DISCONTINUOUS OBSERVABLES AS AN OBSTRUCTION FOR SMALL ESSENTIAL SPECTRAL RADIUS

OLIVER BUTTERLEY AND DANIEL SMANIA

ABSTRACT. We show that for a very wide class of Banach spaces of functions on $[0,1]$ there are intrinsic lower bounds for the essential spectral radius of the transfer operator associated to piecewise smooth expanding maps. The class of Banach spaces studied includes any reasonable space which permits discontinuities.

1. INTRODUCTION

Given a chaotic system with some degree of hyperbolicity it is natural to investigate the statistical properties of the system. This involves finding and studying relevant invariant measures, proving CLT, LLT, large deviations, estimating the decay of correlations, studying zeta functions, etc. Study of the transfer operators associated to dynamical systems is a very convenient and powerful way to investigate statistical properties. Such an idea goes back at least to the use of the Koopman operator by von Neumann to prove the mean ergodic theorem. Subsequently Sinai and others in the Russian school developed theory for the Koopman operator acting on L^2 and the connection with ergodicity and mixing (see e.g., [14]). Soon it was realised that it was useful to study the adjoint of the Koopman operator and this object became known as the transfer operator. Amongst this period of development is the work of Lasota & Yorke [26], Ruelle [31], Keller [23, 22], Baladi & Keller [2], Keller & Liverani [24]. See the books by Baladi [3, 4] for a more complete history and the notes of Liverani [28] for an overview. In many cases the use of transfer operators was done by reducing the system to symbolic dynamics and then using standard function spaces (see Bowen [8] and Parry and Pollicott [30]). On the other hand, in some cases one could choose dynamically sensible choices of Banach space on which to study the transfer operator without the need of coding and potential loss of information. In the case of hyperbolic systems, as opposed to expanding systems, this led to the need of anisotropic Banach spaces of distributions to match with the distinctly different behaviour in different directions (see Blank, Keller, and Liverani[7], Gouëzel and Liverani [20] and Baladi and Tsujii [5]). At present there are many different Banach spaces available for studying many different piecewise expanding maps [32] and the approach has shown many great successes.

The spectrum of the transfer operator will consist of essential spectrum and a set of isolated eigenvalues, The spectrum depends on the choice of Banach space. In order to be useful for studying the system the essential spectrum radius needs to be smaller than the

2020 *Mathematics Subject Classification.* 37A05, 37A25, 37C30 .

Key words and phrases. transfer operator, essential spectrum, spectral gap, piecewise expanding map, discontinuities.

spectral radius. For some systems, those with high degree of regularity, there exists a choice of Banach space so that the essential spectral radius is arbitrarily small. For example this has been shown for smooth expanding maps in any dimension (see Collet and Isola [13], Gundlach and Latushkin [21]) and pseudo Anosov diffeomorphisms by Faure, Gouëzel and Lanneau [18]. This corresponds to being able to give a precise description of decay of correlations in terms of resonances [18, Definition 1.1.]. Choosing a Banach space which is larger than required can lead to the essential spectrum being large. For instance, studying the transfer operator acting on L^p with $1 \leq p < \infty$ an approximate eigenfunction argument can be used to show that the spectrum of the transfer operator is the entire disc, i.e., the essential spectrum is equal to the spectral radius [10, Footnote 8]. One direction of interest is to identify the isolated points of spectrum (see [25, 11, 27] and references within). Indeed the isolated points of spectrum can be shown to be essentially independent on the choice of Banach space [6].

The focus of this present work is in the other direction, on the essential spectrum and understanding if it can be reduced in a given situation. Showing that the essential spectral radius is small is intimately connected with showing a large meromorphic extension of the zeta function (see, e.g., [19, 4]).

Given the huge availability of different Banach spaces and the unlimited creative possibility it is natural to ask if, in a given situation, there is a chance of finding a better Banach space in order to reduce the essential spectral radius. As hinted above, if the Banach space is too large the essential spectral radius is large. On the other hand the Banach space must be sufficiently large in order to be useful, typically one would want it to include at least all smooth observables. It was shown by B., Canestrari & Jain [10] that, for the case of smooth interval maps with discontinuities, the essential spectrum is large for a very large class of observables (see [9] for a higher dimensional extension). The only requirement on the Banach space (apart from containing $\mathscr{C}^\infty$ and being invariant under the dynamics) was that it had to be continuously embedded in L^∞. This, as already noted in that work, is unfortunate since there are many spaces, for example Besov spaces (see Arbieto and S. [1], S. [32] and Nakano and Sakamoto [29]) and Sobolev spaces (see Thomine [34]) which seem like a reasonable choices but are not embedded in L^∞. Discontinuities are natural in physical systems (e.g. see Chernov and Markarian [12]) but, additionally, it can be argued that some physically relevant observables are unbounded and such should therefore be permitted in the analysis. Rectifying this gap is a major motivation in the present work.

Here, we will show that *for a broad class of Banach spaces of observables, satisfying fairly minimal conditions — in particular, allowing simple discontinuities in the observables — the action of the transfer operator has a large essential spectral radius.* These classes of Banach spaces are sufficiently general to include unbounded observables.

In Theorem A we give conditions which imply that Besov space is embedded in a given Banach space and also derive a lower estimate for the essential spectral radius. Theorem B applies to linear expanding maps and again gives a lower bound for the essential spectral radius but with weaker assumptions on the Banach space. On the other hand, Theorem C requires an assumption of the topological pressure in order to give a lower bound (an assumption that holds for C^∞ maps). In Theorem D we apply the ideas to the case where the norm is *natural* (defined in Section 2.2) and we obtain a lower estimate on the essential spectral radius.

Let $I = [0,1]$ and

$$T\colon \cup_i I_i \to I$$

where $\{I_i\}_i$ is a finite partition of I by open intervals, and $T\colon I_i \to T(I^i)$ extends to a C^1 diffeomorphism on $\overline{I}_i$. Then the transfer operator L with respect to the is a bounded operator acting on $L^1(I)$. Denote by $r(L, L^1(I))$ its spectral radius.

1.1. **Growth of derivatives and topological entropy.** Let

$$\mathscr{P}^k = \{I_i^k\}_i$$

be the intervals of monotonicity of T^k. Let

$$\theta_i^k = \sup_{x\in I_i^k} \frac{1}{|Df^k(x)|}.$$

Notice that $\Theta^k(\beta) = \sum_i (\theta_i^k)^\beta$ is sub-multiplicative, $\Theta^{k+j}(\beta) \le \Theta^k(\beta)\Theta^j(\beta)$, so we can define

$$\Theta^\infty(\beta) = \lim_k (\Theta^k(\beta))^{1/k}.$$

Remark 1.1.1. An easy upper bound is

$$\Theta^\infty(\beta) \le \#\mathscr{P}^1 \lim_k \Big(\sup_{x\in I} \frac{1}{|Df^k(x)|^\beta}\Big)^{1/k}.$$

If T is continuous one can be more precise

$$\Theta^\infty(\beta) \le e^{h_{top}(T)} \lim_k \Big(\sup_{x\in I} \frac{1}{|Df^k(x)|^\beta}\Big)^{1/k}.$$

Here $h_{top}(T)$ is the topological entropy of T. If T is an $C^{1+\beta}$ expanding map on the circle we observe that

$$\lim_k \Big(\sup_{x\in I} \frac{1}{|Df^k(x)|^\beta}\Big)^{1/k} = e^{-\beta M},$$

where

$$M = \inf_{\mu inv.prob.T} \int \ln|Df|\, d\mu.$$

Indeed in this case

$$\Theta^\infty(\beta) = e^{P_{top}(-\beta \ln|Df|)},$$

where $P_{top}(\phi)$ denotes the topological pressure of ϕ.

2. Main results

2.1. **Lower bound for essential spectrum radius.** Discontinuities of a dynamical system is a serious obstruction for small essential spectrum radius of the transfer operator L acting on Banach spaces of bounded functions (see B., Canestrari and Jain [10]). Here we are going to see that if the transfer operator acts on a space of functions B that contains the simplest of discontinuous functions, and satisfies a basic norm estimate, then B is indeed quite large and its essential spectrum cannot be small.

Given $s \in (0,1)$, let $B^s_{1,1}$ be the classical space of Besov function on the interval I. Denote by $|A|$ the Lebesgue measure of the set A.

Theorem A. *Suppose that*

- *The map $T\colon I_i \to T(I^i)$ extends to a $C^{1+\beta}$ diffeomorphism on $\overline{I}_i$, for every i, with $\beta > 0$.*
- *The Lebesgue measure m on I is T-invariant.*
- *The transfer operator L associated with T, with respect to the Lebesgue measure m, preserves a Banach space of functions B that is continuously embedded in $L^1(I)$ and the operator $L\colon B \to B$ is bounded.*
- *There are $C \geq 0$ and $s \in (0,1)$ such that for every interval $J \subset I$ we have that $1_J \in B$ and*

$$||1_J||_B \leq C|J|^{1-s} \tag{2.1.2}$$

Then the Besov space $B^s_{1,1}$ is continuously embedded in B. If furthermore $s < \beta$ we have

$$r_{ess}(T,B) \geq 1/\Theta^\infty(1-s)$$

and every $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\Theta^\infty(1-s)$ is an eigenvalue of L on B with an infinite dimensional eigenspace.

In the piecewise linear case, we can weaken the assumption and still obtain a lower bound for the essential spectrum radius.

Theorem B. *Suppose that*

- *The Lebesgue measure m on I is T-invariant.*
- *T is linear on each branch and it has k branches.*
- *The transfer operator L associated with T, with respect to the Lebesgue measure m, preserves a Banach space of functions B that is continuously embedded in $L^1(I)$ and the operator $L\colon B \to B$ is bounded.*
- *There is $C \geq 0$ such that for every interval $J \subset I$ we have that $1_J \in B$ and*

$$||1_J||_B \leq C. \tag{2.1.3}$$

Then

$$r_{ess}(L,B) \geq 1/k$$

and every $\lambda \in \mathbb{C}$ which satisfies $|\lambda| < 1/k$ is an eigenvalue of L on B with an infinite dimensional eigenspace.

Theorem C. *Suppose that*

- *The Lebesgue measure m on I is T-invariant.*
- *T is piecewise C^{r+1} and expanding, for some $r > 0$, Markovian with k branches, all of them onto, and*

$$P_{top}(-(r+1)\log|DT|) < \frac{1}{k}. \tag{2.1.4}$$

- *The transfer operator L associated with T, with respect to the Lebesgue measure m, preserves a Banach space of functions B that is continuously embedded in $L^1(I)$ and the operator $L\colon B \to B$ is bounded.*
- *There is $C \geq 0$ such that for every interval $J \subset I$ we have that $1_J \in B$ and*

$$||1_J||_B \leq C. \tag{2.1.5}$$

Then

$$r_{ess}(L,B) \geq \frac{1}{k}.$$

Remark 2.1.6. Condition (2.1.4) is satisfied

- for every T that is piecewise expanding C^{r+1} and Markovian with k branches, and
$$\sup_{x\in I} \frac{1}{|DT(x)|^r} < \frac{1}{k}.$$
- for every T that is piecewise expanding C^{∞} and Markovian with k branches, provided r is large enough.

Remark 2.1.7. Every C^r Markovian expanding map, $r > 1$, with only full branches, is conjugate by a C^r conjugacy with a C^r Markovian expanding map that preserves the Lebesgue measure, so this assumption is not strong.

Remark 2.1.8. Note that if $L\phi = \lambda\phi$ with $\phi \in B \setminus \{0\}$ and $\psi(x) = \overline{\phi}(x)/|\phi(x)|$ if $\phi(x) \neq 0$, and $\psi(x) = 0$ otherwise, then

$$\int \psi \circ T^n \phi \, dm = \lambda^n \int |\phi| \, dm,$$

so λ-eigenvectors are obstructions for decay of correlations faster than $|\lambda|$.

2.2. **Natural spaces of functions.** Conditions (2.1.2) and (2.1.3) appear challenging to verify. However, most Banach function spaces on $\mathbb{R}$ discussed in the literature exhibit good behaviour with respect to translation and scaling, which will significantly aid our analysis.

Almost Homogeneity and invariance by translations. We will say that a pseudo-norm $n(\cdot)$ on a space of functions on the interval I is *purely natural* if there is $t \in \mathbb{R}$ and $C > 0$ such that if $u\colon \mathbb{R} \to \mathbb{R}$ is an invertible affine transformation and $\phi\colon I \to \mathbb{C}$ satisfies

$$u^{-1}(\text{supp } \phi) \subset I$$

then $\phi \circ u \in B$ and

$$\frac{1}{C}|u'|^t n(\phi) \leq n(\phi \circ u) \leq C|u'|^t n(\phi).$$

The parameter t will be called the degree of homogeneity of u.

A pseudo-norm n is called *natural* it is a finite sum of purely natural pseudo-norms, that is, there are purely natural pseudo-norm n_i, with $i \leq j$, and degree of homogeneity t_i such that

$$n(\phi) = \sum_{i \leq j} u_i(\phi). \tag{2.2.9}$$

Remark 2.2.10. Many norms and pseudo-norms of spaces of function of an interval are natural. The sup norm, the L^p norms, the p-bounded variation pseudo-norm, the Hölder norm, C^k norms, Sobolev norms and Besov norms.

A nice application of the previous results is

Theorem D. *Suppose that*

- T *is piecewise* C^{r+1} *and expanding acting on the interval* $I = [0,1]$, *for some* $r > 0$, *Markovian with* k *branches, all of them onto, and*

$$P_{top}(-(r+1)\log|DT|) < \frac{1}{k}. \tag{2.2.11}$$

- *The Lebesgue measure* m *on* I *is* T*-invariant.*
- B *is a Banach space of functions continuously embedded in* $L^1(I)$ *whose norm* $||\cdot||_B$ *is natural.*
- *The transfer operator* L *of* T *with respect to the Lebesgue measure* m *keeps a Banach space of functions* B *invariant, and* $L\colon B \to B$ *is a bounded operator with spectral gap.*
- *For every interval* $J \subset I$ *we have that* $1_J \in B$.

Then one of the following cases occurs

I. *There is* $C > 0$ *such that*
$$\frac{1}{C} \leq ||1_P||_B \leq C$$
for every interval $P \subset I$, *and*
$$r_{ess}(L,B) \geq 1/k.$$

II. *There is* $s \in (0,1)$ *and* $C > 0$ *such that*
$$\frac{1}{C}|P|^{1-s} \leq ||1_P||_B \leq C|P|^{1-s}$$
for every interval $P \subset I$, *the Besov space* $B^s_{1,1}$ *is continuously embedded in* B *and*
$$r_{ess}(L,B) \geq \frac{1}{\Theta^\infty(1-s)}.$$

Remark 2.2.12. Both cases occur. If B represents the space of bounded variation functions on I and $T(x) = 2x \mod 1$ on $I = [0,1]$, then we are in Case I. On the other hand, if we consider $B^s_{1,1}$ with $s \in (0,1)$, we are in Case II (see Nakano and Sakamoto [29] and S. [32]). Furthermore, note that if $B \subset L^\infty$, we must be in case I, as $B^s_{1,1}$ includes unbounded functions.

3. PRELIMINARIES

3.1. p**-bounded variation.** Let $J = [a,b] \subset I$. The p-variation of a function $\psi\colon I \to \mathbb{C}$ on the interior of J is

$$v_p(\psi, J) = \sup(\sum_{i=0}^{n} |\phi(x_{i+1}) - \phi(x_i)|^p)^{1/p},$$

where the sup runs over all possible finite increasing sequences

$$a < x_0 < x_1 < \cdots < x_{n-1} < x_n < b.$$

Lemma 3.1.13. *The pseudo-norm v_p has the following properties properties*

- *v_p is invariant with respect to continuous change of coordinates: if $u\colon P \to J$ is a homeomorphism then*

$$v_p(\phi, J) = v_p(\phi \circ u, P).$$

- *if J_1 and J_2 are intervals such that $J_1 \cap J_2$ is just a point then*

$$v_p^p(\psi, J_1) + v_p^p(\psi, J_2) \leq v_p^p(\psi, J_1 \cup J_2).$$

3.2. **Besov space** $B^s_{1,1}(I)$**.** Given $s \in (0,1)$, consider the space of all functions $\psi \in L^1(I)$ that can be written as

$$\psi = \sum_{n=0}^{\infty} c_n |Q_n|^{s-1} 1_{Q_n}, \tag{3.2.14}$$

where this series converges in $L^1(I)$ and Q_n are subintervals of I. If we endow $B^s_{1,1}$ with the norm

$$|\phi|_{B^s_{1,1}(I)} = \inf \sum_n |c_n|,$$

where the infimum runs over all possible representations, $B^s_{1,1}$ is a Banach space.

Those spaces were introduced by de Souza [16]. There are indeed many way to describe this space (see de Souza [17]). In particular it coincides with the classical Besov spaces $B^s_{1,1}(I)$. The proof of the following proposition is quite simple.

Proposition 3.2.15. *Let B be as in the Theorem A. There is a continuous embedding of $B^s_{1,1}$ in B, that is, $B^s_{1,1} \subset B$ and there is C such that*

$$||\phi||_B \leq C||\phi||_{B^s_{1,1}}.$$

Proposition 3.2.16. *Given $1/p > s$ there is $C \geq 0$ such that the following holds. For every function $\psi\colon I \to \mathbb{C}$ that vanish outside a closed interval J and $v_p(\psi, J) < \infty$ we have that $\psi \in B^s_{1,1}(I)$ and*

$$|\psi - m(\psi, J) 1_J|_{B^s_{1,1}(I)} \leq C|J|^{1-s} v_p(\psi, J).$$

Here

$$m(\psi, J) = \frac{1}{|J|}\int_J \psi \, dm.$$

Note that C does not depend on J.

Proof. We do a argument similar to the proof of Proposition 16.3 in S. [33]. Let $\mathscr{D}^k$ be the partition of J by intervals of length $|J|/2^{-k}$. Then

$$\psi = \lim_k \psi_k,$$

in $L^1(m)$, where

$$\psi_k = \sum_{P\in\mathscr{D}^k} m(\psi, P)1_P.$$

Note that

$$\psi_0 = \frac{1}{|J|}\int_J \psi \, dm.$$

We claim that sequence converges in $B^s_{1,1}$. Indeed

$$\begin{aligned}
&\psi_{k+1} - \psi_k \\
&= \sum_{P\in\mathscr{D}^{k+1}} m(\psi, P)1_P - \sum_{Q\in\mathscr{D}^k} m(\psi, Q)1_Q \\
&= \sum_{Q\in\mathscr{D}^k} \sum_{\substack{P\in\mathscr{D}^{k+1}\\ P\subset Q}} (m(\psi, P) - m(\psi, Q))1_P
\end{aligned}$$

and

$$|m(\psi, P) - m(\psi, Q)| \leq 2\upsilon_p(\psi, Q),$$

so

$$\begin{aligned}
&\sum_{k\geq 0} |\psi_{k+1} - \psi_k|_{B^s_{1,1}(I)} \\
\leq &\sum_{k\geq 0} \sum_{Q\in\mathscr{D}^k} |Q|^{1-s} 4\upsilon_p(Q) \\
\leq &4\sum_{k\geq 0} \Big(\sum_{Q\in\mathscr{D}^k} |Q|^{\frac{(1-s)p}{p-1}} \Big)^{\frac{p-1}{p}} \Big(\sum_{Q\in\mathscr{D}^k} \upsilon_p^p(Q) \Big)^{1/p} \\
&\leq 4\sum_{k\geq 0} \Big(\sum_{Q\in\mathscr{D}^k} |Q|^{1+\frac{1-sp}{p-1}} \Big)^{\frac{p-1}{p}} \Big(\sum_{Q\in\mathscr{D}^k} \upsilon_p^p(Q) \Big)^{1/p} \\
&\leq C|J|^{1-s}\upsilon_p(Q).
\end{aligned}$$

□

4. Proof of the main results

Proof of Theorem A. Let $\psi \in L^\infty(I)$, with $\psi \neq 0$ and $L\psi = 0$. Note that

$$|\psi \circ T^\ell|_{L^p(I)} \leq |\psi|_{L^\infty}$$

for every $p \in [1, \infty]$. Given $z \in \mathbb{C}$ with $|z| < 1$, define

$$h_z = \sum_{\ell=0}^{\infty} z^\ell \psi \circ T^\ell \in L^p(I), \ with \ p \in [1, \infty]. \tag{4.1.17}$$

Let

$$\mathscr{F}_\ell = \{\phi \circ T^\ell : \ \phi \in L^\infty(I) \ and \ L\phi = 0\}.$$

Then $\mathscr{F}_\ell$ are mutually orthogonal on $L^2(I)$ (see de Lima and S. [15]). In particular $h_z \neq 0$ since

$$\{\psi \circ T^\ell\}_\ell$$

is an orthogonal set in $L^2(I)$. Since m is T-invariant we have that $L(\phi \circ T) = \phi$ for every $\phi \in L^\infty(I)$, and an easy calculation shows that

$$Lh_z = zh_z.$$

That is a classical construction of eigenvalues of L (see Collet and Isola[13]) and indeed it shows that every $|z| < 1$ is an eigenvalue of L with an infinite dimensional eigenspace on $L^p(I)$, for every $p \in [1,\infty]$.

Since T is piecewise $C^{1+\beta}$ we can choose disjoint closed intervals $I_1, I_2 \subset I$ and β-Hölder functions $\phi_i \colon J_i \to \mathbb{R}$, $i = 1,2$, such that

$$\psi = \phi_1 1_{J_1} - \phi_2 1_{J_2} \in L^\infty(I),$$
$$v_{1/\beta}(\phi_i, J_i) < \infty,$$

and $L\psi = 0$, with $\psi \neq 0$. Indeed note that the set of all possible such ψ is infinite dimensional. We have that

$$(\phi_j 1_{I_j}) \circ T^k = \sum_i \phi_j \circ T^k 1_{Q_i^k},$$

where $Q_i^k \subset I_i^k$ is an interval mapped by T^k monotonically to a sub-interval of I_j. In particular

$$|Q_i^k| \le \theta_i^k,$$
$$\sup_i \sup_k v_{1/\beta}(\phi_i \circ T^k, Q_i^k) < \infty,$$

In the last estimate we used Lemma 3.1.13. Since $\beta > s$, due Proposition 3.2.16 we have

$$|\psi \circ T^k|_{B^s_{1,1}} \le C \sum_i (\theta_i^k)^{1-s} = C\Theta^k(1-s).$$

So if $|z| < 1/\Theta^\infty(1-s)$ we have that $h_z \in B^s_{1,1} \subset B$. Consequently

$$r_{ess}(T,B) \ge 1/\Theta^\infty(1-s)$$

and

$$r_{ess}(T, B^s_{1,1}) \ge 1/\Theta^\infty(1-s).$$

□

Proof of Theorem B. We use the same notation as in the proof of Theorem A. Since T is linear on each branch we can choose disjoint intervals I_1 and I_2 in I and *positive constants* c_1 and c_2 such that $\psi = c_1 1_{I_1} - c_2 1_{I_2} \in L^\infty(I)$, $L\psi = 0$ and $\psi \neq 0$. In this case $\psi \circ T^k$ is a linear combinations of characteristic functions of intervals. So for $|z| < 1/k$ we have

$$h_z = \sum_{\ell=0}^{\infty} z^\ell \psi \circ T^\ell. \tag{4.1.18}$$

indeed converges in B since

$$|\psi \circ T^\ell|_B \leq Ck^\ell.$$

So $r_{ess}(L,B) \geq 1/k$. □

Proof of Theorem C. Suppose, for the sake of contradiction, that

$$r_{ess}(L,B) < 1/k. \tag{4.1.19}$$

Since T is a piecewise C^{r+1} markovian map, by Collet and Isola [13] (see also Baladi [3]) we have that

$$r_{ess}(L, C^r(I)) = \exp(P_{top}(-(r+1)\log|DT|)) < \frac{1}{k}.$$

Let $z \in \mathbb{C}$ be such that

$$\max\{\exp(P_{top}(-(r+1)\log|DT|)), r_{ess}(L,B)\} < |z| < \frac{1}{k}. \tag{4.1.20}$$

In particular there is a finite dimensional subspace $E \subset C^r(I)$ and a closed subspace $F \subset C^r(I)$ such that

i. E and F are L^n-invariant for every $n > 0$.
ii. $r(L^n, F) < |z|^n$ for every $n > 0$.
iii. $B = E \oplus F$

and there is a finite dimensional subspace $\hat{E} \subset B$ and a closed subspace $\hat{F} \subset B$ such that

iv. $\hat{E}$ and $\hat{F}$ are L^n-invariant for every $n > 0$.
v. $r(L^n, \hat{F}) < |z|^n$ for every $n > 0$.
vi. $B = \hat{E} \oplus \hat{F}$

Let $\mathscr{P}^n$ be the partition of I by the intervals of monotonicity of T^n and $\mathscr{F}^n \subset B$ be the space of functions that are constant of each element of $\mathscr{P}^n$. Note that if $L^{n_0}\mathscr{F}^{n_0} \subset C^r(I)$.

Claim A. If $n_0 > \dim E + \dim \hat{E} + 1$ *then there is* $\psi \in \mathscr{F}^{n_0}$ *that is not constant,* $\psi \in \hat{F}$, *and* $L^{n_0}\psi \in F$.
Let π_E, π_F be the orthogonal projections associated with the decomposition $C^r(I) = E \oplus F$, and $\pi_{\hat{E}}, \pi_{\hat{F}}$ be the orthogonal projections associated with the decomposition $B = \hat{E} \oplus \hat{F}$. Define

$$H\colon \mathscr{F}^{n_0} \to E \oplus \hat{E}$$

as $H(\psi) = (\pi_E(L^{n_0}\psi), \pi_{\hat{E}}(\psi))$. If $n_0 > \dim E + \dim \hat{E} + 1$ we have that $\dim Ker\ H \geq 2$, so it contains a non-constant function. This finishes the proof of the claim. Choose n_0 and ψ as in Claim A.

Claim B. For every $n > 0$ *we have that*

$$h_{z,n} = \sum_{\ell=0}^{\infty} z^{\ell n} \psi \circ T^{\ell n} \tag{4.1.21}$$

converges in B *and*

$$L^n h_{z,n} = z^n h_{z,n} + L^n \psi.$$

Moreover for large n we have that the image of $h_{z,n}$ is a Cantor set (up to a countable set).

Note that $\psi \circ T^\ell$ is a linear combination of characteristic functions of intervals and we have

$$h_{z,n} = \sum_{\ell=0}^{\infty} z^{\ell n} \psi \circ T^{\ell n}. \tag{4.1.22}$$

indeed converges in B since by (2.1.5)

$$|\psi \circ T^\ell|_B \leq C k^\ell.$$

One can easily check that

$$-z^{-n}\psi = h_{z,n} \circ T^n - z^{-n} h_{z,n}, \tag{4.1.23}$$

and $h_{z,n}$ is the unique bounded function that is a solution of such cohomological equation. Let

$$\mathscr{I} = \{y \in \mathbb{C} \colon \ \psi = y \text{ in some } P \in \mathscr{P}^{n_0}\}.$$

Since ψ is not constant we have that $\#\mathscr{I} \geq 2$. For every $q \in \mathscr{I}$ define the affine map

$$\phi_{q,n} \colon \mathbb{C} \to \mathbb{C}$$

given by

$$\phi_{q,n}(u) = z^{-n} u - z^{-n} q.$$

One can rewrite (4.1.23) as

$$\phi_{\psi(x),n} \circ h_{z,n}(x) = h_{z,n} \circ T^n(x).$$

Observe that the unique fixed point of $\phi_{q,n}$ is

$$x_{q,n} = \frac{q}{1 - z^n},$$

and $\lim_n x_{q,n} = q$. Let $R = 2 diam\, \mathscr{I}$ and choose $q_0 \in \mathscr{I}$. If $n > n_0$ is large enough

- We have

$$\phi_{q,n}^{-1}(B(q_0, R)) \subset B(q_0, R),$$

  for every $q \in \mathscr{I}$,
- $\phi_{q_1,n}^{-1}(B(q_0,R))$ and $\phi_{q_2,n}^{-1}(B(q_0,R))$ are disjoint for every $q_1, q_2 \in \mathscr{I}$ with $q_1 \neq q_2$.

In particular the map

$$G_n \colon \bigcup_{q \in \mathscr{I}} \phi_{q,n}^{-1}(B(q_0,R)) \to B(q_0,R)$$

defined by $G_n(x) = \phi_{q,n}(x)$ for $x \in \phi_{q,n}^{-1}(B(q_0,R))$, is a conformal expanding map and its maximal invariant set

$$\Omega_n = \bigcap_{\ell > 0} G_n^{-\ell} B(q_0, R)$$

is a Cantor set. Let

$$\hat{h}_{z,n} \colon I \to \mathbb{C}$$

be the bounded function defined by

$$\hat{h}_{z,n}(x) = \lim_k \phi^{-1}_{\psi(x),n} \circ \phi^{-1}_{\psi(T^n x),n} \circ \phi^{-1}_{\psi(T^{2n} x),n} \cdots \circ \phi^{-1}_{\psi(T^{kn} x),n}(q_0) \in \Omega_n \cap \phi^{-1}_{\psi(x),n}(B(q_0,R))$$

Since T^n has only full branches the image of $\hat{h}_{z,n}$ is the Cantor set Ω_n. It is easy to see that

$$G_n \circ \hat{h}_{z,n}(x) = \hat{h}_{z,n} \circ T^n(x),$$

that is

$$\phi_{\psi(x),n} \circ \hat{h}_{z,n}(x) = \hat{h}_{z,n} \circ T^n(x),$$

so

$$-z^{-n}\psi = \hat{h}_{z,n} \circ T^n - z^{-n}\hat{h}_{z,n}.$$

and consequently $\hat{h}_{z,n} = h_{z,n}$.

Claim C. For $n \geq n_0$ we have that

$$w_{z,n} = \sum_{\ell=1}^{\infty} z^{-\ell n} L^{\ell n}\psi$$

converges in B and $C^r(I)$, and moreover

$$L^n w_{z,n} = z^n w_{z,n} - L^n \psi. \tag{4.1.24}$$

Since $L^{n_0}\psi \in F \cap \hat{F}$ the above series converges in B and $C^r(I)$. It is easy to verify (4.1.24).

Claim D. For large n we have that $h_{z,n} + w_{z,n} \neq 0$ and

$$L^n(h_{z,n} + w_{z,n}) = z^n(h_{z,n} + w_{z,n}). \tag{4.1.25}$$

The equality (4.1.25) is obvious. Note that due Claim B. we have that for large n the image of $h_{z,n}$ is a Cantor set (up to a countable set), and the image of $-w_{z,n}$ is a (perhaps empty) interval (up to a finite subset). So $h_{z,n} + w_{z,n} \neq 0$ and z^n is an eigenvalue of L^n. In particular there is $\delta \in \mathbb{C}$, with $\delta^n = 1$, such that δz is an eigenvalue of L.

Since z can be an arbitrary complex number satisfying (4.1.20), Claim D. implies that $r_{ess}(L^n, B) \geq 1/k^n$ and $r_{ess}(L, B) \geq 1/k$. □

Proof of Theorem D. Since $n(\phi) = ||\phi||_B$ is natural, it can be written as in (2.2.9), where u_i there are purely natural pseudo-norm n_i, with $i \leq j$, and degree of homogeneity t_i.

Given $\phi \in B$, define

$$t_{\max}(\phi) = \max\{t_i : \ n_i(\phi) > 0\}.$$

Claim I. Let $\mathscr{I}$ be the family of all characteristic functions of intervals in I. Then $t_{\max}$ is constant on $\mathscr{I}$. Let $t_{\max}(\mathscr{I})$ be the valued of $t_{\max}$ on $\mathscr{I}$. There is C_1 such that

$$\frac{1}{C_1}|Q|^{-t_{\max(\mathscr{I})}} \leq ||1_Q||_B \leq C_1|Q|^{-t_{\max(\mathscr{I})}}$$

Given a interval $Q \subset I$ there is an affine map ψ such that $\psi_Q(Q) = I$, with $|\psi'| = |I|/|Q| = 1/|Q|$, so

$$1_Q = 1_I \circ \psi$$

and due the the almost homogeneity of the norm there is $K_i > 0$ such that

$$\frac{1}{k_i}|Q|^{-t_i} n_i(1_I) \le u_i(1_Q) \le K_i |Q|^{-t_i} n_i(1_I)$$

for every interval $Q \subset I$. It follows that $t_{\max}$ is constant on $\mathscr{I}$. Note also that

$$\begin{aligned} ||1_Q||_B &\le K_{t_{\max}}^{-1} |Q|^{-t_{\max}} u_{t_{\max}}(1_I) + \sum_{t_i < t_{\max}} K_{t_i}^{-1} |Q|^{-t_i} u_{t_i}(1_I) \\ &\le K_{t_{\max}}^{-1} |Q|^{-t_{\max}} \Big(u_{t_{\max}}(1_I) + \sum_{t_i < t_{\max}} \frac{K_{t_i}^{-1}}{K_{t_{\max}}^{-1}} |Q|^{t_{\max} - t_i} \frac{u_{t_i}(1_I)}{u_{t_{\max}}(1_I)} \Big) \\ &\le C |Q|^{-t_{\max}} ||1_I||_B. \end{aligned}$$

and the opposite inequality is obtained with a similar argument. This finishes the proof of the claim.

Let $\mathscr{P}_k$ be the Markov partition of T^k. Given an interval $P \in \mathscr{P}^{k+1}$, chose $Q_1, Q_2 \in \mathscr{P}^{k+2}$ such that $Q_i \subset P$, with $i = 1, 2$, and $Q_1 \neq Q_2$. Define

$$a_P = \frac{1_{Q_1}}{|Q_1|} - \frac{1_{Q_2}}{|Q_2|}.$$

Note that

$$\int a_P \, dm = 0.$$

Claim II. There is $C > 0$ and $\lambda \in (0,1)$ such that for every $P \in \mathscr{P}^k$ we have

$$||a_P||_B \ge C\lambda^{-k}.$$

Since L has spectral gap on B there is $C \ge 0$, $\lambda \in (0,1)$ such that for every function $a \in B$ such that

$$\int a \, dm = 0,$$

we have

$$||L^k a||_B \le C\lambda^k ||a||_B,$$

So given $P \in \mathscr{P}^{k+1}$ we have

$$||L^k a_P||_B \le C\lambda^k ||a_P||_B,$$

On the other hand since B is continuously embedded in $L^1(m)$ there is $C > 0$ such

$$||L^k a_P||_B \ge C||L^k a_P||_{L^1(m)} = C||a_P||_{L^1(m)} = 2C > 0,$$

Consequently there is $C > 0$ such that

$$||a_P||_B \ge C\lambda^{-k}.$$

Claim III. We have $t_{\max}(\mathscr{I}) > -1$.

It follows from Claim II. that for every $P \in \mathscr{P}^{k+1}$ there is $Q \in \mathscr{P}^{k+2}$, with $Q \subset P$, satisfying

$$||\frac{1_Q}{|Q|}||_B \geq \frac{C}{2}\lambda^{-k}.$$

Let

$$\alpha = \min \frac{1}{|DT|} < 1.$$

Then

$$|Q| \geq \alpha^{k+1},$$

It follows that

$$||1_Q||_B \geq \frac{C}{2}\lambda^{-k}|Q| \geq \frac{C}{2}\alpha^{-k\frac{\ln\lambda}{\ln\alpha}}|Q| \geq C|Q|^{1-\beta},$$

with

$$\beta = \frac{\ln\lambda}{\ln\alpha} > 0.$$

Due Claim II. that implies $1 + t_{\max}(\mathscr{I}) \geq \beta > 0$.

Claim IV. We have $t_{\max}(\mathscr{I}) \leq 0$.

Suppose that $t_{\max}(\mathscr{I}) > 0$. Let C_1 be as in Claim I. Choose θ such that $\theta \in (0, 1/4)$, $\theta^{t_{\max}(\mathscr{I})} \in (0, 1/4)$ and such that

$$\frac{1}{C_1}\theta^{-t_{\max}(\mathscr{I})} > 2C_1.$$

Let $Q_i = [x_i, x_{i+1}]$, with $x_0 = 0$ and

$$|Q_i| = \theta^i$$

Then by Claim I

$$\begin{aligned}
C_1|Q_0|^{-t_{\max}(\mathscr{I})} &\geq C_1|[0, x_{k+1}]|^{-t_{\max}(\mathscr{I})} \\
&\geq ||1_{[0,x_{k+1}]}||_B = ||\sum_{i=0}^{k} 1_{[x_i,x_{i+1}]}||_B \\
&\geq ||1_{Q_k}||_B - \sum_{i=0}^{k-1} ||1_{Q_i}||_B \\
&\geq \frac{1}{C_1}\theta^{-k t_{\max}(\mathscr{I})} - \sum_{i=0}^{k-1} C_1\theta^{-i t_{\max}(\mathscr{I})} \\
&\geq \frac{1}{C_1}\theta^{-k t_{\max}(\mathscr{I})} - C_1\theta^{-(k-1) t_{\max}(\mathscr{I})}\sum_{i=0}^{k-1}\theta^{i t_{\max}(\mathscr{I})} \\
&\geq \frac{1}{C_1}\theta^{-k t_{\max}(\mathscr{I})} - C_1\theta^{-(k-1) t_{\max}(\mathscr{I})} \geq C_1\theta^{-(k-1) t_{\max}(\mathscr{I})}.
\end{aligned}$$

so if k is sufficiently large, we arrive at a contradiction. This concludes the proof of Claim IV.

Now we can conclude the proof of Theorem D. We know that $-1 < t_{\max}(\mathscr{I}) \leq 0$. If $t_{\max}(\mathscr{I}) = 0$ we can apply Theorem C and obtain Case I. Otherwise we can take $s = 1 + t_{\max}(\mathscr{I}) \in (0, 1)$. The remaining conclusions follow from Theorem A.

□

ACKNOWLEDGEMENTS

O.B. was partially supported by PRIN Grant "Regular and stochastic behaviour in dynamical systems" (PRIN 2017S35EHN). O.B. acknowledges the MIUR Excellence Department Projects awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006, CUP E83C2300033000s6. This research is part of the O.B.'s activity within the UMI Group "DinAmicI" and the INdAM group GNFM. D.S. was financed by the São Paulo Research Foundation (FAPESP), Brasil, Process Number 2017/06463-3, and Bolsa de Produtividade em Pesquisa CNPq-Brazil 311916/2023-6.

REFERENCES

[1] Alexander Arbieto and Daniel Smania. Transfer operators and atomic decomposition, 2020.

[2] V. Baladi and G. Keller. Zeta functions and transfer operators for piecewise monotone transformations. *Comm. Math. Phys.*, 127(3):459–477, 1990.

[3] Viviane Baladi. *Positive transfer operators and decay of correlations*, volume 16 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[4] Viviane Baladi. *Dynamical zeta functions and dynamical determinants for hyperbolic maps*, volume 68 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2018. A functional approach.

[5] Viviane Baladi and Masato Tsujii. Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms. *Ann. Inst. Fourier (Grenoble)*, 57(1):127–154, 2007.

[6] Viviane Baladi and Masato Tsujii. Dynamical determinants and spectrum for hyperbolic diffeomorphisms. In *Geometric and probabilistic structures in dynamics*, volume 469 of *Contemp. Math.*, pages 29–68. Amer. Math. Soc., Providence, RI, 2008.

[7] Michael Blank, Gerhard Keller, and Carlangelo Liverani. Ruelle-Perron-Frobenius spectrum for Anosov maps. *Nonlinearity*, 15(6):1905–1973, 2002.

[8] Rufus Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. Edited by Chazottes, Jean-René. With a preface by David Ruelle.

[9] Oliver Butterley, Giovanni Canestrari, and Roberto Castorrini. Discontinuities cause essential spectrum on surfaces. *Annales Henri Poincaré*, 2024.

[10] Oliver Butterley, Giovanni Canestrari, and Sakshi Jain. Discontinuities cause essential spectrum. *Comm. Math. Phys.*, 398(2):627–653, 2023.

[11] Oliver Butterley, Niloofar Kiamari, and Carlangelo Liverani. Locating Ruelle-Pollicott resonances. *Nonlinearity*, 35(1):513–566, 2022.

[12] Nikolai Chernov and Roberto Markarian. *Chaotic billiards*, volume 127 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[13] Pierre Collet and Stefano Isola. On the essential spectrum of the transfer operator for expanding Markov maps. *Comm. Math. Phys.*, 139(3):551–557, 1991.

[14] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai. *Ergodic theory*, volume 245 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskiĭ.

[15] Amanda de Lima and Daniel Smania. On infinitely cohomologous to zero observables. *Ergodic Theory Dynam. Systems*, 33(2):375–399, 2013.

[16] Geraldo Soares de Souza. The atomic decomposition of Besov-Bergman-Lipschitz spaces. *Proceedings of the American Mathematical Society*, 94(4):682–686, 1985.

[17] Geraldo Soares De Souza, Richard O'Neil, and G. Sampson. Several characterizations for the special atom spaces with applications. *Rev. Mat. Iberoamericana*, 2(3):333–355, 1986.

[18] Frédéric Faure, Sébastien Gouëzel, and Erwan Lanneau. Ruelle spectrum of linear pseudo-Anosov maps. *J. Éc. polytech. Math.*, 6:811–877, 2019.

[19] Leopold Flatto, Jeffrey C. Lagarias, and Bjorn Poonen. The zeta function of the beta transformation. *Ergodic Theory Dynam. Systems*, 14(2):237–266, 1994.

[20] Sébastien Gouëzel and Carlangelo Liverani. Banach spaces adapted to Anosov systems. *Ergodic Theory Dynam. Systems*, 26(1):189–217, 2006.

[21] V. M. Gundlach and Y. Latushkin. A sharp formula for the essential spectral radius of the Ruelle transfer operator on smooth and Hölder spaces. *Ergodic Theory Dynam. Systems*, 23(1):175–191, 2003.

[22] G. Keller. Markov extensions, zeta functions, and Fredholm theory for piecewise invertible dynamical systems. *Trans. Amer. Math. Soc.*, 314(2):433–497, 1989.

[23] Gerhard Keller. On the rate of convergence to equilibrium in one-dimensional systems. *Comm. Math. Phys.*, 96(2):181–193, 1984.

[24] Gerhard Keller and Carlangelo Liverani. Stability of the spectrum for transfer operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(1):141–152, 1999.

[25] Gerhard Keller and Hans Henrik Rugh. Eigenfunctions for smooth expanding circle maps. *Nonlinearity*, 17(5):1723–1730, 2004.

[26] A. Lasota and James A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.*, 186:481–488 (1974), 1973.

[27] Carlangelo Liverani. Rigorous numerical investigation of the statistical properties of piecewise expanding maps. A feasibility study. *Nonlinearity*, 14(3):463–490, 2001.

[28] Carlangelo Liverani. Invariant measures and their properties. A functional analytic point of view. In *Dynamical systems. Part II*, Pubbl. Cent. Ric. Mat. Ennio Giorgi, pages 185–237. Scuola Norm. Sup., Pisa, 2003.

[29] Yushi Nakano and Shota Sakamoto. Spectra of expanding maps on Besov spaces. *Discrete Contin. Dyn. Syst.*, 39(4):1779–1797, 2019.

[30] William Parry and Mark Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque*, 187-188:284, 1990.

[31] David Ruelle. The thermodynamic formalism for expanding maps. *Comm. Math. Phys.*, 125(2):239–262, 1989.

[32] Daniel Smania. Transfer operators, atomic decomposition and the Bestiary. Arxiv preprint 1903.06976, 2021.

[33] Daniel Smania. Besov-ish spaces through atomic decomposition. *Anal. PDE*, 15(1):123–174, 2022.

[34] Damien Thomine. A spectral gap for transfer operators of piecewise expanding maps. *Discrete Contin. Dyn. Syst.*, 30(3):917–944, 2011.

(OLIVER BUTTERLEY) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROME TOR VERGATA, VIA DELLA RICERCA SCIENTIFICA 1, 00133 ROMA, ITALY

Email address: butterley@mat.uniroma2.it

URL: https://www.mat.uniroma2.it/butterley/

(DANIEL SMANIA) DEPARTAMENTO DE MATEMÁTICA, INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DE COMPUTAÇÃO (ICMC), UNIVERSIDADE DE SÃO PAULO (USP), AVENIDA TRABALHADOR SÃO-CARLENSE, 400, SÃO CARLOS, SP, CEP 13566-590, BRAZIL

Email address: smania@icmc.usp.br

URL: https://sites.icmc.usp.br/smania/