquad (2.8)$$

for L_3. The explicit expression of the sum in (2.5) involving the coefficients (2.4) as well as the Legendre polynomials will be needed to perform the normal form reduction of Sections 5 and 6. We also remark that the series at the right hand side of (2.5) is a sum of homogeneous polynomials (with coefficients $c_n(\mu)$), say $T_n(x, y, z) \equiv \rho^n \mathcal{P}_n\left(\frac{x}{\rho}\right)$, which can be iteratively computed by means of the following formulae:

$$T_0 = 1 , \qquad T_1 = x , \qquad T_n = \frac{2n-1}{n} x T_{n-1} - \frac{n-1}{n}(x^2 + y^2 + z^2)T_{n-2} .$$

3. Reduction of the Hamiltonian

Linearizing (2.5) around a given equilibrium point, we obtain that the quadratic part of the Hamiltonian is of the form:

$$H_1^{(q)}(p_x, p_y, p_z, x, y, z) = \frac{1}{2}\left(p_x^2 + p_y^2\right) + yp_x - xp_y - c_2 x^2 + \frac{c_2}{2}y^2 + \frac{p_z^2}{2} + \frac{c_2}{2}z^2 , \qquad (3.1)$$

where the coefficient c_2 provides the frequency ω_z of the z-direction, being $\omega_z = \sqrt{c_2}$. We remind that the coefficient c_2 takes different expressions (see (2.4)) according to which equilibrium point we are investigating.

We now aim at transforming (3.1) through a standard procedure that we sketch here for self–consistency (we refer to [17] for full details). Since the (p_z, z) components are already diagonalized, let us focus on the remaining variables. To this end, we define the vector $\xi \equiv (x, y, p_x, p_y)^T$ and we write the equations of motion as

$$\dot{\xi} = J\nabla H_1^{(q)} = M\xi , \qquad (3.2)$$

where J (the symplectic matrix) and M are defined as

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \qquad M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 2c_2 & 0 & 0 & 1 \\ 0 & -c_2 & -1 & 0 \end{pmatrix} .$$

We now have the following

Proposition 1. *Given Hamilton's equations* (3.2)*, there exists a symplectic change of variables, say* $\tilde{\xi} = C\,\xi$*, where* C *is a* 4×4 *real matrix, such that*

$$\frac{d\tilde{\xi}}{dt} = \widetilde{M}\,\tilde{\xi} , \tag{3.3}$$

where

$$\widetilde{M} = \begin{pmatrix} \lambda_x & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_y \\ 0 & 0 & -\lambda_x & 0 \\ 0 & -\omega_y & 0 & 0 \end{pmatrix} \tag{3.4}$$

for some $\lambda_x \in \mathbb{R}_+$*,* $\omega_y \in \mathbb{R}_+$*.*

Proof. The characteristic polynomial associated to M is

$$p(\lambda) = \lambda^4 + (2 - c_2)\lambda^2 + (1 + c_2 - 2c_2^2) ;$$

the equation $p(\lambda) = 0$ admits the solutions given by the square roots of the quantities η_1, η_2, defined as

$$\eta_1 = \frac{c_2 - 2 - \sqrt{9c_2^2 - 8c_2}}{2}, \qquad \eta_2 = \frac{c_2 - 2 + \sqrt{9c_2^2 - 8c_2}}{2} . \tag{3.5}$$

Let $\omega_y \equiv \sqrt{-\eta_1}$, $\lambda_x = \sqrt{\eta_2}$; according to [17], we proceed to implement a symplectic change of variables, defined through the matrix

$$C = \begin{pmatrix} \frac{a_{+\lambda_x}}{s_1} & \frac{a_{\omega_y}}{s_2} & \frac{1}{\sqrt{\omega_z}} e_3 & \frac{b_{-\lambda_x}}{s_1} & \frac{b_{\omega_y}}{s_2} & \sqrt{\omega_z} e_6 \end{pmatrix} , \tag{3.6}$$

where e_j are the unit vectors of the canonical basis, $a_{\omega_y} + i b_{\omega_y}$ is the eigenvector associated to ω_y, the quantities $a_{+\lambda_x}$, $b_{-\lambda_x}$ are the eigenvectors associated to $\pm\lambda_x$, while

$$s_1 = \sqrt{2\lambda_x((4 + 3c_2)\lambda_x^2 + 4 + 5c_2 - 6c_2^2)} , \qquad s_2 = \sqrt{\omega_y((4 + 3c_2)\omega_y^2 - 4 - 5c_2 + 6c_2^2)}$$

are normalizing factors which make the transformation symplectic (see [17]). With this change of coordinates we obtain (3.3) with $\widetilde{M}$ as in (3.4). $\square$

Since $c_2 > 1$, from (3.5) we have $\eta_1 < 0$ and $\eta_2 > 0$, which shows that the equilibrium point is of the type saddle × center × center; thanks to the change of variables $\tilde{\xi} = C\xi$ with C as in (3.6), the quadratic part of the Hamiltonian is reduced to

$$H_1^{(qd)}(\tilde{p}_x, \tilde{p}_y, \tilde{p}_z, \tilde{x}, \tilde{y}, \tilde{z}) = \lambda_x \tilde{x} \tilde{p}_x + \frac{\omega_y}{2}(\tilde{y}^2 + \tilde{p}_y^2) + \frac{\omega_z}{2}(\tilde{z}^2 + \tilde{p}_z^2) \ , \tag{3.7}$$

where we denote by $(\tilde{p}_x, \tilde{p}_y, \tilde{p}_z, \tilde{x}, \tilde{y}, \tilde{z})$ the new variables.

4. Center manifold reduction

Given the saddle × center × center character of the equilibria as shown in Section 3, we proceed to implement a center manifold reduction, which consists in reducing the study to the center directions and in eliminating the hyperbolic component through a suitable canonical transformation, which will be obtained by means of a Lie series. Full details of this procedure are given in [17]; for self-consistency of the present work we report here the outline of the method.

We start by writing the Hamiltonian (3.7) in complex form through the change of coordinates $(\tilde{p}_x, \tilde{p}_y, \tilde{p}_z, \tilde{x}, \tilde{y}, \tilde{z}) \to (p_1, p_2, p_3, q_1, q_2, q_3)$, defined by

$$\begin{aligned} \tilde{x} &= q_1 \ , & \tilde{p}_x &= p_1 \ , \\ \tilde{y} &= \frac{q_2 + i p_2}{\sqrt{2}} \ , & \tilde{p}_y &= \frac{i q_2 + p_2}{\sqrt{2}} \ , \\ \tilde{z} &= \frac{q_3 + i p_3}{\sqrt{2}} \ , & \tilde{p}_z &= \frac{i q_3 + p_3}{\sqrt{2}} \ , \end{aligned}$$

so that the Hamiltonian (3.7) to which we add the remainder is given by

$$H_2^{(c)}(p_1, p_2, p_3, q_1, q_2, q_3) = \lambda_x q_1 p_1 + i\omega_y q_2 p_2 + i\omega_z q_3 p_3 + \sum_{n \geq 3} H_n(p, q) \ , \tag{4.1}$$

where H_n are homogenoeus polynomials of degree n. The goal of this section is the center manifold reduction, which is the content of the following result.

Proposition 2. *Given the Hamiltonian* (4.1)*, there exists a canonical transformation* $(p, q) \to (P, Q)$*, such that* (4.1) *is transformed to*

$$\begin{aligned} H_3^{(N)}(P_1, P_2, P_3, Q_1, Q_2, Q_3) &= \lambda_x Q_1 P_1 + i\omega_y Q_2 P_2 + i\omega_z Q_3 P_3 \\ &+ \sum_{n=3}^{N} \tilde{H}_n(Q_1 P_1, P_2, P_3, Q_2, Q_3) + R_{N+1}(P, Q) \ , \end{aligned} \tag{4.2}$$

where $R_{N+1}(P, Q)$ *is the remainder function of degree* $N+1$*, which in turn might depend on* Q_1*,* P_1 *separately, while the homogeneous polynomials* $\tilde{H}_n$*,* $n = 3, ..., N$*, depend on* Q_1*,* P_1 *through their product.*

For completeness we provide the proof of the above result in Appendix A; it is based on the implementation of a Lie series (see, e.g., [13]), which allows us to eliminate suitable monomials of the Hamiltonian in order to obtain an invariant manifold, tangent to the elliptic directions of the quadratic part. We refer the reader to [17] for an exhaustive description of the center manifold reduction.

Remark 3. *The explicit construction of the generating function* $G = \sum_{k\geq 3} G_k$*, for some polynomial functions* $G_k = G_k(P, Q)$ *of order* k*, provided in Appendix A involves the appearance of small divisors. Precisely, setting*

$$S_k = \left\{(k_p, k_q) \in \mathbb{Z}^3 \times \mathbb{Z}^3 : \ k_{p1} \neq k_{q1} \ , \quad |k_p| + |k_q| = k\right\} \ ,$$

the function G_k *will involve divisors of the form* $\langle k_p - k_q, \omega\rangle$ *with* $\omega = (\lambda_x, i\omega_y, i\omega_z)$ *for* $(k_p, k_q) \in S_k$*. One can verify that* $|\langle k_p - k_q, \omega\rangle| \geq \lambda_x$ *with* λ_x *defined as* $\lambda_x = \sqrt{\eta_2}$ *and* η_2 *as in* (3.5)*.*

Finally, let us introduce the action variable $I_x = Q_1 P_1$; from (4.2) we immediately recognize that I_x is a constant of motion, whenever the remainder is neglected. As a consequence, given an initial condition such that $I_x(0) = 0$ and neglecting R_{N+1}, we obtain a Hamiltonian function with two degrees of freedom, which provides the dynamics in the center manifold within an approximation of order N.

5. Lissajous tori

The aim of this section is to propose an analytical investigation of Lissajous tori through a suitable implementation of a non-resonant normal form and a parametric representation of the tori. All techniques presented below are constructive and can be conveniently implemented to get an effective description of the Lissajous tori.

Let us write the Hamiltonian after the center manifold reduction of Section 4 in the form

$$\begin{aligned} H_4^{(cm)}(P_2, P_3, Q_2, Q_3) &= i\omega_y Q_2 P_2 + i\omega_z Q_3 P_3 + \sum_{n\geq 3} \tilde{H}_n(P_2, P_3, Q_2, Q_3) \\ &= H_4^{(q)}(P_2, P_3, Q_2, Q_3) + \sum_{n\geq 3} \tilde{H}_n(P_2, P_3, Q_2, Q_3) \ , \end{aligned} \tag{5.1}$$

where $H_4^{(q)}$ is the quadratic part and the functions $\tilde{H}_n$ are homogeneous polynomials of degree n. Let us assume that the frequencies ω_y, ω_z are non–resonant, namely that

$$|n_1\omega_y + n_2\omega_z| > 0 \tag{5.2}$$

for any $(n_1, n_2) \in \mathbb{Z}^2 \backslash \{0\}$.

We introduce action–angle coordinates for the quadratic part by introducing the variables $(I_y, I_z, \theta_y, \theta_z)$ as

$$\begin{cases} Q_2 = \sqrt{I_y}(\sin\theta_y - i\cos\theta_y) = -i\sqrt{I_y}e^{i\theta_y} \\ Q_3 = \sqrt{I_z}(\sin\theta_z - i\cos\theta_z) = -i\sqrt{I_z}e^{i\theta_z} \\ P_2 = \sqrt{I_y}(\cos\theta_y - i\sin\theta_y) = \sqrt{I_y}e^{-i\theta_y} \\ P_3 = \sqrt{I_z}(\cos\theta_z - i\sin\theta_z) = \sqrt{I_z}e^{-i\theta_z} \ , \end{cases} \tag{5.3}$$

so that the quadratic part of the Hamiltonian (5.1) is transformed as

$$H_4^{(q)}(I_y, I_z) = \omega_y I_y + \omega_z I_z \ . \tag{5.4}$$

Our next step is to implement a first–order non–resonant perturbation theory (see, e.g., [5, 12]), taking $\tilde{H}_3$ as perturbing function. To this end, let us start by writing $\tilde{H}_3$ as

$$\tilde{H}_3(P_2, P_3, Q_2, Q_3) = \sum_{a,b,c,d=0}^{3} \alpha_{abcd}\, Q_2^a P_2^b Q_3^c P_3^d$$

for suitable coefficients α_{abcd} with $a+b+c+d=3$. Then, transforming $\tilde{H}_3$ through the action–angle variables introduced in (5.3) we obtain:

$$\tilde{H}_3(I_y, I_z, \theta_y, \theta_z) = \sum_{a,b,c,d=0}^{3} \alpha_{abcd}\, (-i)^{a+c} (I_y)^{\frac{a+b}{2}} (I_z)^{\frac{c+d}{2}} e^{i(a-b)\theta_y} e^{i(c-d)\theta_z} \ . \tag{5.5}$$

It is readily seen that $\tilde{H}_3$ is of order 3/2 in the actions I_y, I_z; in general the functions $\tilde{H}_n$ in (5.1) are of order $n/2$. Therefore, we consider the quadratic part (5.4) as the unperturbed Hamiltonian, while the terms $\tilde{H}_n$ with $n \geq 3$ contribute to the perturbing function. Under the non–resonance condition (5.2), we determine a canonical change of variables, such that the perturbation is removed to higher orders. The new unperturbed Hamiltonian will be given by the sum of the original unperturbed Hamiltonian and the average of the perturbation over the angles θ_y, θ_z. However, it is easy to see that the average of $\tilde{H}_3$ is identically zero; in fact, from (5.5) we would require that $a = b$, $c = d$ with $(a, b, c, d) \in \mathbb{Z}_+^4$, such that their sum is equal to 3.

Therefore we are led to make a further perturbative step by considering $\tilde{H}_4$ as perturbing function. To this end, we start by writing $\tilde{H}_4$ as

$$\tilde{H}_4(I_y, I_z, \theta_y, \theta_z) = \sum_{a,b,c,d=0}^{4} \alpha_{abcd}\, (-i)^{a+c} (I_y)^{\frac{a+b}{2}} (I_z)^{\frac{c+d}{2}} e^{i(a-b)\theta_y} e^{i(c-d)\theta_z}$$

with $a+b+c+d=4$. In this case the average is obtained by solving the system of equations $a = b$, $c = d$ with $a+b+c+d=4$. This leads to the solutions $(a, b, c, d) = (2, 2, 0, 0)$,

$(0, 0, 2, 2)$, $(1, 1, 1, 1)$. Renaming for brevity the constants as

$$\alpha_{2200} = -\alpha \ , \qquad \alpha_{0022} = -\beta \ , \qquad \alpha_{1111} = -\gamma \ ,$$

the new unperturbed Hamiltonian $H_5^{(un)}$ is given by the expression:

$$H_5^{(un)}(I_y, I_z) = \omega_y I_y + \omega_z I_z + \alpha I_y^2 + \beta I_z^2 + \gamma I_y I_z \ , \tag{5.6}$$

where the values of the coefficients α, β, γ in two concrete cases (the so–called, barycenter–Sun and Earth–Moon systems introduced below) are given in Table 1.

Notice that the mass–ratio between the barycenter of the Earth–Moon system and the Sun is equal to

$$\mu = 3.0404 \ 10^{-6} \ ;$$

we will refer to such value as the *barycenter-Sun* mass–ratio. For the Earth–Moon case this value is much larger, amounting to

$$\mu = 1.2154 \ 10^{-2} \ .$$

Other quantities of interest, precisely the c_2 coefficient and the frequencies, are reported in Table 1. We observe that for the case of L_3 in the barycenter–Sun system, the frequencies ω_y, ω_z are equal to one just within the approximation given in Table 1.

	L_1 (BS)	L_2 (BS)	L_3 (BS)	L_1 (EM)	L_2 (EM)	L_3 (EM)
c_2	4.06107	3.94052	1.00000	5.14771	3.19041	1.01069
λ_1	2.53266	2.48432	0.00283	2.93209	2.15867	0.17787
ω_y	2.08645	2.05701	1.00000	2.33441	1.86264	1.01042
ω_z	2.01521	1.98507	1.00000	2.26886	1.78617	1.00533
α	-0.09897	-0.09237	$-1.5914 \cdot 10^{-6}$	-0.16211	-0.05470	-0.00566
β	-0.08098	-0.07430	$-5.34608 \cdot 10^{-8}$	-0.14489	-0.03612	-0.00021
γ	0.02562	0.03552	0	-0.07263	0.08829	0.00021

TABLE 1. Data for the barycenter–Sun (BS) and Earth–Moon (EM) systems.

As a product of the normal form procedure outlined before, we obtain the complete non–resonant Hamiltonian as

$$H_5^{(nr)}(I_y, I_z, \theta_y, \theta_z) = H_5^{(un)}(I_y, I_z) + R^{(5)}(I_y, I_z, \theta_y, \theta_z) \ , \tag{5.7}$$

where $R^{(5)}$ is the perturbing function, whose explicit expression can be obtained through a direct implementation of perturbation theory. From (5.6) the unperturbed frequencies are defined by

$$\Omega_y(I_y, I_z) = \omega_y + 2\alpha I_y + \gamma I_z, \qquad \Omega_z(I_y, I_z) = \omega_z + 2\beta I_z + \gamma I_y \ ,$$

while the unperturbed flow associated to (5.6) is explicitly given by

$$\begin{aligned} I_y(t) &= I_y(0) \\ I_z(t) &= I_z(0) \\ \theta_y(t) &= \Omega_y(I_y(0), I_z(0))t + \theta_y(0) \\ \theta_z(t) &= \Omega_z(I_y(0), I_z(0))t + \theta_z(0) \ . \end{aligned}$$

In order to define a KAM torus associated to (5.7) we need to introduce the following notion of Diophantine vectors.

Definition 4. *Let $\Omega \in \mathbb{R}^\ell$, $\ell \geq 2$; we say that Ω is a Diophantine vector, if there exists $C > 0$, $\tau \geq \ell - 1$, such that*

$$|\Omega \cdot n|^{-1} \leq C |n|_2^\tau \tag{5.8}$$

for any $n \equiv (n_1, ..., n_\ell) \in \mathbb{Z}^\ell \backslash \{0\}$, where $|n|_2 \equiv (\sum_{k=1}^\ell |n_k|^2)^{\frac{1}{2}}$. We denote by $\mathcal{D}_\ell(C, \tau)$ the set of Diophantine vectors satisfying (5.8) *with constants C and τ. For $\tau > \ell - 1$ the union over all $C > 0$ of the sets $\mathcal{D}_\ell(C, \tau)$ has full Lebesgue measure in $\mathbb{R}^\ell$.*

For arbitrary initial conditions $(I_y(0), I_z(0))$, let us assume that the frequency vector

$$(\Omega_y^0, \Omega_z^0) \equiv (\Omega_y(I_y(0), I_z(0)), \Omega_z(I_y(0), I_z(0)))$$

is such that $(\Omega_y^0, \Omega_z^0) \in \mathcal{D}_2(C, \tau)$ for some positive constants C, τ. Finally, we can formulate the following definition of Lissajous KAM tori.

Definition 5. *Let (Ω_1^0, Ω_2^0) be a frequency vector satisfying* (5.8)*; a Lissajous KAM torus for the Hamiltonian* (5.7) *is an invariant 2–dimensional surface which can be parametrized by the equations*

$$\begin{aligned} \theta_y &= \phi_y + u_y(\phi_y, \phi_z) \\ \theta_z &= \phi_z + u_z(\phi_y, \phi_z) \\ I_y &= v_y(\phi_y, \phi_z) \\ I_z &= v_z(\phi_y, \phi_z) \ , \end{aligned} \tag{5.9}$$

where u_y, u_z, v_y, v_z are regular, periodic functions such that the flow in the parametric coordinates is given by $(\phi_y(t), \phi_z(t)) = (\phi_y(0) + \Omega_y^0\, t, \phi_z(0) + \Omega_z^0\, t)$. Moreover, the vector *function* $u \equiv (u_y, u_z)$ *satisfies*

$$\mathrm{Id} + \frac{\partial u}{\partial \phi} \neq 0 \qquad \forall \phi = (\phi_y, \phi_z) \in \mathbb{T}^2$$

(where Id *is the* 2×2 *identity matrix).*

Remark 6. *Given that the perturbing function* $R^{(5)}$ *in* (5.7) *is a homogeneous polynomial of degree greater than 5/2 in the action variables, we can assume that the perturbing function is multiplied by a small parameter, say* η *with* $0 < \eta < 1$*, which is related to the distance of the initial conditions to the origin* $(I_y, I_z) = (0, 0)$*. In other words, we scale the variables as* $(I_y, I_z) \to (\eta I_y, \eta I_z)$*, where we keep the same name for the scaled variables; dividing the transformed Hamiltonian by* η *and introducing a new parameter as* $\rho \equiv \eta^{3/2}$*, we obtain the Hamiltonian*

$$H_5^{(nr)}(I_y, I_z, \theta_y, \theta_z) = \omega_y I_y + \omega_z I_z + \alpha\eta I_y^2 + \beta\eta I_z^2 + \gamma\eta I_y I_z + \rho\ R^{(5)}(I_y, I_z, \theta_y, \theta_z)\ . \quad (5.10)$$

As a consequence, the functions (u_y, u_z, v_y, v_z) *in Definition 5 are themselves depending upon the small parameter* ρ*. We refer to* ρ *as the perturbing parameter, since for* $\rho = 0$ *the Hamiltonian* (5.10) *is integrable.*

Next, we outline a procedure which allows us to determine the vector functions u, v, defining the Lissajous KAM torus.

Let $I = (I_y, I_z)$, $\theta = (\theta_y, \theta_z)$ and let $\Omega(I) = (\Omega_y(I_y, I_z), \Omega_z(I_y, I_z))$ be the frequency vector associated to the linear and quadratic part of (5.10); write Hamilton's equations associated to (5.10) as

$$\begin{aligned} \dot\theta &= \Omega(I) + \rho \frac{\partial R^{(5)}}{\partial I}(I, \theta) \\ \dot I &= -\rho \frac{\partial R^{(5)}}{\partial \theta}(I, \theta)\ . \end{aligned} \quad (5.11)$$

Let $(I_y(0), I_z(0))$ be the initial condition and let $\Omega^0 = (\Omega_y^0, \Omega_z^0)$ be the corresponding value of the frequency vector. Inserting the definition (5.9) into (5.11), we obtain that the functions (u, v) must satisfy the following homological equations:

$$\begin{aligned} \Omega^0 + Du(\phi) &= \Omega(v(\phi)) + \rho \frac{\partial R^{(5)}}{\partial I}(v(\phi), \phi + u(\phi)) \\ Dv(\phi) &= -\rho \frac{\partial R^{(5)}}{\partial \theta}(v(\phi), \phi + u(\phi))\ , \end{aligned} \quad (5.12)$$

where D is the partial differential operator

$$D \equiv \Omega^0 \cdot \frac{\partial}{\partial \phi}\ .$$

Remark 7. *The solution of* (5.12) *provides the functions* (u, v). *However, to solve* (5.12) *we need to invert the operator* D *and this implies the appearance of the so–called small divisors. These quantities can be controlled by using a non–resonance condition or a stronger assumption, like the Diophantine condition* (5.8).

We denote by (u_a, v_a) an approximate solution, which solves the invariance equations (5.12) with an error term $(\varepsilon_u, \varepsilon_v)$, namely

$$\begin{aligned} \Omega^0 + Du_a(\phi) - \Omega(v_a(\phi)) - \rho\, \frac{\partial R^{(5)}}{\partial I}(v_a(\phi), \phi + u_a(\phi)) &= \varepsilon_u \\ Dv_a(\phi) + \rho\, \frac{\partial R^{(5)}}{\partial \theta}(v_a(\phi), \phi + u(\phi)) &= \varepsilon_v \ . \end{aligned} \tag{5.13}$$

To obtain u_a, v_a, let us introduce a finite truncation $u_a^{(M)}$, $v_a^{(M)}$ of the series expansions of u and v in powers of ρ up to a given order $M > 0$:

$$u_a^{(M)}(\phi) = \sum_{j=1}^{M} \rho^j u_j(\phi) \ , \qquad v_a^{(M)}(\phi) = v_0 + \sum_{j=1}^{M} \rho^j v_j(\phi) \ . \tag{5.14}$$

Inserting the expansions (5.14) in (5.12) and equating same orders in ρ, one can compute iteratively the functions u_j and v_j. We are thus led to define an approximate formal solution of (5.12) as follows.

Definition 8. *We say that* $(u_a^{(M)}(\phi), v_a^{(M)}(\phi)) = (\sum_{j=1}^M \rho^j u_j(\theta), v_0 + \sum_{j=1}^M \rho^j v_j(\theta))$, *where all the* (u_j, v_j) *are analytic functions, is an approximate solution of* (5.12) *to order* M, *if* (5.13) *holds with* $|(\varepsilon_u, \varepsilon_v)| = O(|\rho|^{M+1})$.

The existence of an approximate solution of (5.12) is provided by the following result, whose proof gives an explicit algorithm to construct approximations of the KAM Lissajous tori.

Proposition 9. *Let us consider the Hamiltonian* (5.10) *and let* $I_0 = I(0)$ *be an initial condition such that the frequency vector* $\Omega(I_0) = \Omega^0$ *satisfies the following non–resonance condition:*

$$|\Omega^0 \cdot k| > 0 \qquad \forall k \in \mathbb{Z}^2 \ .$$

Let $R^{(5)}$ *in* (5.10) *be a real–analytic function in a neighborhood of* I_0 *for all* $\phi \in \mathbb{T}^2$. *Let* $H_5^{(un)}$ *be the quadratic part in* (5.10) *and assume that the non–degeneracy condition*

$$\det \Big(\frac{\partial^2 H_5^{(un)}}{\partial I^2}(I_0) \Big) \neq 0 \tag{5.15}$$

is satisfied. Then, for any integer $M > 0$ *there exists an approximate solution to order* M.

Proof. Let us expand u and v in Fourier–Taylor series as

$$\begin{aligned} u(\phi) &= \sum_{j=1}^{\infty} \rho^j \sum_{k\in\mathbb{Z}^2} \hat{u}_{jk}\ e^{ik\cdot\phi} \\ v(\phi) &= v_0 + \sum_{j=1}^{\infty} \rho^j \sum_{k\in\mathbb{Z}^2} \hat{v}_{jk}\ e^{ik\cdot\phi}\ ; \end{aligned} \tag{5.16}$$

inserting (5.16) in (5.12) and equating same orders of ρ, we obtain recursive relations defining the functions (u_j, v_j) in (5.14) in terms of the functions at the previous orders. Precisely, we have that v_0 is determined by solving the equation

$$\Omega(v_0) = \Omega^0\ ;$$

the function v_1 is obtained by solving the equation

$$Dv_1 = -\frac{\partial R^{(5)}}{\partial \theta}(v_0, \phi)\ , \tag{5.17}$$

whose solution gives the non–average part of v_1; the equation is well defined, since the right hand side of (5.17) has zero average.

Next we proceed to solve the following equation (the prime denotes the derivative with respect to I), defining u_1:

$$Du_1 = \Omega'(v_0)v_1 + \frac{\partial R^{(5)}}{\partial I}(v_0, \phi)\ . \tag{5.18}$$

First we take the average of the right hand side which we impose to be zero; this determines the average of v_1, say $\bar{v}_1$, provided that the non-degeneracy condition $\det(\Omega'(v_0)) \neq 0$, which is equivalent to (5.15), is satisfied. Then, we solve the non-average part of (5.18) to get u_1 (as usual, the average of u_1 can be taken to be zero, as it corresponds to a shift of the phase of the angles).

For any $n \leq M$ we obtain that the function (u_n, v_n) can be computed as follows. The non-zero average part of the function v_n is obtained by solving an equation of the form

$$Dv_n = Q(v_0, v_1, ..., v_{n-1}, u_1, ..., u_{n-1})\ , \tag{5.19}$$

where Q is a function with zero-average, depending on the functions obtained at the previous iterative steps. The average of v_n, say $\bar{v}_n$, is computed by solving the equation

$$\Omega'(v_0)\bar{v}_n = F(v_0, \bar{v}_1, ..., \bar{v}_{n-1})\ , \tag{5.20}$$

for a suitable function F, provided (5.15) is satisfied. Finally, the function u_n is obtained by solving an equation of the form:

$$Du_n = \Omega'(v_0)(v_n - v_0) + G(v_0, v_1, ..., v_{n-1}, u_1, ..., u_{n-1})\ , \tag{5.21}$$

for a suitable function G. In conclusion, by solving (5.19)-(5.20)-(5.21) for $n \leq M$ we obtain an approximate solution to order M. □

The existence of Lissajous KAM tori is provided by the KAM theorem for the Hamiltonian system (5.10). To state the theorem we need to extend the Hamiltonian to a complex space, according to the following definition.

Definition 10. *Given $\rho > 0$, let $\mathbb{T}^2_\rho$ be the set*

$$\mathbb{T}^2_\rho \equiv \{\theta = (\theta_y, \theta_z) \in \mathbb{C}^2 \backslash (2\pi\mathbb{Z})^2 : \ Re(\theta) \in \mathbb{T}^2, \ |Im(\theta_y)| \leq \rho, \ |Im(\theta_z)| \leq \rho\} \ .$$

We denote by $\mathcal{A}_\rho$ the set of analytic functions f in the interior of $\mathbb{T}^2_\rho$ with the norm

$$\|f\|_\rho \equiv \sup_{(\theta_y,\theta_z)\in\mathbb{T}^2_\rho} |f(\theta_y, \theta_z)| \ .$$

We state now a KAM theorem which provides the existence of Lissajous KAM tori with Diophantine frequency.

Theorem 11. *Consider the Hamiltonian* (5.10) *and let $(I_y(0), I_z(0))$ be such that the corresponding frequency vector (Ω^0_y, Ω^0_z) is Diophantine in the sense of Definition 4 for some $C > 0$, $\tau > 0$. Let $(u_a, v_a) \in \mathcal{A}_\rho$ for some $\rho > 0$ be an approximate solution of* (5.12) *with error term $(\varepsilon_u, \varepsilon_v)$. Assume that the non-degeneracy condition* (5.15) *is satisfied. Then, if $\varepsilon \equiv \|(\varepsilon_u, \varepsilon_v)\|_\rho$ is sufficiently small, there exists an exact solution of* (5.12)*, say (u_e, v_e), such that for $0 < \delta < \rho/2$, one has*

$$\|(u_e, v_e) - (u_a, v_a)\|_{\rho-2\delta} < C_e\varepsilon$$

for some constant $C_e > 0$.

We observe that, given the expression of the quadratic part in (5.10), the non–degeneracy condition (5.15) amounts to require that

$$\det(H''_{nr}) = \det\begin{pmatrix} 2\alpha\eta & \gamma\eta \\ \gamma\eta & 2\beta\eta \end{pmatrix} = (4\alpha\beta - \gamma^2)\eta^2 \neq 0 \ , \tag{5.22}$$

which is satisfied for $\eta \neq 0$, whenever

$$\gamma \neq \pm 2\sqrt{\alpha\beta} \ .$$

Remark 12. *It is important to keep in mind that the goodness of the procedure of approximating a Lissajous torus depends on several factors: the order of the remainder in* (5.10)*, the order M of the formal approximate solution constructed in Proposition 9 (affecting the size of the error of the approximate solution), the choice of the frequency vector (hence, the size of the Diophantine constant appearing in* (5.8)*). Of course, the domain* *in which the tori are defined depends also on the region in which the non–degeneracy condition is satisfied.*

The proof of Theorem 11 is rather technical and we refer the reader to the specialized literature (see [20, 1, 26], compare also with [4, 7, 8] for applications to Celestial Mechanics).

The proof is based on the solution of (5.12) by a Newton's approach. More precisely, one starts with an approximate solution (u_a, v_a) satisfying (5.13) with an error $(\varepsilon_u, \varepsilon_v)$ sufficiently small. Then, one constructs a new approximate solution, which satisfies the analogous of equations (5.13) with an error quadratically smaller. By defining the approximations on a suitable scale of Banach spaces, one can get the convergence of the sequence of approximate solutions to the true solution of equation (5.12). The procedure converges provided the initial error is sufficiently small, which in turn, by expanding (u, v) as in (5.16), amounts to require a smallness condition on the perturbing parameter ρ.

We mention that the proof of the KAM theorem provides an explicit algorithm, which can be efficiently implemented to construct the tori; however, this task requires a long set of explicit estimates on the different steps needed to prove the existence of a KAM torus: bounds on the approximate solution and the corresponding error, estimates on the correction needed to construct a quadratically smaller approximate solution, bounds on the Newton's step, an iteration procedure providing a sequence of approximate solutions, a proof of the convergence of the sequence of solutions on a non-empty domain. To provide efficient results, it is convenient to develop a proof adapted to the model at hand, without using very general estimates (compare, for example, with the results obtained for the standard map in [6], [23], or the - quite long - estimates for the three-body problem performed in [8]). Providing explicit estimates might represent a future development of the present work; however, we profit here of the fact that we computed explicitly the normal form (5.6) to make a preliminary study on the non–degeneracy assumption (5.15), which is mandatory to prove Theorem 11. Precisely, we provide in Figure 1 the graph of the quantity $4\alpha\beta - \gamma^2$ appearing in (5.22) versus the mass parameter μ, where α, β, γ have been introduced in the quadratic part (5.6) of the non–resonant normal form.

The behavior of the quantity $4\alpha\beta - \gamma^2$ is completely different in the three cases. For L_1 the non-degeneracy condition $4\alpha\beta - \gamma^2 \neq 0$ is satisfied for any value of the mass parameter. In the case of L_2 we observe that $4\alpha\beta - \gamma^2 = 0$ for a value slightly bigger

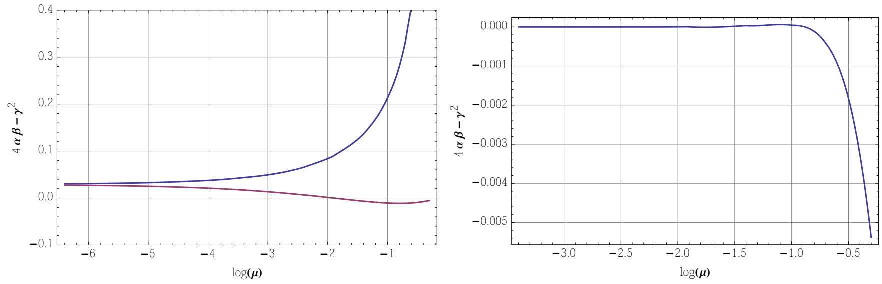

Figure 1. Graph of the quantity $4\alpha\beta-\gamma^2$ appearing in (5.22) versus the mass parameter μ. Left panel: the blue curve represents L_1, while the red curve refers to L_2; right panel: graph for L_3.

than the Earth–Moon mass-ratio, precisely for $\mu \simeq 0.01239$. For L_3 the quantity $4\alpha\beta-\gamma^2$ is positive, but very close to zero, for small μ, while it changes sign at $\mu = 0.123$.

We have to keep in mind that, of course, a higher order normal form would contribute to the expression (5.6) by adding higher order terms; however, such terms would typically modify slightly the shape of the curves in Figure 1. In conclusion, these results provide preliminary, though essential, information in view of concrete applications of KAM theory. Of course, the investigation of the existence of invariant manifolds in the degenerate case requires the study of *meandering* and *shearless* tori, which would certainly deserve a dedicated work based on a different approach than that outlined in this section.

6. Halo orbits

In this section we concentrate on some special resonant orbits for which we have that the frequency of the planar Lyapunov orbit is in resonance with (actually is equal to) the frequency of its vertical perturbation. From the Hamiltonian (5.1) (after the center manifold reduction), we implement the change of variables (5.3) to get the Hamiltonian in action–angle variables, thus obtaining that the Hamiltonian is given by the quadratic part (5.4) plus the remainder function, say

$$\tilde{H}_4^{(cm)}(I_y, I_z, \theta_y, \theta_z) = \omega_y I_y + \omega_z I_z + \tilde{H}^{(3)}(I_y, I_z, \theta_y, \theta_z) + \tilde{H}^{(4)}(I_y, I_z, \theta_y, \theta_z) \ , \qquad (6.1)$$

where $\tilde{H}^{(j)}$ is a homogeneous polynomial of degree $j/2$ in the actions. Next we proceed to perform a resonant perturbation theory in the neighborhood of the synchronous resonance $\omega_y = \omega_z$ (see [5, 12]) by constructing a canonical transformation of variables, which conjugates (6.1) to the following form (for simplicity of notation we keep the same name for the new coordinates):

$$H^{(res)}(I_y, I_z, \theta_y, \theta_z) = h_0(I_y, I_z) + h_r(I_y, I_z, \theta_y - \theta_z) + R^{(r)}(I_y, I_z, \theta_y, \theta_z) \ ,$$

where h_0 depends only on the actions; h_r is the resonant part depending on the actions as well as on the angles, but just through the combination $\theta_y - \theta_z$, which corresponds to the synchronous resonance; $R^{(r)}$ represents the remainder function. This procedure will lead to have, by construction, that $\dot{I}_y + \dot{I}_z = 0$ up to the remainder.

The third degree Hamiltonian $\tilde{H}_3$ does not contain resonant terms, which are instead found in $\tilde{H}_4$, thus leading to the following resonant normal form $H_6^{(r)}$ (up to a remainder):

$$H_6^{(r)}(I_y, I_z, \theta_y, \theta_z) = \omega_y I_y + \omega_z I_z + [\alpha I_y^2 + \beta I_z^2 + I_y I_z(\gamma + 2\tilde{\gamma}\cos(2(\theta_y - \theta_z)))] \qquad (6.2)$$

for suitable coefficients α, β, γ, $\tilde{\gamma}$. In the case of the collinear points of the barycenter–Sun system, we have the following values:

$$\begin{array}{lllll} L_1 : & \alpha = -0.09897 \, , & \beta = -0.08098 \, , & \gamma = 0.02562 \, , & \tilde{\gamma} = -0.10214 \, , \\ L_2 : & \alpha = -0.09237 \, , & \beta = -0.07430 \, , & \gamma = 0.03552 \, , & \tilde{\gamma} = -0.10045 \, , \\ L_3 : & \alpha = -1.77963 \cdot 10^{-6} \, , & \beta = -5.9765 \cdot 10^{-8} \, , & \gamma = 0 \, , & \tilde{\gamma} = -5.31247 \cdot 10^{-7} \, , \end{array} \qquad (6.3)$$

while for the Earth–Moon system we obtain the values below:

$$\begin{array}{lllll} L_1 : & \alpha = -0.16211 \, , & \beta = -0.14489 \, , & \gamma = -0.07263 \, , & \tilde{\gamma} = -0.11654 \, , \\ L_2 : & \alpha = -0.05470 \, , & \beta = -0.03612 \, , & \gamma = 0.08829 \, , & \tilde{\gamma} = -0.08898 \, , \\ L_3 : & \alpha = -0.00566 \, , & \beta = -0.00020 \, , & \gamma = 0.00021 \, , & \tilde{\gamma} = -0.00186 \, . \end{array} \qquad (6.4)$$

6.1. **First order estimate of the bifurcation value.** Hamilton's equations associated to the resonant Hamiltonian (6.2) are given by

$$\begin{aligned} \dot{I}_y &= 4\tilde{\gamma} I_y I_z \sin(2(\theta_y - \theta_z)) \\ \dot{I}_z &= -4\tilde{\gamma} I_y I_z \sin(2(\theta_y - \theta_z)) \\ \dot{\theta}_y &= \omega_y + [2\alpha I_y + \gamma I_z + 2\tilde{\gamma} I_z \cos(2(\theta_y - \theta_z))] \\ \dot{\theta}_z &= \omega_z + [2\beta I_z + \gamma I_y + 2\tilde{\gamma} I_y \cos(2(\theta_y - \theta_z))] \, . \end{aligned}$$

We immediately recognize that $\dot{I}_y + \dot{I}_z = 0$, so that the quantity $I_y + I_z$ becomes a constant of motion, up to the (resonant) normal form order. Following [24, 29], let us implement the canonical change of variables

$$\begin{aligned} \mathcal{E} &= I_y + I_z \\ \mathcal{R} &= I_y \\ \nu &= \theta_z \\ \psi &= \theta_y - \theta_z \ . \end{aligned} \tag{6.5}$$

We introduce the quantity

$$\delta = \omega_z \tilde{\delta} \equiv \omega_y - \omega_z$$

to which we refer as the *detuning,* which provides a measure of the distance in the frequency from the synchronous resonance. We assume that, in the neighborhood of the resonance, δ is a small quantity. Writing (6.2) in the transformed variables (6.5) and rescaling time by dividing the Hamiltonian by ω_z, we obtain

$$H_6^{(tr)}(\mathcal{E}, \mathcal{R}, \nu, \psi) = \mathcal{E} + \tilde{\delta}\mathcal{R} + a\mathcal{R}^2 + b\mathcal{E}^2 + c\mathcal{E}\mathcal{R} + d(\mathcal{R}^2 - \mathcal{E}\mathcal{R})\cos(2\psi) \ , \tag{6.6}$$

with $a = (\alpha + \beta - \gamma)/\omega_z$, $b = \beta/\omega_z$, $c = (\gamma - 2\beta)/\omega_z$, $d = -2\tilde{\gamma}/\omega_z$. Hamilton's equations associated to (6.6) become:

$$\begin{aligned} \dot{\mathcal{E}} &= 0 \\ \dot{\mathcal{R}} &= 2d\mathcal{R}(\mathcal{R} - \mathcal{E})\sin(2\psi) \\ \dot{\nu} &= 1 + 2b\mathcal{E} + c\mathcal{R} - d\mathcal{R}\cos(2\psi) \\ \dot{\psi} &= \tilde{\delta} + 2a\mathcal{R} + c\mathcal{E} + d(2\mathcal{R} - \mathcal{E})\cos(2\psi) \ . \end{aligned} \tag{6.7}$$

The main result of this section is the following proposition.

Proposition 13. *The energy level at which a bifurcation to halo orbits occurs is given, to first order in the detuning* δ*, by*

$$E = \frac{\omega_z \delta}{\gamma - 2(\alpha + \tilde{\gamma})} \ ,$$

where α*,* γ*,* $\tilde{\gamma}$ *have been introduced in* (6.2) *(and take the values* (6.3)*,* (6.4) *for the barycenter–Sun, Earth–Moon case, respectively).*

Proof. Having fixed a level value for $\mathcal{E}$, we consider the second and last equations in (6.7), which provide a one degree of freedom system in the variables $(\mathcal{R}, \psi)$, whose fixed points yield periodic orbits in the original system.

For $\mathcal{R} = \mathcal{E}$ and $\mathcal{R} = 0$ we obtain the normal modes, namely periodic orbits along just one of the axes. Precisely, if $\mathcal{R} = \mathcal{E}$, then the motion takes place along the y-axis, while if

$\mathcal{R} = 0$ the motion takes place along the z-axis. In terms of the original coordinates, the first normal mode provides a first-order approximation to the planar 'Lyapunov' periodic orbit; the second normal mode provides the approximation of the 'vertical' periodic orbit.

Assuming that $\psi \in (-\pi, \pi]$, we obtain also the following equilibrium positions of the system (6.7): we have $\dot{\mathcal{R}} = 0$ for $\psi = 0, \pi$ as well as for $\psi = \pm\frac{\pi}{2}$. Borrowing the terminology from galactic dynamics or molecular physics, we call these solutions *inclined* ($\psi = 0, \pi$) and *loop* ($\psi = \pm\frac{\pi}{2}$) orbits (see [24]). These trajectories arise from bifurcations of the normal modes, when entering the synchronous resonance; at that point the normal modes lose stability, though they can get again stable through a second bifurcation ([15]).

For $\psi = 0, \pi$ we obtain

$$\mathcal{R} = -\frac{\tilde{\delta} + (c-d)\mathcal{E}}{2(a+d)} \ ;$$

for $\psi = \pm\frac{\pi}{2}$ we get

$$\mathcal{R} = -\frac{\tilde{\delta} + (c+d)\mathcal{E}}{2(a-d)} \ .$$

Next, we observe that the transformation (5.3) is well defined for $I_y \geq 0$ and $I_z \geq 0$. On the other hand, from the first of (6.5) we obtain $0 \leq I_y \leq \mathcal{E}$ and $0 \leq I_z \leq \mathcal{E}$. As a consequence we get that $0 \leq \mathcal{R} \leq \mathcal{E}$; this inequality translates into the following constraints, which provide the existence of resonant orbits, bifurcating from the normal modes:

$$\mathcal{E} \geq \mathcal{E}_{iy} \equiv -\frac{\tilde{\delta}}{2a+c+d} \qquad \text{or} \qquad \mathcal{E} \geq \mathcal{E}_{iz} \equiv \frac{\tilde{\delta}}{-c+d} \tag{6.8}$$

for the inclined orbits and

$$\mathcal{E} \geq \mathcal{E}_{\ell y} \equiv -\frac{\tilde{\delta}}{2a+c-d} \qquad \text{or} \qquad \mathcal{E} \geq \mathcal{E}_{\ell z} \equiv \frac{\tilde{\delta}}{-c-d}$$

for the loop orbits: the quantity $\mathcal{E}_{\ell y}$ marks the occurrence of the bifurcation of the halo family from the planar Lyapunov orbit, which becomes unstable.

To determine the energy level at which the bifurcation takes place, we write the energy $\mathcal{E}_{\ell y}$ as a power series in $\tilde{\delta}$, namely

$$\mathcal{E}_N = \sum_{k=1}^{N} C_k \ \tilde{\delta}^k$$

for some real coefficients C_k and we look for a relation on the bifurcating normal mode between $\mathcal{E}$ and E. The estimate to first order is simply

$$E_1 = \omega_z \mathcal{E}_1 = \omega_z C_1 \tilde{\delta} \ ,$$

which, coming back to the original coefficients, gives the bifurcation value

$$E_1 = \frac{\omega_z \delta}{\gamma - 2(\alpha + \tilde{\gamma})} \ . \tag{6.9}$$

□

	barycenter–Sun	Earth–Moon
L_1 first order	0.3356	0.3069
L_2 first order	0.3391	0.3636
L_3 first order	0.3218	0.3354
L_1 second order	0.332612	0.306870
L_2 second order	0.335602	0.355552
L_3 second order	0.2871	0.2965
L_1 numerical	0.332820	0.306857
L_2 numerical	0.335743	0.355733
L_3 numerical	0.091962	0.298520

TABLE 2. Results for the bifurcation values of the barycenter–Sun and Earth–Moon systems. We report first and second order analytical estimates, as well as the numerical values obtained in [16, 14, 15].

For the collinear point L_1 of the barycenter–Sun system we obtain the first–order analytic estimate of the bifurcation threshold

$$E = 0.3356$$

to be compared with the value 0.332820 obtained numerically in [14]. Table 2 provides the results for the cases barycenter–Sun and Earth–Moon, and for all collinear points. For comparison the table provides also the numerical values: for the Earth-Moon system they are taken from [15] and converted by means of (2.6)–(2.7); for the barycenter–Sun system the numerical value for L_1 is taken from [14], while for L_2 and L_3 we refer to [16].

Remark 14. *i) A second bifurcation may occur when the Lyapunov orbit regains stability. From the point of view of numerical simulations, we recall that a second bifurcation is observed in* [15] *in the case of the Earth–Moon system. We stress that the value provided by the first of* (6.8) *is typically far from the numerical expectation and a higher order computation would be necessary to get reliable results. As an example, we mention that the value of $\mathcal{E}_{iy}$ in* (6.8) *amounts for L_1 to about 8.03 for the Earth–Moon system, while numerical experiments show that the second bifurcation occurs at 3.6640.*

A careful study of the occurrence of the second bifurcations requires different techniques and it is deferred to a later work.

ii) The discussion of the bifurcation around L_3 of the barycenter–Sun system is qualitatively different from the analysis of the bifurcation around L_1 and L_2, due to two main factors: first, the two frequencies are almost identical (indeed, in Table 1 the correct values are $\omega_y = 1.00000266$ and $\omega_z = 1.00000133$) and the coefficients of the normal form are quite small numbers (see the third line of (6.3)); second, the limit problem characterized by a vanishing perturbing parameter is drastically different from the problem with a non-zero perturbing parameter. This makes more difficult both the analytical estimates and the numerical computations (indeed, the value provided in Table 2 is obtained from halo orbits with very small amplitudes ([16])). The agreement between the analytical and numerical values is not satisfactory and further investigation is required for this special case, see also next item.

iii) In the Earth–Moon case the bifurcation of the halo orbits around L_3 is predicted with fairly good accuracy by the second order normal form theory (see Table 2). To evaluate the efficacy of a higher order normal form theory, we need a careful investigation of the optimal order of normalization around L_3 (which might be a very low order), but this task goes beyond the scopes of the present work and it might be demanding for small mass-ratios, like in the barycenter–Sun case.

6.2. **Second order estimate of the bifurcation value.** To get a higher order estimate of the bifurcation value, we need an explicit expression of the perturbing function to sixth order. To this end, we reduce also the term $\tilde{H}_5$ to the center manifold and we implement perturbation theory. Let us start from a generic expression of the Hamiltonian function written as

$$H(p,q) = H_2(p,q) + \sum_{n\geq 3} H_n(p,q) \ ,$$

where $H_2(p,q)$ denotes the quadratic part. Define a generating function $G = G(P,Q)$, that we expand up to the order six as

$$G(P,Q) = \sum_{i=3}^{6} G_i(P,Q) \ ,$$

where the G_i are homogeneous polynomials of degree i. Using Lie series ([12]) up to the order six the transformed Hamiltonian is given by

$$\hat{H} = \hat{H}_2 + \hat{H}_3 + \hat{H}_4 + \hat{H}_5 + \hat{H}_6 \ ,$$

where the quadratic term is unaltered, while the other functions can be computed through simple formulae providing $\hat{H}_j$ for $j \geq 3$ in terms of H_k with $k = 2, ..., j$ and G_ℓ with $\ell = 3, ..., j$. Again, we recall that the generating function is chosen so that the new Hamiltonian depends upon P_1 and Q_1 only through the product Q_1P_1; then, the center manifold corresponds to choose $Q_1P_1 = 0$. We finally obtain that the order five does not contribute to the average as well as to the resonant part, while at the order six we obtain the following expression:

$$\hat{H}_6 = \sum_{a,b,c,d=0}^{6} \alpha_{abcd}\, Q_2^a P_2^b Q_3^c P_3^d = \sum_{a,b,c,d=0}^{6} \alpha_{abcd}\, (-i)^{a+c} (I_y)^{\frac{a+b}{2}} (I_z)^{\frac{c+d}{2}} e^{i(a-b)\theta_y} e^{i(c-d)\theta_z}$$

for some coefficients α_{abcd} with $a + b + c + d = 6$. Up to the sixth order, the resonant Hamiltonian is finally given by

$$\begin{aligned} H_7^{(r)}(I_y, I_z, \theta_y, \theta_z) &= \omega_y I_y + \omega_z I_z + [\alpha I_y^2 + \beta I_z^2 + I_y I_z(\gamma + 2\tilde{\gamma}\cos(2(\theta_y - \theta_z)))] \\ &+ \alpha_{3300} I_y^3 + \alpha_{0033} I_z^3 + \alpha_{1122} I_y I_z^2 + \alpha_{2211} I_y^2 I_z \\ &+ 2\alpha_{2013} I_y I_z^2 \cos(2(\theta_y - \theta_z)) + 2\alpha_{3102} I_y^2 I_z \cos(2(\theta_y - \theta_z)) \,. \end{aligned} \tag{6.10}$$

The coefficients for the relevant cases associated to the Earth–Moon and barycenter–Sun systems are listed in Table 3.

	L_1 (BS)	L_2 (BS)	L_3 (BS)	L_1 (EM)	L_2 (EM)	L_3 (EM)
α_{3300}	-0.01761	-0.01779	$3.82055 \cdot 10^{-7}$	-0.01327	-0.01731	0.00021
α_{0033}	-0.01200	-0.01208	$-2.10074 \cdot 10^{-8}$	-0.00843	-0.01086	-0.00007
α_{1122}	-0.00045	0.00060	$1.53621 \cdot 10^{-7}$	-0.00231	0.01142	0.00052
α_{2211}	-0.00269	-0.00182	$8.68859 \cdot 10^{-7}$	-0.00295	0.00808	0.00237
α_{2013}	-0.02023	-0.02091	$-5.61015 \cdot 10^{-7}$	-0.01367	-0.02475	-0.00166
α_{3102}	-0.02188	-0.02252	$-9.1274 \cdot 10^{-7}$	-0.01576	-0.02621	-0.00254

TABLE 3. Coefficients of the second order normal forms associated to L_1, L_2, L_3 of the barycenter–Sun (BS) and Earth–Moon (EM) systems.

The main result of this section is the content of the following proposition.

Proposition 15. *The energy level at which a bifurcation to halo orbits occurs is given, to second order in the detuning* δ, *by*

$$E = \frac{\omega_z \delta}{\gamma - 2(\alpha + \tilde{\gamma})} + \left[\frac{\gamma - \alpha - 2\tilde{\gamma}}{(\gamma - 2(\alpha + \tilde{\gamma}))^2} - \omega_z \frac{\alpha_{2211} - 3\alpha_{3300} - 2\alpha_{3102}}{(\gamma - 2(\alpha + \tilde{\gamma}))^3} \right] \delta^2 \ ,$$

where α, γ, $\tilde{\gamma}$ *have been introduced in* (6.2) *(and take the values* (6.3), (6.4) *for the barycenter–Sun, Earth–Moon case, respectively), while* α_{2211}, α_{3300}, α_{3102} *have been introduced in* (6.10) *(and take the values given in Table 3).*

Proof. The procedure to get the second order estimate is the following. We express the Hamiltonian in terms of the variables (6.5) and we define the quantity $\mathcal{R}_s = \mathcal{R}_1 + \varepsilon^2 \mathcal{R}_2$, where ε is a book–keeping parameter (see, e.g., [18, 11]), which will be set to one. We implement now the change of coordinates (6.5); recalling (6.7), we have that $\mathcal{E}$ is constant and that the transformed Hamiltonian does not depend on ν, but just on $\mathcal{R}$ and ψ. Keeping the same name $H_7^{(r)}$ for the transformed Hamiltonian and following [18], we compute $\mathcal{R}_1$ as the solution of

$$\frac{\partial H_7^{(r)}}{\partial \mathcal{R}} \left(\mathcal{R}_1, \frac{\pi}{2} \right) = 0 \ ,$$

where the derivative has been computed for $\psi = \frac{\pi}{2}$. Then we compute $\frac{\partial H_7^{(r)}}{\partial \mathcal{R}}$ for $\psi = \frac{\pi}{2}$ and using $\mathcal{R}_s = \mathcal{R}_1 + \varepsilon^2 \mathcal{R}_2$, we expand it in series of ε^2 up to the fourth order. By setting the result to zero, we obtain the value of $\mathcal{R}_2$. Finally, restoring $\varepsilon = 1$ we obtain the solution as $\mathcal{R}_s = \mathcal{R}_1 + \mathcal{R}_2$, that we can express as a function of δ as $\mathcal{R} = A_1 + A_2\delta + A_3\delta^2$ for some real coefficients A_1, A_2, A_3. The existence condition $\mathcal{E} - \mathcal{R} > 0$ provides the inequality

$$\mathcal{E} > \mathcal{E}_2 \ ,$$

where the second order bifurcation value is (compare also with [25])

$$\mathcal{E}_2 = \mathcal{E}_1 - \frac{\alpha_{2211} - 3\alpha_{3300} - 2\alpha_{3102}}{(\gamma - 2(\alpha + \tilde{\gamma}))^3} \, \delta^2 \ . \tag{6.11}$$

Finally, after evaluating (6.10) for $\mathcal{R} = \mathcal{E}$ and $\psi = \frac{\pi}{2}$, we obtain the following expression for the energy on the normal mode:

$$E = \omega_z \left(1 + \tilde{\delta} \right) \mathcal{E} + \alpha \mathcal{E}^2 .$$

Using (6.11) we get the bifurcation energy of the halo at second order as

$$E_2 = E_1 + \left[\frac{\gamma - \alpha - 2\tilde{\gamma}}{(\gamma - 2(\alpha + \tilde{\gamma}))^2} - \omega_z \frac{\alpha_{2211} - 3\alpha_{3300} - 2\alpha_{3102}}{(\gamma - 2(\alpha + \tilde{\gamma}))^3} \right] \delta^2$$

with E_1 as in (6.9). □

For L_1 of the barycenter-Sun system we get

$$E_2 = 0.332612 \ .$$

This value must be compared with the numerical result obtained in [14]:

$$E_{bif}^{(num)} = 0.332820 \ ,$$

according to which for values less than this threshold the Lyapunov orbit is stable, while from such value halo orbits arise.

Similar computations have been performed for the other collinear point and for the Earth–Moon case, leading to the results shown in Table 2. We remark the striking agreement between the analytical results and the numerical estimates. We conclude by mentioning that, starting from our approach, we believe it is possible to extend the methods to encompass second bifurcations as well as secondary resonances, which have been studied numerically in [15].

Appendix A: proof of Proposition 2

The proof of Proposition 2 can be found in several papers (see, e.g., [17]). For self-consistency we give in this Appendix some details of the proof.

Let $P = (P_1, P_2, P_3)$, $Q = (Q_1, Q_2, Q_3)$ and let us introduce a generating function $G = G(P, Q)$, that we expand as a sum of homogeneous polynomials $G = \sum_{k \geq 3} G_k$ with

$$G_k(P, Q) = \sum_{k_p, k_q \in \mathbb{Z}^3, \ |k_p|+|k_q|=k} g_{k_p,k_q} P^{k_p} Q^{k_q} \ ,$$

where $|k_p| = \sum_{j=1}^{3} |k_{pj}|$ (similarly for k_q), while P^{k_p} stands for $P_1^{k_{p1}} P_2^{k_{p2}} P_3^{k_{p3}}$ (similarly for Q^{k_q}). At each order k the terms G_k are defined in order to separate the center and hyperbolic directions, so to obtain a first integral which admits the center manifold as level surface. This can be achieved by eliminating all monomials such that the first component of k_p is different from the first component of k_q, say $k_{p1} \neq k_{q1}$. Precisely, denote by H_{2q} the quadratic part in (4.1). The generating function G induces a transformation of coordinates, such that the new Hamiltonian $\hat{H}$ is given by

$$\hat{H} = H_2^{(c)} + \{H_2^{(c)}, G\} + \frac{1}{2!}\{\{H_2^{(c)}, G\}, G\} + ... \ ,$$

where $\{\cdot, \cdot\}$ denotes the Poisson brackets. Let us start to determine the third order term G_3 of G. Let $\hat{H} = \sum_{k \geq 2} \hat{H}_k$, where $\hat{H}_k$ are homogeneous polynomials of degree k. Equating terms of the same degree in P, Q, we obtain that

$$\begin{aligned}
\hat{H}_2 &= H_{2q} \\
\hat{H}_3 &= H_3 + \{H_{2q}, G_3\} \\
\hat{H}_4 &= H_4 + \{H_3, G_3\} + \frac{1}{2!}\{\{H_{2q}, G_3\}, G_3\} \ , ... \qquad (6.12)
\end{aligned}$$

We determine G_3 is such a way to eliminate all monomials of the form $P^{k_p}Q^{k_q}$ with $k_{p1} \neq k_{q1}$. Expanding H_3 as

$$H_3(P,Q) = \sum_{k_p,k_q\in\mathbb{Z}^3,\ |k_p|+|k_q|=3} h^{(3)}_{k_p,k_q}\ P^{k_p}Q^{k_q}\ ,$$

then from the second of (6.12) we obtain that G_3 is given by

$$G_3(P,Q) = -\sum_{k_p,k_q\in\mathbb{Z}^3,\ |k_p|+|k_q|=3,\ k_{p1}\neq k_{q1}} \frac{h^{(3)}_{k_p,k_q}}{\langle k_p - k_q, \omega\rangle}\ P^{k_p}Q^{k_q}\ ,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product and $\omega \equiv (\lambda_x, i\omega_y, i\omega_z)$. Thus, we have obtained that the new Hamiltonian has the desired form (4.2) up to the third order. Iterating the procedure up to the order N and determining G_4, ..., G_N as we did for G_3, we obtain the Hamiltonian (4.2), where the polynomials $\tilde{H}_n$ will depend on Q_1, P_1, only through the product Q_1P_1.

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References

[1] V. I. Arnold, *Proof of a Theorem by A.N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian,* Russ. Math. Surveys **18**, 13–40 (1963)
[2] D. Brouwer, G. M. Clemence, *Methods of Celestial Mechanics,* Academic Press, New York and London (1961)
[3] R. Calleja, A. Celletti, R. de la Llave, *A KAM theory for conformally symplectic systems: efficient algorithms and their validation*, J. Differential Equations **255**, n. 5, 978–1049 (2013)
[4] A. Celletti, *Analysis of resonances in the spin-orbit problem in Celestial Mechanics: The synchronous resonance (Part I),* Journal of Applied Mathematics and Physics (ZAMP), **41**, 174–204 (1990)
[5] A. Celletti, *Stability and Chaos in Celestial Mechanics*, Springer-Verlag, Berlin; published in association with Praxis Publishing Ltd., Chichester, ISBN: 978-3-540-85145-5 (2010)
[6] A. Celletti, L. Chierchia, *A Constructive Theory of Lagrangian Tori and Computer–assisted Applications*, Dynamics Reported (C.K.R.T. Jones, U. Kirchgraber, H.O. Walther Managing Editors), Springer–Verlag, **4** (New Series), 60–129 (1995)
[7] A. Celletti, L. Chierchia, *KAM tori for N-body problems: a brief history,* Cel. Mech. Dyn. Astr. **95**, 117-139 (2006)
[8] A. Celletti, L. Chierchia, *KAM Stability and Celestial Mechanics,* Memoirs American Mathematical Society **187**, no. 878 (2007)

[9] A. Celletti, C. Lhotka, *Transient times, resonances and drifts of attractors in dissipative rotational dynamics*, Commun. Nonlinear Sci. Numer. Simulat. **19**, n. 9, 3399-3411 (2014)
[10] C. C. Conley, *Low energy transit orbits in the restricted three-body problem,* SIAM J. Appl. Math. **16**, n. 4, 732-746 (1968)
[11] J. J. Duistermaat, *Bifurcation of periodic solutions near equilibrium points of Hamiltonian systems,* Lecture Notes in Mathematics, **1057**, 57–105, Springer-Verlag (1984)
[12] S. Ferraz–Mello, *Canonical Perturbation Theories,* Springer–Verlag (2007)
[13] A. Giorgilli, L. Galgani, *Rigorous estimates for the series expansions of Hamiltonian perturbation theory,* Cel. Mech. **37**, 95–112 (1985)
[14] G. Gómez, À. Jorba, J. Masdemont, C. Simó, *Study refinement of semi-analytical Halo orbit theory,* ESOC Contract 8625/89/D/MD(SC), Final Report (1991)
[15] G. Gómez, J. M. Mondelo, *The dynamics around the collinear equilibrium points of the RTBP,* Physica D **157**, 283–321 (2001)
[16] G. Gómez, J. M. Mondelo, Private communication (2014)
[17] À. Jorba, J. Masdemont, *Dynamics in the center manifold of the collinear points of the restricted three body problem,* Physica D **132**, 189–213 (1999)
[18] J. Henrard, *Periodic orbits emanating from a resonant equilibrium,* Cel. Mech. **1**, 437–466 (1970)
[19] E. Kolemen, N. J. Kasdin, P. Gurfil, *Multiple Poincaré sections method for finding the quasiperiodic orbits of the restricted three body problem,* Cel. Mech. Dyn. Astr. **112**, 47–74 (2012)
[20] A. N. Kolmogorov, *On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian,* Dokl. Akad. Nauk. SSR **98**, 527–530 (1954)
[21] W. S. Koon, M. W. Lo, J. E. Marsden, S. D. Ross, *Heteroclinic connections between periodic orbits and resonance transitions in Celestial Mechanics,* Chaos **10**, n. 2, 427-469 (2000)
[22] R. de la Llave, A. Gonzàlez, À. Jorba, J. Villanueva, *KAM theory without action–angle variables,* Nonlinearity **18**, no. **2**, 855-895 (2005)
[23] R. de la Llave, D. Rana, *Accurate strategies for small divisor problems,* Bull. Amer. Math. Soc. (N. S.) **22**, 1, 85–90 (1990)
[24] A. Marchesiello, G. Pucacco, *Relevance of the 1:1 resonance in galactic dynamics,* Eur. Phys. J. Plus **126**: 104 (2011)
[25] A. Marchesiello, G. Pucacco, *Equivariant singularity analysis of the 2:2 resonance*, Nonlinearity, **27**, 43–66 (2014)
[26] J. Moser, *On invariant curves of area–preserving mappings of an annulus,* Nach. Akad. Wiss. Göttingen, Math. Phys. Kl. II **1**, 1–20 (1962)
[27] C. D. Murray, S. F. Dermott, *Solar system dynamics*, Cambridge University Press (1999)
[28] G. Pucacco, D. Boccaletti, C. Belmonte, *Periodic orbits in the logarithmic potential,* Astr. & Astroph. **489**, 1055–1063 (2008)
[29] G. Pucacco, A. Marchesiello, *An energy-momentum map for the time-reversal symmetric 1:1 resonance with* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *symmetry*, Physica D **271**, 10–18 (2014)
[30] D. L. Richardson, *Analytic construction of periodic orbits about the collinear points,* Celestial Mechanics **22**, 241–253 (1980)
[31] C. Simó, *Effective Computations in Celestial Mechanics and Astrodynamics,* in "Modern Methods of Analytical Mechanics and their Applications", V. V. Rumyantsev and A. V. Karapetyan eds., CISM Courses and Lectures 387, 55-102, Springer, Vienna (1998)
[32] V. Szebehely, *Theory of Orbits,* Academic Press, New York and London (1967)

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