MA2 – Call 4 – 01/07/2022 Name:

Fill the blanks with the correct **integer** (possibly zero or negative), select the correct options in multiple choices.

Question 1. The Taylor series expansion, with initial point $x_0 = 1$, of the function

$$f(x) = 2(x-1)\ln(x^2 - 2x + 2),$$

is $\sum_{n=0}^{\infty} a_n(x-1)^n$ where $a_0 = \boxed{}$, $a_1 = \boxed{}$, $a_2 = \boxed{}$, $a_3 = -2$, $a_4 = 0$, $a_5 = 2$. The series is convergent when $\boxed{\text{a}} < x < \boxed{\text{b}}$ and divergent when $x < \boxed{\text{a}}$ or when $\boxed{\text{b}} < x$ where $\boxed{\text{a}} = \boxed{}$ and $\boxed{\text{b}} = \boxed{}$. For $x = \boxed{\text{a}}$ and for $x = \boxed{\text{b}}$ the series ☐ is / ☐ is not convergent. (1 pt each part.) *Hint:* $x^2 - 2x + 2 = ? + (x-1)^2$.

Solution. We write $x^2 - 2x + 2 = 1 + (x-1)^2$. We recall or calculate that

$$\log(1+y) = \sum_{n=1}^{\infty} (-1)^n \frac{y^n}{n},$$

with radius of convergence equal to 1. Putting $y = (x-1)^2$ we have that

$$\log(1+(x-1)^2) = \sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^{2n}}{n},$$

and this converges when $|x-1| < 1$ and diverges when $|x-1| > 1$. Multiplying by $2(x-1)$ doesn't change the radius of convergence and we obtain

$$2(x-1)\log(1+(x-1)^2) = \sum_{n=1}^{\infty} 2(-1)^n \frac{(x-1)^{2n+1}}{n}.$$

Calculating at $x = 2$ we obtain $f(2) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ which is an alternating series and convergent by the Leibniz criterion.

Question 2. Let $(a_1, b_1) = (1, 2)$, $(a_2, b_2) = (2, 3)$, $(a_3, b_4) = (-1, 1)$. Find and classify all the stationary points $(x, y) \in \mathbb{R}^2$ of the function $f(x, y) = \sum_{n=1}^{3} (\frac{a_n}{2}x - y - b_n)^2$. The number of stationary points is $\boxed{}$ (2pts). The stationary point in the upper right quadrant is at $\left(\frac{\boxed{\text{a}}}{7}, \frac{\boxed{\text{b}}}{7}\right)$ where $\boxed{\text{a}} = \boxed{}$ and $\boxed{\text{b}} = \boxed{}$ (1 pt each). The Hessian at this point is a 2×2 matrix with determinant equal to 14 and trace equal to $\boxed{}$ (2 pts). Consequently this point is a minimum.

Solution. We calculate the gradient of the scalar field. $\nabla f = \binom{A}{B}$ where $A(x,y) = \sum_{n=1}^{3} a_n(\frac{a_n}{2}x - y - b_n)$ and $B(x,y) = \sum_{n=1}^{3} -2(\frac{a_n}{2}x - y - b_n)$. Substituting the values for a_n and b_n we obtain

$$\begin{aligned}
A(x,y) &= 1(\frac{x}{2} - y - 2) + 2(x - y - 3) + -1(-\frac{x}{2} - y - 1) \\
&= 3x - 2y - 7 = 0 \\
B(x,y) &= -2(\frac{x}{2} - y - 2) + -2(x - y - 3) + -2(-\frac{x}{2} - y - 1) \\
&= -2x + 6y + 12 = 0.
\end{aligned}$$

Solving these equations we obtain a single solution $x = \frac{9}{7}, y = \frac{-11}{7}$. The Hessian matrix is

$$\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$$

and so has trace equal to 9 and determinant equal to 14. This means that both eigenvalues are positive and so the point in a minimum.

Question 3. Let C be the curve $\{(x,y) : x^2 + y^2 = 1, x \geq 0\}$ in $\mathbb{R}^2$ And let

$$\mathbf{f}(x,y) = \binom{2y}{x^2}$$

be a vector field on $\mathbb{R}^2$. Find a parametrization $\boldsymbol{\alpha}(t)$ of C which starts at $(0,-1)$ and ends at $(0,1)$. Then show that $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \frac{1}{3}\left(\boxed{\text{a}} + \boxed{\text{b}}\pi\right)$ where $\boxed{\text{a}} = \boxed{}$ and $\boxed{\text{b}} = \boxed{}$ (3 pts each). *Hint:* $\int \sin^2 t\, dt = \frac{t}{2} - \frac{1}{2}\sin t + C$, $\int \cos^3 t\, dt = \sin t - \frac{1}{3}\sin^3 t + C$.

Solution. A usable parametrization of the path is given by

$$\boldsymbol{\alpha}(t) = (2\cos t, \sin t), \quad t \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}).$$

Consequently $\mathbf{f}(\boldsymbol{\alpha}(t)) = (\sin t + 1, \cos^2 t)$ and $\boldsymbol{\alpha}'(t) = \binom{-\sin t}{\cos t}$. The line integral is then

$$\begin{aligned}
\int \mathbf{f} \cdot d\boldsymbol{\alpha} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)\, dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2\sin^2 t + \cos^3 t\, dt = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t\, dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 t\, dt \\
&= -2\left[\frac{t}{2} - \frac{1}{2}\sin t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left[\sin t - \frac{1}{3}\sin^3 t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= -2\left(\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{4} - \frac{1}{2}\right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = -\pi + \frac{10}{3}.
\end{aligned}$$

Question 4. Let S denote the region determined by $y \leq 4 - x^2, x \geq 0, y \geq 0$. Sketch this region and calculate that the area of S is $A = \iint_S \, dxdy = \boxed{}/3$. The two moments are $M_y = \iint_S y \, dxdy = \boxed{}/15$ and $M_x = \iint_S x \, dxdy = 4$. Using this we conclude that the centre of mass of S is $(\frac{3}{4}, \frac{\boxed{a}}{5})$ where $\boxed{a} = \boxed{}$. (2 pts each part.) *Hint: centre of mass* (x_0, y_0) *satisfies* $x_0 A = M_x$ *and* $y_0 A = M_y$.

Solution. See tutorial.math.lamar.edu.

Question 5. Let S be the bounded portion of the paraboloid $x^2 + y^2 = z$ which is cut off by the plane $z = 4$. Sketch S and choose a parametric representation of this surface. A possible parametric representation is $\mathbf{r}(u, v) = (u \cos v, u \sin v, u^2)$, $(u, v) \in T$ for a suitable choice of $T \subset \mathbb{R}^2$. Use this to compute that the area of S is equal to $\frac{\pi}{6}\left(\boxed{a}^{\frac{3}{2}} + 1\right)$ where $\boxed{a} = \boxed{}$ (6 pts).

Solution. We use the parametric representation is $\mathbf{r}(u, v) = (u \cos v, u \sin v, u^2)$, $(u, v) \in T$, where $T = \{(u, v) : 0 \leq u \leq 2, 0 \leq v \leq 2\pi\}$. Firstly we calculate the fundamental vector product.

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) = \begin{pmatrix} \cos v \\ \sin v \\ 2u \end{pmatrix}, \qquad \frac{\partial \mathbf{r}}{\partial v}(u, v) = \begin{pmatrix} -u \sin v \\ u \cos v \\ 0 \end{pmatrix},$$

and so

$$\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)(u, v) = \begin{pmatrix} -2u^2 \cos v \\ -2u^2 \sin v \\ u(\sin^2 v + \sin^2 v) \end{pmatrix}.$$

This means that

$$\left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\|(u, v) = u\left(4u^2 + 1\right)^{\frac{1}{2}}.$$

Finally we can calculate the area.

$$\begin{aligned} \text{Area} &= \int_0^2 \int_0^{2\pi} u\left(4u^2 + 1\right)^{\frac{1}{2}} \, dv \, du \\ &= 2\pi \int_0^2 u\left(4u^2 + 1\right)^{\frac{1}{2}} \, du = 2\pi \left[\frac{1}{12}\left(4u^2 + 1\right)^{\frac{3}{2}}\right]_0^2 \\ &= \frac{\pi}{6}\left(17^{\frac{3}{2}} + 1\right). \end{aligned}$$