Mathematical Analysis 2

Call 4 – 20/07/2023

<u>Do **not** turn this sheet over until instructed to do so.</u>

Name: **De Nym Sue**,
Mat: **'0000000'** Seat: **0**

This exam consists of 5 questions, each are worth 6 points (no negative scoring), you have 3 hours to attempt the exam. Solutions available online when the exam finishes.

Fill the blanks with the correct **integer**, $-\infty$ or ∞.
Select the correct option for ☑ is / ☐ is not questions.

- Students are permitted to bring only the following items to their desk in the exam room:
  - Two A4 sheets of paper with course notes (writing permitted on both sides, or equivalent area);
  - Pens and pencils;
  - An identity document;
  - Drinking bottle.
- Paper for rough calculations will be provided in the exam room. After the exam the paper used during the exam remains in the exam room.
- During the exam it is forbidden to communicate, using any means, with anyone except the exam invigilators. If you want to communicate, raise your hand and an invigilator will come to you.
- It is forbidden to have anything that can be used for communication (e.g., telephone, smartwatch, computer, headphones, earbuds, messenger pigeon) on your person or close to you during the exam.
- Under penalty of exclusion, during written tests the use of electronic devices and applications, is not allowed. Calculators are not permitted.
- You may choose to leave the exam early but only after receiving confirmation from an invigilator.

I affirm that I will not give or receive any unauthorized help on this exam. If I am offered unauthorized help I will notify an invigilator.

Name: De Nym Sue, Signature:

Question 1. Calculate the radius of convergence for each of the following power series.

Series	Radius
$\sum_{n=0}^{\infty} \frac{(x+4)^n}{(n^3+2)3^n}$	$\boxed{\mathbf{3}}$
$\sum_{n=0}^{\infty} \frac{n!(x+2)^n}{(n-1)^2+3n}$	$\boxed{\mathbf{0}}$
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\boxed{\infty}$
$\sum_{n=0}^{\infty} \frac{x^{3n}}{(-27)^n}$	$\boxed{\mathbf{3}}$

(6)

Question 2. Optimize $f(x,y,z) = yz + xy$ subject to the constraints $xy = 1, y^2 + z^2 = 1$.

$$\text{Maximum:} \quad \frac{\boxed{\mathbf{3}}}{2} = f\left(\sqrt{\boxed{\mathbf{2}}}, \frac{1}{2}\sqrt{\boxed{\mathbf{2}}}, \frac{1}{2}\sqrt{\boxed{\mathbf{2}}}\right),$$

$$\text{Minimum:} \quad \frac{\boxed{\mathbf{1}}}{2} = f\left(\sqrt{\boxed{\mathbf{2}}}, \frac{1}{2}\sqrt{\boxed{\mathbf{2}}}, -\frac{1}{2}\sqrt{\boxed{\mathbf{2}}}\right).$$

(6)

Question 3. Determine which of the following is a parametrization of the path

$$C = \{(x,y) : x^2 + y^2 = 1, x \geq 0\} \subset \mathbb{R}^2,$$

starting at $(0,-1)$ and finishing at $(0,1)$:

$\boldsymbol{\alpha}_1(t) = (1-t^2, t),$	$t \in [-1,1]$	☐ is / ☑ is not
$\boldsymbol{\alpha}_2(t) = (-\cos t, \sin t),$	$t \in [-\pi, 0]$	☐ is / ☑ is not
$\boldsymbol{\alpha}_3(t) = \left(\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1}\right),$	$t \in [-1,1]$	☑ is / ☐ is not
$\boldsymbol{\alpha}_4(t) = (\sqrt{1-t^2}, t),$	$t \in [-1,1]$	☑ is / ☐ is not
$\boldsymbol{\alpha}_5(t) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right),$	$t \in [-1,1]$	☐ is / ☑ is not
$\boldsymbol{\alpha}_6(t) = (-\sin t, \cos t),$	$t \in [-\pi, 0]$	☑ is / ☐ is not

(6)

Question 4. Let $T \subset \mathbb{R}^2$ be the region in the upper right quadrant bounded by the curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. Evaluate[1]

$$\iint_T y^2 - 3x \, dx dy = \boxed{\mathbf{-26}} + \boxed{\mathbf{5}}\,\pi.$$

Hint: the sum of these numbers is equal to -21.

(6)

Question 5. Consider the cone $V = \{(x,y,z) : x^2 + y^2 \leq 4, 0 \leq z \leq 2 - \sqrt{x^2+y^2}\}$ and the cylinder $W = \{(x,y,z) : (x-1)^2 + y^2 \leq 1\}$. Let $D \subset \mathbb{R}^3$ be the subset of the cone V which is contained within the cylinder W. The volume of D is equal to[2]

$$\operatorname{Vol}(D) = \boxed{\mathbf{2}}\,\pi + \frac{\boxed{\mathbf{-32}}}{9}.$$

Hint: the sum of these numbers is equal to -30.

(6)

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[1] Identities: $\int_0^{\pi/2} \cos\theta \, d\theta = 1$, $\int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{\pi}{4}$, $\int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{\pi}{4}$, $\int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{2}{3}$.

[2] We may write $D = \{(x,y,z) : (x,y) \in S, 0 \leq z \leq \varphi(x,y)\}$ where $S = \{(x,y) : (x-1)^2 + y^2 \leq 1\}$. Consequently $\operatorname{Vol}(D) = \iint_S \varphi(x,y) \, dx dy$. Sketch the region S and use polar coordinates to evaluate the integral.

$\boxed{Q1}$ (a) Fix x and let $a_n = \dfrac{(x+4)^n}{(n^3+2)3^n}$.

$$\frac{a_{n+1}}{a_n} = \frac{(x+4)}{3} \cdot \frac{n^3+2}{(n+1)^3+2}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{x+4}{3} . \Rightarrow \text{radius} = 3$$

(b) radius = $\cancel{\infty}$ 0

(c) radius = ∞

(d) Fix x and let $a_n = \dfrac{x^{3n}}{(-27)^n}$.

$$\frac{a_{n+1}}{a_n} = \frac{x^{3(n+1)}}{(-27)^{n+1}} \cdot \frac{(-27)^n}{x^{3n}} = \frac{x^3}{-27}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^3}{27} . \quad \frac{|x|^3}{27} < 1$$

$$\Updownarrow$$

$$|x|^3 < 27 \Leftrightarrow |x| < 3$$

radius = 3.

$\boxed{Q2}$ $f(x,y,z) = yz + xy$.

Let $g_1(x,y,z) = xy$, $g_2(x,y,z) = y^2+z^2$.

$\nabla f(x,y,z) = \begin{pmatrix} y \\ z+x \\ y \end{pmatrix}$, $\nabla g_1(x,y,z) = \begin{pmatrix} y \\ x \\ 0 \end{pmatrix}$, $\nabla g_2(x,y,z) = \begin{pmatrix} 0 \\ 2y \\ 2z \end{pmatrix}$

System of equations:

$$\begin{cases} y = \lambda_1 y \\ z + x = \lambda_1 x + 2\lambda_2 y \\ y = 2\lambda_2 z \\ xy = 1 \\ y^2+z^2 = 1 \end{cases}$$

$y = \lambda_1 y \Rightarrow \lambda_1 = 1$ or $y = 0$.
But $y=0$ is impossible so $\boxed{\lambda_1 = 1}$.

$\Rightarrow x = \frac{1}{y}$ and remains to solve

$$\begin{cases} z = 2\lambda_2 y \\ y = 2\lambda_2 z \\ y^2+z^2 = 1 \end{cases}$$

$\Rightarrow z = (2\lambda_2)^2 z$

$\Rightarrow z = 0$ or $(2\lambda_2)^2 = 1$

<u>Case $z=0$</u>: $y^2 = 1 \Rightarrow$ Solutions: $(1, 1, 0)$, $(-1, -1, 0)$.

<u>Case $\lambda_2 = \frac{1}{2}$</u>: $y = z$, $\Rightarrow y^2 = \frac{1}{2}$.

Solutions: $(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$

<u>Case $\lambda_2 = -\frac{1}{2}$</u>: $y = -z \Rightarrow y^2 = \frac{1}{2}$.

Solutions: $(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Q2 cont.

Calculate f at these 6 potential extrema points:

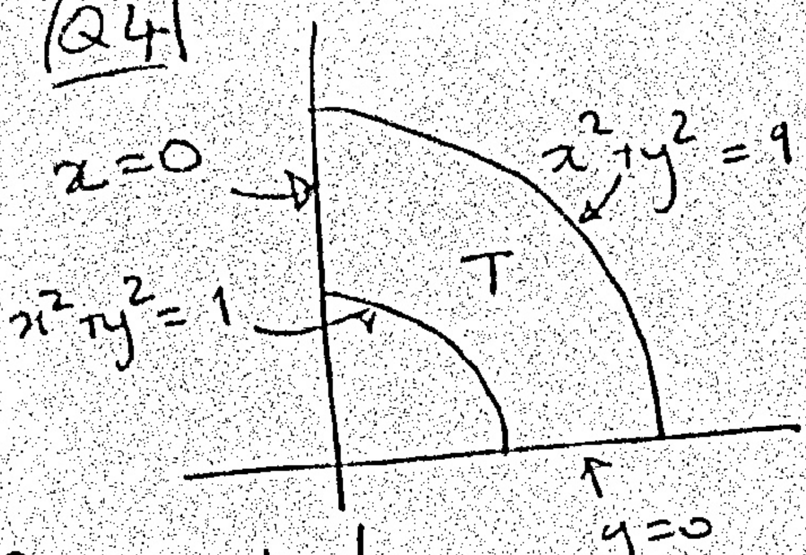
$0+1=1$, $0+1=1$, $\frac{1}{2}+1=\frac{3}{2}$, $\frac{1}{2}+1=\frac{3}{2}$,

$-\frac{1}{2}+1=\frac{1}{2}$, $-\frac{1}{2}+1=\frac{1}{2}$.

Maximum of $\frac{3}{2}$ at $(\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\sqrt{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

Minimum of $\frac{1}{2}$ at $(\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(-\sqrt{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$\boxed{Q4}$



Polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad 1 \le r \le 3,\ 0 \le \theta \le \pi/2$$

Jacobian $= r$.

$$\iint_T y^2 - 3x \, dx\,dy = \int_1^3 \left[\int_0^{\pi/2} r\left(r^2 \sin^2\theta - 3r\cos\theta\right) d\theta \right] dr$$

$$= \left(\int_1^3 r^3\,dr\right)\left(\int_0^{\pi/2} \sin^2\theta\,d\theta\right) - 3\left(\int_1^3 r^2\,dr\right)\left(\int_0^{\pi/2} \cos\theta\,d\theta\right)$$

$$= \left[\frac{r^4}{4}\right]_1^3 \left(\frac{\pi}{4}\right) - 3\left[\frac{r^3}{3}\right]_1^3 (1)$$

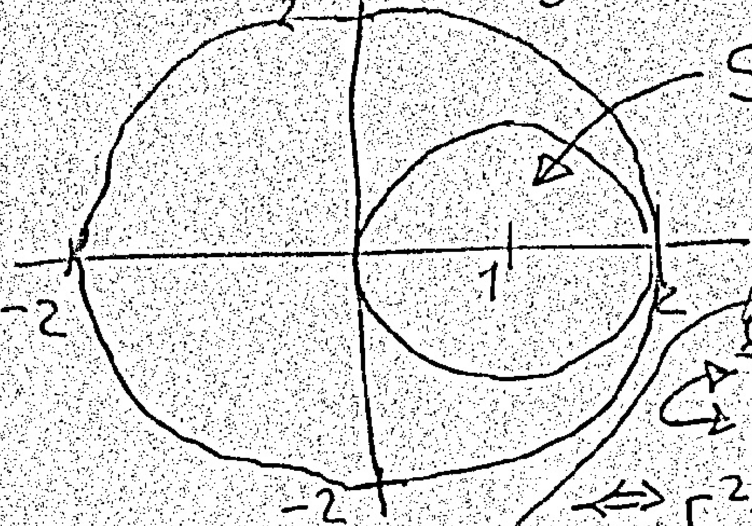
$$= \frac{81-1}{4}\left(\frac{\pi}{4}\right) - (27-1) = 5\pi - 26$$

$$\boxed{\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{\pi}{4}, \quad \int_0^{\pi/2} \cos\theta\,d\theta = 1}$$

Q5 let $S = \{(x,y) : (x-1)^2 + y^2 \leq 1\}$ and

so $D = \{(x,y,z) : (x,y) \in S,\ 0 \leq z \leq 2 - \sqrt{x^2+y^2}\}$.

Consequently $\mathrm{Vol}(D) = \iint_S 2 - \sqrt{x^2+y^2}\, dx\, dy$



$z = 2 - \sqrt{x^2+y^2}$

$(x-1)^2 + y^2 = 1$

$\Leftrightarrow (r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$

$\Leftrightarrow r^2 - 2r\cos\theta = 0 \Leftrightarrow r = 0$ or $r = 2\cos\theta$

Use polar coordinates: S corresponds to T

where $T = \{(r,\theta) : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos\theta\}$

$\Rightarrow \mathrm{Vol}(D) = \iint_T r(2-r)\, dr\, d\theta$

$$= \int_{-\pi/2}^{\pi/2} \left[\int_0^{2\cos\theta} 2r - r^2\, dr \right] d\theta$$

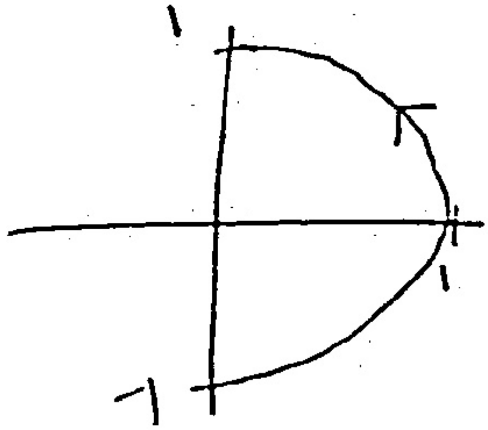
$$= \int_{-\pi/2}^{\pi/2} \left[r^2 - \frac{r^3}{3} \right]_0^{2\cos\theta} d\theta$$

$$= 2 \int_0^{\pi/2} \left(4\cos^2\theta - \frac{8}{3}\cos^3\theta\right) d\theta = 4 \cdot 2\left(\frac{\pi}{4}\right) - \frac{8}{3} \cdot 2\left(\frac{2}{3}\right) = 2\pi - \frac{32}{9}$$

$\boxed{\alpha_1}$ $\boxed{Q3}$

$(1-t^2)^2 + (t)^2 = 1-2t^2+t^4+t^2 = 1-t^2+t^4$

but should be equal to 1. for all $t \in [-1,1]$.



$\boxed{\alpha_2}$ $\alpha_2(0) = (-1,0)$ but should be equal to $(0,1)$

$\boxed{\alpha_3}$ $\alpha_3(-1) = \left(0, \frac{-2}{1+1}\right) = (0,-1)$ ✓

$\alpha_3(1) = \left(0, \frac{2}{1+1}\right) = (0,1)$ ✓

$$\left(\frac{1-t^2}{t^2+1}\right)^2 + \left(\frac{2t}{t^2+1}\right)^2 = \frac{1-2t^2+t^4+4t^2}{(t^2+1)^2}$$

$$= \frac{1+2t^2+t^4}{(t^2+1)^2} = \frac{(1+t^2)^2}{(t^2+1)^2} = 1$$ ✓

$\boxed{\alpha_4}$ $\alpha_4(-1) = (\sqrt{1-1}, -1) = (0,-1)$ ✓

$\alpha_4(1) = (\sqrt{1-1}, 1) = (0,1)$ ✓

$(\sqrt{1-t^2})^2 + (t)^2 = 1 - t^2 + t^2 = 1$ ✓

$\boxed{\text{Q3 cont.}}$

$\boxed{\alpha_5}$ $\alpha_5(0) = (-1, 0)$ but C requires "$x \geq 0$".

$\boxed{\alpha_6}$ $\alpha_6(-\pi) = (-0, -1) = (0, -1)$ ✓

$\alpha_6(0) = (-0, 1) = (0, 1)$ ✓

$(-\sin t)^2 + (\cos t)^2 = 1$ ✓