Mathematical Analysis 2

Call 1 – 24/01/2023

Do **not** turn this sheet over until instructed to do so.

Name: **Sue de Nym**,
Mat: **12345** Seat: **0**

This exam consists of 5 questions, each are worth 6 points (no negative scoring), you have 3 hours to attempt the exam. Solutions available online when the exam finishes.

Fill the blanks with the correct **integer**, $-\infty$ or ∞.

- Students are permitted to bring only the following items to their desk in the exam room:
  - Two A4 sheets of paper with course notes (writing permitted on both sides, or equivalent area);
  - Pens and pencils;
  - An identity document;
  - Drinking bottle.
- Paper for rough calculations will be provided in the exam room. After the exam the paper used during the exam remains in the exam room.
- During the exam it is forbidden to communicate, using any means, with anyone except the exam invigilators. If you want to communicate, raise your hand and an invigilator will come to you.
- It is forbidden to have anything that can be used for communication (e.g., telephone, smartwatch, computer, headphones, earbuds, messenger pigeon) on your person or close to you during the exam.
- Under penalty of exclusion, during written tests the use of electronic devices and applications, is not allowed. Calculators are not permitted.
- You may choose to leave the exam early but only after receiving confirmation from an invigilator.

I affirm that I will not give or receive any unauthorized help on this exam. If I am offered unauthorized help I will notify an invigilator.

Name: Sue de Nym, Signature:

Question 1. Consider the differential equation $(x^2-1)y''+6xy'+4y=-4$. Suppose a solution of the form $y(x)=\sum_{n=0}^{\infty} a_n x^n$ and calculate that

$$\sum_{n=0}^{\infty}\left[(n+\square)(n+1)a_n-(n+2)(n+\square)a_{n+2}\right]x^n=-4$$ (2)

Consequently, $a_2=2a_0+\square$ and $a_{n+2}=a_n(n+\square)/(n+\square)$ for all $n\in\mathbb{N}$. (3)
Furthermore, writing even and odd terms separately, for all $k\in\mathbb{N}$, (1)

$$a_{2k}=(k+1)(a_0+1),\qquad a_{2k+1}=(2k+\square)a_1/3.$$

Question 2. We wish to find the extrema of $f(x,y,z)=xyz$ subject to the constraint $x+9y^2+z^2=4$ under the assumption that $x\geq 0$ and $y\geq 0$. Using the Lagrange multiplier method we find 6 possible extrema points,

$(\square,\square,-2),\quad(0,0,2),\quad(\square,0,0),\quad(0,\frac{2}{3},0),\quad(2,\frac{1}{3},-1),\quad(\square,\frac{1}{3},\square).$ (5)

The absolute maximum obtained by f is $\square/3$. (1)

Question 3. Let $R=[0,3]\times[0,3]$, let α denote a path anticlockwise around the boundary of R and consider the vector field,

$$\mathbf{F}(x,y)=\begin{pmatrix}-y^4+2y\\6x-4xy^3\end{pmatrix},\qquad(x,y)\in R.$$

Evaluate $\int\mathbf{F}\cdot d\alpha=\square$. *Hint: Green's theorem is useful and the answer is a square number.* (6)

Question 4. Let $T\subset\mathbb{R}^2$ denote the region bounded by $y^2=1-x$ and $y^2=1+x$. We write $T=\left\{(x,y):\square\leq y\leq\square,-(1-y^2)\leq x\leq(1-y^2)\right\}$ and calculate (2)

$$\int_{-(1-y^2)}^{(1-y^2)}(x^2-y^2)\,dx=\tfrac{2}{3}-4y^2+\square y^4-\tfrac{2}{3}y^6,\qquad\iint_T(x^2-y^2)\,dxdy=\frac{2^3}{3\cdot5\cdot\square}.$$ (4)

Question 5. Let $S=\mathbf{r}(T)$ be the parametric surface defined by $\mathbf{r}(u,v)=(u,v^2-u,u+v)$, $T=[0,2]\times[0,3]$. Let $\mathbf{F}(x,y,z)=\begin{pmatrix}-y\\x\\0\end{pmatrix}$ and write $\mathbf{n}$ for the unit normal. Then

$$\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)(1,2)=\begin{pmatrix}-5\\-1\\\square\end{pmatrix},\qquad\left|\iint_{\mathbf{r}(T)}\mathbf{F}\cdot\mathbf{n}\,dS\right|=\square$$ (2+4)

Hint: Sum of digits of the final answer equals 15.

$\boxed{Q1}$ Solve $(x^2-1)y'' + 6xy' + 4y = -4$

Suppose $y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$,

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

$$\Rightarrow (x^2-1)\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + 6x \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$+ 4\sum_{n=0}^{\infty} a_n x^n = -4$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1) x^n - \sum_{n=2}^{\infty} n(n-1) x^{n-2} + 6\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = -4$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n n(n-1) x^n - \sum_{n=0}^{\infty} (n+2)(n+1) \overset{a_{n+2}}{\downarrow} x^n + 6\sum_{n=0}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = -4$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n \left[a_n\left(n(n-1) + 6n + 4\right) - a_{n+2}(n+2)(n+1) \right] = -4$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n \left[a_n (n+1)(n+4) - a_{n+2}(n+1)(n+2) \right] = -4$$

$\Rightarrow a_0(4) - a_2(2) = -4 \Rightarrow a_2 = 2 + 2a_0 \qquad (n=0)$

and $\quad a_n(n+4) - a_{n+2}(n+2) = 0 \qquad (n \in \mathbb{N})$

$$\Rightarrow \begin{cases} a_{2k} = (k+1)(a_0+1) \\ a_{2k+1} = (2k+3)\dfrac{a_1}{3} \end{cases} \qquad k \in \mathbb{N}$$

$\boxed{Q2}$ $f(x,y,z) = xyz$,

$g(x,y,z) = x + 9y^2 + z^2 - 4 = 0$

$$\nabla f(x,y,z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}, \quad \nabla g(x,y,z) = \begin{pmatrix} 1 \\ 18y \\ 2z \end{pmatrix}$$

If (x,y,z) is extrema point then

$$\begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 18y \\ 2z \end{pmatrix} \Leftrightarrow \begin{cases} yz = \lambda \\ xz = 18\lambda y \\ xy = 2\lambda z \\ x + 9y^2 + z^2 = 4 \end{cases}$$

$g(x,y,z) = 0$

$$\Rightarrow \begin{cases} xyz = \lambda x \\ xyz = 18\lambda y^2 \\ xyz = 2\lambda z^2 \end{cases} \Rightarrow \lambda = 0 \text{ or } x = 18y^2 = 2z^2$$

Case $\lambda = 0$

$\Rightarrow y = 0$ or $z = 0$, $x = 0$ or $z = 0$, $x = 0$ or $y = 0$

Case $\lambda = 0, y = 0, x = 0$

$\Rightarrow 0 + 9(0) + z^2 = 4 \Rightarrow z \pm 2$ $\quad (0,0,-2)$ $(0,0,2)$

Case $\lambda = 0, z = 0, x = 0$

$\Rightarrow 0 + \cancel{9(y)^2 +} 9y^2 + 0 = 4 \Rightarrow y = \frac{2}{3}$ $\quad (0, \frac{2}{3}, 0)$

Case $\lambda = 0, y = 0, z = 0$

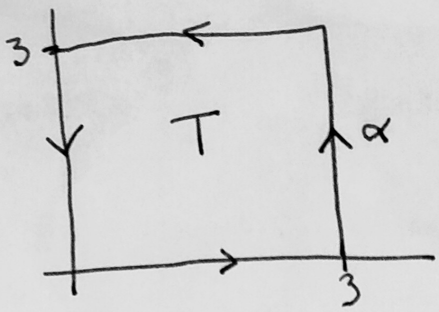
$\Rightarrow x + 0 + 0 = 4 \Rightarrow x = 4$ $\quad (4,0,0)$

Case $x = 18y^2 = 2z^2$

$\Rightarrow 18y^2 + 9y^2 + 9y^2 = 4 \Rightarrow 9y^2 = 1 \Rightarrow y = \frac{1}{3}$

$\Rightarrow x = \frac{18}{9} = 2$, $z^2 = 1$ $\quad (2, \frac{1}{3}, \pm 1)$

$\boxed{Q3}$



$$F(x,y) = \begin{pmatrix} -y^4 + 2y \\ 6x - 4xy^3 \end{pmatrix}$$

let $P(x,y) = -y^4 + 2y$, $Q(x,y) = 6x + 4xy^3$ so $F = \begin{pmatrix} P \\ Q \end{pmatrix}$

Green's theorem:

$$\iint_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int F \cdot d\alpha$$

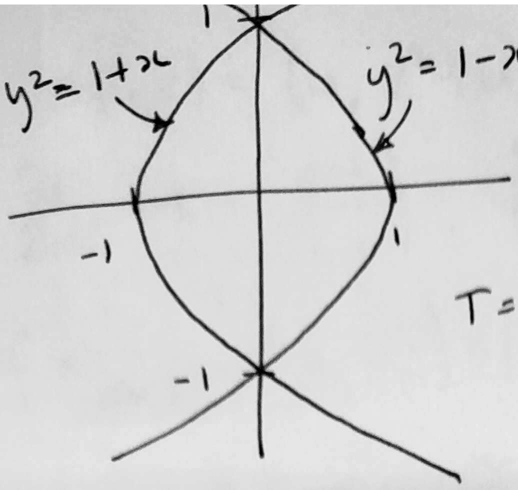
$$\frac{\partial P}{\partial y} = -4y^3 + 2, \quad \frac{\partial Q}{\partial x} = 6 - 4y^3 .$$

$$\Rightarrow \int F \cdot d\alpha = \iint_T \left(6 - 4y^3 + 4y^3 - 2 \right) dx\,dy$$

$$= \iint_T (6-2)\, dx\,dy = 4 \iint_T dx\,dy = 4 \,\text{Area}(T)$$

$$= 4\,(9) = 36$$

$\boxed{Q4}$



$T = \{(x,y) : -1 \le y \le 1, -(1-y^2) \le x \le (1-y^2)\}$

$$\iint_T (x^2-y^2)\,dx\,dy = \int_{-1}^{1}\left[\int_{-(1-y^2)}^{(1-y^2)} (x^2-y^2)\,dx\right]dy$$

$$\int_{-(1-y^2)}^{(1-y^2)} (x^2-y^2)\,dx = 2\int_0^{1-y^2}(x^2-y^2)\,dx = 2\left[\frac{x^3}{3} - xy^2\right]_{x=0}^{x=1-y^2}$$

$$= 2\left(\frac{(1-y^2)^3}{3} - (1-y^2)y^2\right)$$

$$= 2\left(\tfrac{1}{3} - y^2 + y^4 - \frac{y^6}{3}\right) - 2(1-y^2)y^2$$

$$= \tfrac{2}{3} - 2y^2 + 2y^4 - \tfrac{2}{3}y^6 - 2y^2 + 2y^4$$

$$= \tfrac{2}{3} - 4y^2 + 4y^4 - \tfrac{2}{3}y^6 \quad [= A(y)]$$

$$\int_{-1}^{1} A(y)\,dy = 2\int_0^1 A(y)\,dy = 2\left[\tfrac{2}{3}y - \tfrac{4}{3}y^3 + \tfrac{4}{5}y^5 - \frac{2}{(3)(7)}y^7\right]_0^1$$

$$= 2\left(\tfrac{2}{3} - \tfrac{4}{3} + \tfrac{4}{5} - \frac{2}{(3)(7)}\right) = 4\left(-\tfrac{1}{3} + \tfrac{2}{5} - \frac{1}{(3)(7)}\right)$$

$$= \frac{4}{(3)(3)(5)(7)}\left(-3\cdot5\cdot7 + 2\cdot3\cdot3\cdot7 - 3\cdot5\right)$$

$$= \frac{4}{3\cdot3\cdot5\cdot7}\left(3\cdot7(-5+6) - 15\right) = \frac{4}{3\cdot3\cdot5\cdot7}(21-15) = \frac{4\cdot6}{3\cdot3\cdot5\cdot7}$$

$$= \frac{2^3}{3\cdot5\cdot7}$$

$\boxed{Q5}$ $r(u,v) = (u,\ v^2 - u,\ u+v)$

$$\frac{\partial r}{\partial u}(u,v) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \frac{\partial r}{\partial v}(u,v) = \begin{pmatrix} 0 \\ 2v \\ 1 \end{pmatrix}$$

$$\left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right)(u,v) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2v \\ 1 \end{pmatrix} = \begin{pmatrix} -1-2v \\ -1 \\ 2v \end{pmatrix} \ (= N(u,v))$$

$$F(r(u,v)) = \begin{pmatrix} u - v^2 \\ u \\ 0 \end{pmatrix}, \quad F(r(u,v)) \cdot N(u,v) = (1+2v)(v^2-u) - u$$

$$= v^2 - 2u + 2v^3 - 2uv$$

$$\iint_S F \cdot n \, dS = \iint_T (v^2 - 2u + 2v^3 - 2uv)\, du\, dv$$

$$= \int_0^3 \left[\int_0^2 (v^2 - 2u + 2v^3 - 2uv)\, du \right] dv$$

$$\int_0^2 (v^2 - 2u + 2v^3 - 2uv)\, du = \int_0^2 (v^2 + 2v^3) - 2u(1+v)\, du$$

$$= \Big[(v^2+2v^3)u - u^2(1+v) \Big]_{u=0}^{2} = 2(v^2+2v^3) - 4(1+v)$$

$$\iint_S F \cdot n\, dS = \int_0^3 (-4 - 4v + 2v^2 + 4v^3)\, dv$$

$$= \left[-4v - 2v^2 + \frac{2}{3}v^3 + v^4 \right]_0^3$$

$$= -4 \cdot 3 - 2 \cdot 3^2 + 2 \cdot 3^2 + 3^4 \qquad = -12 + 81$$

$$= \cancel{-12 + 27} = \cancel{-(39)} \qquad = 69$$