MA2 – Call 6 – 15/09/2022

Fill the blanks with the correct **integer** (possibly zero or negative), $-\infty$ or ∞, select the correct options in multiple choices.

Question 1. Consider the differential equation $y' = 2x^2 - y^2$ with initial condition $y(0) = 1$. We observe that $y'' = \boxed{\mathbf{4}}\, x - \boxed{\mathbf{2}}\, yy'$ and the differential equation has the following power series solution (each part 1 pt.).

$$y(x) = \boxed{\mathbf{1}} + \boxed{\mathbf{-1}}\, x + \frac{\boxed{\mathbf{2}}}{2!}x^2 + \frac{(-2)}{3!}x^3 + \frac{\boxed{\mathbf{16}}}{4!}x^4 + \cdots$$

Solution. We start with the given equation and differentiate repeatedly.

$$\begin{aligned} y' &= 2x^2 - y^2 \\ y'' &= 4x - 2yy' \\ y''' &= 4 - 2(y')^2 - 2yy'' \\ y'''' &= -4y'y'' - 2y'y'' - 2yy'''. \end{aligned}$$

Using the initial condition $y(0) = 1$ and the above equations we calculate that

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 - 1^2 = -1 \\ y''(0) &= 0 - 2 \cdot 1 \cdot (-1) = 2 \\ y'''(0) &= 4 - 2 \cdot (-1)^2 - 2 \cdot 1 \cdot 2 = -2 \\ y''''(0) &= -4 \cdot (-1) \cdot 2 - 2 \cdot (-1) \cdot 2 - 2 \cdot 1 \cdot (-2) = 8 + 4 + 4 = 16. \end{aligned}$$

We then apply Taylor's formula.

Question 2. For each $n \in \mathbb{N}$ consider the functions defined for $x \in \mathbb{R}$ by

$$f_n(x) = \frac{n}{\sqrt{\pi}} e^{-(nx)^2}, \qquad g_n(x) = \frac{1}{1 + e^{-nx}}.$$

Integrating we find that $\int_{-\infty}^{\infty} f_1(x)\, dx = 1$ and $\int_{-\infty}^{\infty} f_{10}(x)\, dx = \boxed{\mathbf{1}}$. *Hint: change variables in the integral.* We observe that $f_n(0) \to \infty$ as $n \to \infty$ and, if $x < 0$, $f_n(x) \to \boxed{\mathbf{0}}$ as $n \to \infty$. The sequence $\{f_n\}_n$ $\boxed{\square \text{ is } / \square \text{ is not}}$ pointwise convergent on the domain $x > 0$. If $x > 0$, $g_n(x) \to \boxed{\mathbf{1}}$ as $n \to \infty$, if $x < 0$, $g_n(x) \to \boxed{\mathbf{0}}$ as $n \to \infty$. The sequence $\{g_n\}_n$ $\boxed{\square \text{ is } / \square \text{ is not}}$ pointwise convergent on $\mathbb{R}$. (Each part 1pt.)

Solution. Changing variables in the integral we see that the integral does not depend on n. The sequence f_n converges pointwise to 0 on $\{x > 0\}$. In a loose sense the sequence f_n converges to the Dirac delta "function". The sequence g_n converges pointwise to the step function

$$g(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0. \end{cases}$$

Question 3. Use Lagrange's multipliers to find the maximum and minimum values of $f(x, y, z) = y^2 - 10z - 30$ subject to the constraint $x^2 + y^2 + z^2 = 36$. The absolute minimum is -90 and the absolute maxima is $\boxed{\mathbf{31}}$ (6 pts). *Hint: The answer is a prime.*

Solution. We calculate the gradient of $f(x, y, z) = y^2 - 10z$ and the gradient of the constraint $g(x, y, z) = x^2+y^2+z^2$. We then introduce λ according to Lagrange's multiplier method and arrive at the following series of equations.

$$\begin{aligned} 0 &= 2x\lambda \\ 2y &= 2y\lambda \\ -10 &= 2z\lambda \\ 36 &= x^2 + y^2 + z^2. \end{aligned}$$

From the first equation we see that either $x = 0$ or $\lambda = 0$. However, if $\lambda = 0$, then the third wouldn't be satisfied and so it must be that $x = 0$. The second equation can be written as $2y(1 - \lambda) = 0$ and so we see that this implies that either $y = 0$ or $\lambda = 1$. We need to separately check these two options. Case $y = 0$: Using also that $x = 0$ the forth equation implies that $z^2 = 36$ and so $z = \pm 6$. This gives two possible extrema points:

$$(0, 0, -6), \quad (0, 0, 6).$$

Case $\lambda = 1$: From the third equation this implies that $-10 = 2z$ and so $z = -5$. Now using the forth equation we have that $y^2 + 25 = 36$ and so $y^2 = 11$ and consequently $y = \pm\sqrt{11}$. This gives two possible extrema points:

$$(0, -\sqrt{11}, -5), \quad (0, \sqrt{11}, -5).$$

We now check the value of $f(x, y, z)$ at each of these points.

$$\begin{aligned} f(0, 0, -6) &= 30, & f(0, 0, 6) &= -90, \\ f(0, -\sqrt{11}, -5) &= 31, & f(0, \sqrt{11}, -5) &= 31. \end{aligned}$$

The absolute maximum is 31.

Question 4. Let $\mathbf{f}(x, y) = (xy, 1 + 3y)$ be a vector field on $\mathbb{R}^2$ and consider the paths: $\boldsymbol{\alpha}_1(t) = (-2t, -4), t \in (0, 1)$; $\boldsymbol{\alpha}_2(t)$ follows curve $y = -x^2$ from $(-2, -4)$ to $(2, -4)$; $\boldsymbol{\alpha}_3(t)$ is straight line from $(2, -4)$ to $(5, 1)$. The integrals are (each part 2 pts.).

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha}_1 = \boxed{\mathbf{-8}}, \quad \int \mathbf{f} \cdot d\boldsymbol{\alpha}_2 = \boxed{\mathbf{0}}, \quad \int \mathbf{f} \cdot d\boldsymbol{\alpha}_3 = \frac{\boxed{\mathbf{-59}}}{2}.$$

Hint: Integral along the path $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3$ *is equal to* $-\frac{75}{2}$.

Solution. We have the paths

$$\begin{aligned}
\boldsymbol{\alpha}_1(t) &= (-2t, -4), \quad t \in (0, 1), \\
\boldsymbol{\alpha}_2(t) &= (t, -t^2), \quad t \in (-2, 2), \\
\boldsymbol{\alpha}_3(t) &= (2 + 3t, -4 + 5t), \quad t \in (0, 1).
\end{aligned}$$

The line integrals are therefore

$$\begin{aligned}
\int \mathbf{f} \cdot d\boldsymbol{\alpha}_1 &= \int_0^1 -16t \, dt = -8, \\
\int \mathbf{f} \cdot d\boldsymbol{\alpha}_2 &= \int_{-2}^2 5t^3 - 2t \, dt = 0, \\
\int \mathbf{f} \cdot d\boldsymbol{\alpha}_3 &= \int_0^1 45t^2 + 69t - 79 \, dt = -\frac{59}{2}.
\end{aligned}$$

Question 5. Let S be the triangle with vertices $(0, 0), (4, 4), (0, 12)$. The center of mass is at the point $\left(4/\boxed{\text{a}}, \boxed{\text{b}}/3\right)$ where $\boxed{\text{a}} = \boxed{\mathbf{3}}$, $\boxed{\text{b}} = \boxed{\mathbf{16}}$ (3 pts each). *Hint:* $\boxed{a} + \boxed{b} = 19$; $M = \iint_S dxdy$, $M_x = \iint_S x \, dxdy$ *and* $M_y \iint_S y \, dxdy$ *(area, x-moment and y-moment), are all even numbers; centre of mass is the point* $(M_x/M, M_y/M)$.

Solution. It can be helpful to sketch the triangle. This region can be considered as "projected" onto the x-axis, onto the interval $[4, 0]$ and bounded below by the curve $y = x$, above by the curve $y = 12 - 2x$. We first calculate the area of the triangle (mass with unit density),

$$M = \int_0^4 (12 - 2x) - (x) \, dx = \int_0^4 -3x + 12 \, dx = \left[\frac{3}{2}x^2 + 12x\right]_0^4 = 24.$$

Now we calculate the two moments (assuming constant density),

$$
\begin{aligned}
M_x &= \int_0^4 \left(\int_x^{12-2x} x \, dy \right) \, dx = \int_0^4 x \left((12 - 2x) - (x) \right) \, dx \\
&= \int_0^4 12x - 3x^2 \, dx = \left[-x^3 + 6x^2 \right]_0^4 = 32,
\end{aligned}
$$

and

$$
\begin{aligned}
M_y &= \int_0^4 \left(\int_x^{12-2x} y \, dy \right) \, dx = \int_0^4 \tfrac{(12-2x)^2}{2} - \tfrac{x^2}{2} \, dx \\
&= \int_0^4 \frac{3x^2}{2} - 24x + 72 \, dx = \left[\frac{x^3}{2} - 12x^2 + 72x \right]_0^4 = 128.
\end{aligned}
$$

The centre of mass is at the point $(M_x/M, M_y/M) = (4/3, 16/3)$.