Mathematical Analysis 2

Call 2 – 28/02/2023

<u>Do **not** turn this sheet over until instructed to do so.</u>

Name: **Sue de Nym**,
Mat: **0**   Seat: **0**

This exam consists of 5 questions, each are worth 6 points (no negative scoring), you have 3 hours to attempt the exam. Solutions available online when the exam finishes.

Fill the blanks with the correct **integer**, $-\infty$ or ∞.

- Students are permitted to bring only the following items to their desk in the exam room:
  - Two A4 sheets of paper with course notes (writing permitted on both sides, or equivalent area);
  - Pens and pencils;
  - An identity document;
  - Drinking bottle.
- Paper for rough calculations will be provided in the exam room. After the exam the paper used during the exam remains in the exam room.
- During the exam it is forbidden to communicate, using any means, with anyone except the exam invigilators. If you want to communicate, raise your hand and an invigilator will come to you.
- It is forbidden to have anything that can be used for communication (e.g., telephone, smartwatch, computer, headphones, earbuds, messenger pigeon) on your person or close to you during the exam.
- Under penalty of exclusion, during written tests the use of electronic devices and applications, is not allowed. Calculators are not permitted.
- You may choose to leave the exam early but only after receiving confirmation from an invigilator.

I affirm that I will not give or receive any unauthorized help on this exam. If I am offered unauthorized help I will notify an invigilator.

Name: Sue de Nym,   Signature:

Question 1. Consider the differential equation together with initial conditions:

$$y'' - xy = \frac{1}{1-x}, \quad y(0) = 0, \quad y'(0) = 0.$$

Supposing a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, the first few terms are,

$$y(x) = \square + \square\, x + \frac{1}{\square}x^2 + \frac{1}{6}x^3 + \frac{1}{\square}x^4 + \frac{3}{\square}x^5 + \frac{7}{180}x^6 + \frac{\square}{504}x^7 + \cdots$$ (6)

The radius of convergence of this series is 1. *Hint: Recall the geometric series.*

Question 2. The scalar field, defined on $\mathbb{R}^2$ as,

$$f(x,y) = \tfrac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$$

has two stationary points:[1]

- $(-1, \square/2)$ where f takes the value $\square/12$ and is a $\square$ (3)
- $(\square, \square/2)$ where f takes the value $-29/4$ and is a $\square$ (3)

Question 3. Let α denote the path along the curve $C = \{(x,y) : x^2 + y^2 = 1, x, y \geq 0\} \subset \mathbb{R}$, from $(1,0)$ to $(0,1)$. Let

$$F(x,y) = \begin{pmatrix} 5y \\ 7x^2 \end{pmatrix}$$

be a vector field and let

$$g(x,y) = e^{x(11+y)} - x^2 + 23y^3$$

be a scalar field. Evaluate the line integrals

$$\int_C F \cdot d\alpha = \frac{\square}{3} - \square\frac{\pi}{4}, \qquad \int_C \nabla g \cdot d\alpha = \square - \exp(\square).$$ (6)

Hint: $\int_0^{\pi/2} \sin^2 t\, dt = \frac{\pi}{4}$, $\int_0^{\pi/2} \cos^3 t\, dt = \frac{2}{3}$.

Question 4. We evaluate the repeated integral,

$$\int_0^3 \left[\int_{-\sqrt{9-x^2}}^{0} \left(\exp(x^2+y^2)\right) dy \right] dx = (\exp(\square) - 1)\frac{\pi}{\square}.$$ (6)

Hint: Sketch the region of integration and use polar coordinates.

Question 5. We consider parametric surface $S = \mathbf{r}(T)$ defined as

$$T = \{(u,v) : 0 \leq u \leq 3, 0 \leq v \leq 4\}, \quad \mathbf{r}(u,v) = (u, v^2 - u, u + v)$$

and where $\mathbf{n}$ denotes a unit normal. Letting $G(x,y,z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ be a vector field we evaluate,

$$\left| \iint_S G \cdot \mathbf{n}\, dS \right| = \square.$$ (6)

Hint: The answer is a 3 digit number and the sum of the digits equals 7.

[1] Options are: "1. saddle", "2. local (not global) minimum", "3. local (not global) maximum", "4. global min.", "5. global max.".

$\boxed{Q1}$ Geometric series: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Suppose series solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow y'(x) = \sum_{n=1}^{\infty} a_n(n)x^{n-1}, \Rightarrow y''(x) = \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$$

Differential equation: $y'' - xy = \frac{1}{1-x}$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n-1}x^n - \sum_{n=0}^{\infty} x^n = 0$$

$$\Rightarrow [2a_2 - 1] + \sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+2} - (1 + a_{n-1})]x^n = 0$$

$\Rightarrow 2a_2 - 1 = 0$ (and so $\boxed{a_2 = 1/2}$)

and, for all $n \in \mathbb{N}$, $a_{n+2} = \frac{1 + a_{n-1}}{(n+1)(n+2)}$ ✶

Initial conditions: $y(0) = y'(0) = 0$

$$\Rightarrow \boxed{a_0 = 0}, \boxed{a_1 = 0}$$

Using ✶:

$$a_3 = \frac{1 + a_0}{2 \cdot 3} = \frac{1}{6}, \quad a_4 = \frac{1 + a_1}{3 \cdot 4} = \frac{1}{12}, \quad a_5 = \frac{1 + a_2}{4 \cdot 5} = \frac{3}{40}$$

$$a_6 = \frac{1 + a_3}{5 \cdot 6} = \frac{7}{5 \cdot 6 \cdot 6} = \frac{7}{180}, \quad a_7 = \frac{1 + a_4}{6 \cdot 7} = \frac{13}{6 \cdot 7 \cdot 12} = \frac{13}{504}$$

$\boxed{Q2}$ $f(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Find critical points:

$$\nabla f(x,y) = \begin{pmatrix} x^2 + 2y - 6 \\ 2y + 2x - 3 \end{pmatrix}$$

$$\nabla f(x,y) = \underline{0} \Rightarrow y = \frac{3-2x}{2}$$

$$\Rightarrow x^2 + (3 - 2x) - 6 = 0$$

$$\Rightarrow (x+1)(x-3) = 0 \quad \Rightarrow x = -1 \text{ or } x = 3.$$

$$\Rightarrow (-1, 5/2), \quad (3, -3/2)$$

Hessian:

$$Hf(x,y) = \begin{pmatrix} 2x & 2 \\ 2 & 2 \end{pmatrix}$$

$$\Rightarrow Hf(-1, 5/2) = \begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}, \quad Hf(3, -3/2) = \begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$$

$(-1, 5/2)$ is a saddle because $\det\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix} < 0$.

$(3, -3/2)$ is a local minimum because $\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$ has two positive eigenvalues. $\left[(6-\lambda)(2-\lambda) - 4 = 0\right.$

$\left.\Rightarrow \lambda^2 - 8\lambda + 8 = 0 \Rightarrow \lambda = 4 \pm \sqrt{16-8} > 0\right]$.

$\boxed{Q3}$ let $\alpha(t) = (\cos t, \sin t)$, $t \in [0, \pi/2]$

$$\Rightarrow \alpha'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \quad F(\alpha(t)) = \begin{pmatrix} 5 \sin t \\ 7 \cos^2 t \end{pmatrix}$$

$$\Rightarrow F(\alpha(t)) \cdot \alpha'(t) = -5 \sin^2 t + 7 \cos^3 t$$

$$\Rightarrow \int_C F \cdot d\alpha = \int_0^{\pi/2} (-5 \sin^2 t + 7 \cos^3 t)\, dt$$

$$= -5\left(\frac{\pi}{4}\right) + 7\left(\frac{2}{3}\right) = \frac{14}{3} - 5\frac{\pi}{4}$$

$$\int_C \nabla g \cdot d\alpha = g(0,1) - g(1,0) \qquad g(x,y) = e^{x(11+y)} - x^2 + 16y^3$$

$$= (e^0 - 0 + A) - (e^{11} - 1 + 0)$$

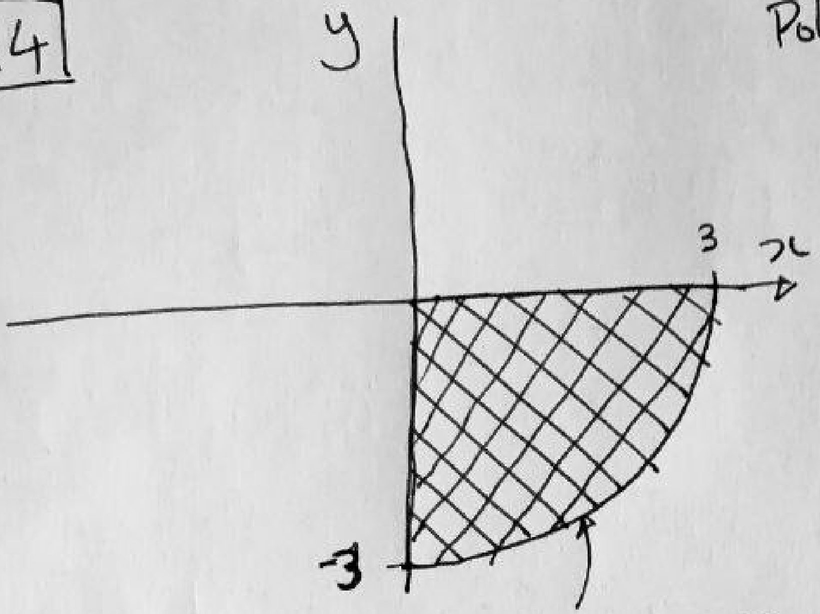
$$\cancel{= 5 - e^{11}} \quad 1 + A - e^{11} + 1$$

$$= 2 + A - e^{11}$$

Q4

Polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$y = -\sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2$$

$$\Rightarrow 9 = x^2 + y^2$$

$$\Rightarrow r = 3$$

$$-\frac{\pi}{2} \leq \theta \leq 0$$

$$0 \leq r \leq 3$$

$$\int_0^3 \left[\int_{-\sqrt{9-x^2}}^{0} \exp(x^2+y^2)\,dy\right] dx = \int_{-\frac{\pi}{2}}^{0} \left[\int_0^3 r e^{r^2}\,dr\right] d\theta$$

$$= \frac{\pi}{2} \left[\frac{e^{r^2}}{2}\right]_{r=0}^{3} = \frac{\pi}{2}\left(\frac{e^9}{2} - \frac{e^0}{2}\right)$$

$$= \frac{\pi}{4}\left(e^9 - 1\right)$$

$\boxed{Q5}$ <u>F.V.P</u>: $\dfrac{\partial r}{\partial u}(u,v) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \dfrac{\partial r}{\partial v}(u,v) = \begin{pmatrix} 0 \\ 2v \\ 1 \end{pmatrix}$

$$\Rightarrow \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right)(u,v) = \begin{pmatrix} -1-2v \\ -1 \\ 2v \end{pmatrix} \quad \left(= N(u,v)\right)$$

<u>Surface integral</u>:

$$\iint_S G \cdot n \, dS = \iint_T G(r(u,v)) \cdot N(u,v) \, du\,dv$$

$$\left[\begin{aligned} G(r(u,v)) &= \begin{pmatrix} -(v^2-u) \\ u \\ 0 \end{pmatrix} \\ G(r(u,v)) \cdot N(u,v) &= (u-v^2)(-1-2v) + u(-1) \\ &= -u + v^2 - 2uv + 2v^3 - u \\ &= 2v^3 + v^2 - 2uv - 2u \end{aligned}\right]$$

$$\iint_S G \cdot n \, dS = \int_0^4 \underbrace{\left[\int_0^3 (2v^3 + v^2 - 2uv - 2u)\, du\right]}_{\shortparallel} dv$$

$$\left[(2v^3+v^2)u - u^2(v+1)\right]_{u=0}^{3}$$

$$= 3(2v^3+v^2) - 9(v+1)$$

$$= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\, dv = \left[\frac{3}{2}v^4 + v^3 - \frac{9}{2}v^2 - 9v\right]_0^4$$

$$= \frac{3\cdot 4^4}{2} + 4^3 - \frac{9\cdot 4^2}{2} - 9\cdot 4 = 3\cdot 4^3\cdot 2 - 8 - 9\cdot 4$$

$$= 3\cdot 4^3\cdot 2 + 4^3 - 9\cdot 4\cdot 2 - 9\cdot 4 = 3\cdot 4^3 \cdot 2 - 4(2+9) = 4(96-11)$$

$$= 3\cdot 4^3\cdot 2 + 8(8-9) - 9\cdot 4 = 3\cdot 4^3\cdot 2 - 4\cdot 11 = 4(2\cdot 3\cdot 4^2 - 11) = 4\cdot 85 = 340$$

$$\cancel{= 3\cdot 128 + 64 - 9\cdot 8 - 9\cdot 4}$$

$$= \cancel{3\cdot 128 + 8(8-9)\, 9\cdot 4 = 3\cdot 128 - 8 - 9\cdot 4}$$