Mathematical Analysis 2

Call 3 – 03/07/2023

Do **not** turn this sheet over until instructed to do so.

Name: **Sue de Nym**,
Mat: **000000** Seat: **0**

This exam consists of 5 questions, each are worth 6 points (no negative scoring), you have 3 hours to attempt the exam. Solutions available online when the exam finishes.

Fill the blanks with the correct **integer**, $-\infty$ or ∞.

- Students are permitted to bring only the following items to their desk in the exam room:
  - Two A4 sheets of paper with course notes (writing permitted on both sides, or equivalent area);
  - Pens and pencils;
  - An identity document;
  - Drinking bottle.
- Paper for rough calculations will be provided in the exam room. After the exam the paper used during the exam remains in the exam room.
- During the exam it is forbidden to communicate, using any means, with anyone except the exam invigilators. If you want to communicate, raise your hand and an invigilator will come to you.
- It is forbidden to have anything that can be used for communication (e.g., telephone, smartwatch, computer, headphones, earbuds, messenger pigeon) on your person or close to you during the exam.
- Under penalty of exclusion, during written tests the use of electronic devices and applications, is not allowed. Calculators are not permitted.
- You may choose to leave the exam early but only after receiving confirmation from an invigilator.

I affirm that I will not give or receive any unauthorized help on this exam. If I am offered unauthorized help I will notify an invigilator.

Name: Sue de Nym, Signature:

Question 1. Calculate the radius of convergence for each of the following power series.

Series	Radius
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\boxed{\infty}$
$\sum_{n=0}^{\infty} \frac{(3x+4)^n}{(n^3+2)3^n}$	$\boxed{\mathbf{1}}$
$\sum_{n=0}^{\infty} \frac{n!(5x+3)^n}{(n+1)^2+4n}$	$\boxed{\mathbf{0}}$
$\sum_{n=0}^{\infty} \frac{x^{2n}}{(-4)^n}$	$\boxed{\mathbf{2}}$

(6)

Question 2. Consider the scalar field $f(x,y) = xy - x^3 - y^2$. The two stationary points are $\mathbf{a} = (0, \boxed{\mathbf{0}})$ and $\mathbf{b} = (1/\boxed{\mathbf{6}}, 1/\boxed{\mathbf{12}})$. The Hessian at these points are (2)

$$Hf(\mathbf{a}) = \begin{pmatrix} \boxed{0} & \boxed{\mathbf{1}} \\ \boxed{\mathbf{1}} & \boxed{\mathbf{-2}} \end{pmatrix}, \quad Hf(\mathbf{b}) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}.$$ (2)

The point $\mathbf{a}$ $\boxed{\square \text{ is } / \square \text{ is not}}$ a saddle point and the point $\mathbf{b}$ $\boxed{\square \text{ is } / \square \text{ is not}}$ a local minimum. (2)

Question 3. Let α denote the path along the curve $C = \{(x,y) : x^2 + y^2 = 1, x, y \geq 0\} \subset \mathbb{R}$, from $(1,0)$ to $(0,1)$. Let

$$\mathbf{F}(x,y) = \begin{pmatrix} 16y \\ 9x^2 \end{pmatrix}$$

be a vector field and let

$$g(x,y) = \exp(x(11+y)) - x^2 + Ay^3$$

be a scalar field. Evaluate the line integrals

$$\int_C \mathbf{F} \cdot d\alpha = \boxed{\mathbf{-4}}\,\pi + \boxed{\mathbf{6}}, \qquad \int_C \nabla g \cdot d\alpha = \boxed{\mathbf{A+2}} - \exp(\boxed{\mathbf{11}}).$$ (3+3)

Hint: $\int_0^{\pi/2} \sin^2 t\, dt = \frac{\pi}{4}$, $\int_0^{\pi/2} \cos^3 t\, dt = \frac{2}{3}$.

Question 4. Let $T \subset \mathbb{R}^2$ denote the region bounded by the curves $xy = 1$, $xy = 3$, $y = 2$, $y = 6$. Calculate

$$\iint_T xy^3\, dx dy = \boxed{\mathbf{64}}.$$ (6)

Hint: The answer is a square number.

Question 5. Let $S = \left\{(x,y,z) : x^2 + y^2 + z^2 = r^2\right\}$, the surface of the sphere of radius $r > 0$, and consider the vector field on $\mathbb{R}^3$,

$$\mathbf{G}(x,y,z) = (x^2 + y^2 + z^2) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{n}$ denote the outgoing normal vector on S and compute the surface integral

$$\iint_S \mathbf{G} \cdot \mathbf{n}\, dS = \boxed{\mathbf{4}}\,\pi r^{\boxed{\mathbf{5}}}.$$ (6)

Q1 (A) Let $a_n = \frac{x^n}{n!}$ and consider ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = \frac{|x|}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \quad \text{for any } x \text{ and so } r = \infty$$

(B) Let $a_n = \frac{(3x+4)^n}{(n^3+2)3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3x+4)^{n+1}}{((n+1)^3+2)3^n} \cdot \frac{(n^3+2)3^n}{(3x+4)^n}\right| = \left|x+\frac{4}{3}\right|\frac{n^3+2}{(n+1)^3+2}$$

$$\lim_{n\to\infty}\frac{n^3+2}{(n+1)^3+2} = 1 \text{ and so } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|x+\frac{4}{3}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \iff \left|x+\frac{4}{3}\right| < 1 \text{ so } r = 1$$

(C) Let $a_n = \frac{n!\,(5x+3)^n}{(n+1)^2+4n}$. $\quad \frac{a_{n+1}}{a_n} = \frac{(n+1)!\,(5x+3)^{n+1}}{(n+2)^2+4(n+1)}\frac{(n+1)^2+4n}{n!\,(5x+3)^n}$

$$= (n+1)(5x+3)\frac{n^2+6n+1}{n^2+8n+8}$$

$$\lim_{n\to\infty}\frac{n^2+6n+1}{n^2+8n+8} = 1$$

and so $\frac{a_{n+1}}{a_n} \to \infty$ as $n\to\infty$ unless $5x+3=0$

$\Rightarrow r = 0$.

(D) Let $a_n = \frac{x^{2n}}{(-4)^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2(n+1)}}{(-4)^{n+1}} \cdot \frac{(-4)^n}{x^{2n}}\right| = \frac{x^2}{4}.$$

$\frac{x^2}{4} < 1 \iff |x| < 2$

$\Rightarrow r = 2$.

$\boxed{Q2}$ $f(x,y) = xy - x^3 - y^2$

$$\Rightarrow \nabla f(x,y) = \begin{pmatrix} y - 3x^2 \\ x - 2y \end{pmatrix}$$

If $\nabla f(x,y) = \underline{0}$ then

$$\begin{cases} y - 3x^2 = 0 \\ x - 2y = 0 \end{cases} \Rightarrow x = 2y \Rightarrow x^2 = 4y^2$$

$$\Rightarrow y - 12y^2 = 0$$

$$\Rightarrow y = 0 \text{ or } y = 1/12$$

$y = 0 \Rightarrow x = 0$, $y = 1/12 \Rightarrow x = 1/6$.

Stationary points $(0,0)$ and $(1/6, 1/12)$.

$$Hf(x,y) = \begin{pmatrix} -6x & 1 \\ 1 & -2 \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \quad Hf(1/6, 1/12) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

$\det(Hf(0,0)) = -1 \Rightarrow$ saddle

$\det(Hf(1/6, 1/12)) = 1$, $\operatorname{tr}(Hf(1/6, 1/12)) = -3$

$\Rightarrow$ local maximum.

$\boxed{Q3}$ Let $\alpha(t) = (\cos t, \sin t)$, $t \in [0, \pi/2]$

$$\Rightarrow \alpha'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \quad F(\alpha(t)) = \begin{pmatrix} 16 \sin t \\ 9 \cos^2 t \end{pmatrix}$$

$$\Rightarrow F(\alpha(t)) \cdot \alpha'(t) = -16 \sin^2 t + 9 \cos^3 t$$

$$\Rightarrow \int_C F \cdot d\alpha = \int_0^{\pi/2} (-16 \sin^2 t + 9 \cos^3 t)\, dt$$

$$= -16 \left(\frac{\pi}{4}\right) + 9 \left(\frac{2}{3}\right)$$

$$= -4\pi + 6$$

$$\int_C \nabla g \cdot d\alpha = g(0,1) - g(1,0) \quad \left(g(x,y) = e^{x(11+y)} - x^2 + Ay^3\right)$$

$$= (e^0 + A) - (e^{11} - 1) = 2 + A - \exp(11)$$

$\boxed{Q4}$

$$S = \{(x,y) : 2 \le y \le 6,\ 1/y \le x \le 3/y\}$$

$$\iint_S xy^3\,dx\,dy = \int_2^6 \underbrace{\left[\int_{1/y}^{3/y} xy^3\,dx\right]}\,dy$$

$$\int_{1/y}^{3/y} xy^3\,dx = \left[\frac{x^2y^3}{2}\right]_{1/y}^{3/y} = \frac{9}{y^2}\cdot\frac{y^3}{2} - \frac{1}{y^2}\cdot\frac{y^3}{2}$$

$$= \left(\frac{9}{2} - \frac{1}{2}\right)y = 4y$$

$$\iint_S xy^3\,dx\,dy = \int_2^6 (4y)\,dx = 4\left[\frac{y^2}{2}\right]_2^6$$

$$= 2(36-4) = 2(32)$$

$$= 64$$

$\boxed{Q5}$

In this special case we notice that $G(x,y,z)$ is parallel to the normal. Indeed,

$$n(x,y,z) = (x^2+y^2+z^2)^{-1/2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Consequently $$(G \cdot n)(x,y,z) = (x^2+y^2+z^2)^{1-1/2+1}$$
$$= (x^2+y^2+z^2)^{3/2}$$

On the surface, $x^2+y^2+z^2 = r^2$ and so here $(G \cdot n)(x,y,z) = r^3$

Consequently $$\iint_S G \cdot n \, dS = \iint_S r^3 \, dS = r^3 \iint_S dS$$

$$\iint_S dS = 4\pi r^2.$$