MA2 – Call 3 – 14/06/2022 Name:

Fill each blank with the correct **integer** (possibly zero or negative), select the correct option in each multiple choice.

Question 1. The Taylor series expansion, with initial point $x_0 = 1$, of the function

$$f(x) = \frac{x}{(x-2)(x^2-2x+2)},$$

is $\sum_{n=0}^{\infty} a_n (x-1)^n$ where $a_0 = -1$, $a_1 = \boxed{}$, $a_2 = \boxed{}$, $a_3 = \boxed{}$, $a_4 = \boxed{}$, $a_5 = -2$. The radius of convergence is $r = \boxed{}$. At $x = 1 - r$ the series is not convergent, at $x = 1 + r$ the series ☐ is / ☐ is not convergent. (Each part 1 pt.)

Solution. See MA2 2018-19 Call2 Q1.

Question 2. Use Lagrange's multiplier method to find the extremal values of the function $f(x, y, z) = x + 2y + 2z$ on the surface S defined by $x^2 + y^2 + z^2 = 1$. The maximum is $\boxed{}$ and the minimum is $\boxed{}$ (3 pts each). *Hint: The maximum is obtained at a point (x_0, y_0, z_0) where $x_0 + y_0 + z_0 = \frac{5}{3}$.*

Solution. See MA2 2018-19 Call2 Q2.

Question 3. Consider the vector field on $\mathbb{R}^2$,

$$\mathbf{F}(x, y) = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix} = \begin{pmatrix} \cos(xy) - xy\sin(xy) \\ -x^2\sin(xy) + x^3 \end{pmatrix}.$$

Calculate $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$. In particular $(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})(3, 4) = \boxed{}$ (2 pts). Let $\boldsymbol{\alpha}(t)$ be the closed path defined as

$$\boldsymbol{\alpha}(t) = \begin{cases} (t, 0) & \text{if } 0 \le t \le 1 \\ (2-t, t-1) & \text{if } 1 \le t \le 2 \\ (0, 3-t) & \text{if } 2 \le t \le 3. \end{cases}$$

Show that $\int \mathbf{F} \cdot d\boldsymbol{\alpha} = \frac{1}{\boxed{\text{a}}}$ where $\boxed{\text{a}} = \boxed{}$ (4pts). *Hint: $\boxed{a} + 3$ is prime.*

Solution. See MA2 2018-19 Call3 Q3.

Question 4. Let D be the set of all $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 \leq 1$ and $z \geq \sqrt{x^2 + y^2}$. We wish to compute the integral

$$\iiint_D z(x^2 + y^2 + z^2)e^{-(x^2+y^2)}\, dxdydz.$$

Since D is xy-projectable we can write

$$D = \left\{(x, y, z) : (x, y) \in D_0, \sqrt{x^2 + y^2} \leq z \leq \sqrt{1 - (x^2 + y^2)}\right\},$$

$D_0 = \left\{(x, y) : x^2 + y^2 \leq 1/\boxed{\text{a}}\right\}$ where $\boxed{\text{a}} = \boxed{}$ (2 pts). Integrating with respect to z and then changing to polar coordinates we obtain an integral of the form $\frac{\pi}{?} \int_0^? e^{-r^2}(r+r^?)\, dr$ (a change in variables $t = r^2$ could be useful for evaluating the integral involving e^{-r^2}). Evaluating this we find that the volume integral is equal to

$$\frac{\pi}{\boxed{\text{b}}}\left(3 - \frac{\boxed{\text{c}}}{4}e^{-\frac{1}{2}}\right)$$

where $\boxed{\text{b}} = \boxed{}$ and $\boxed{\text{c}} = \boxed{}$ (2 pts each). *Hint: Both* $\boxed{b}$ *and* $\boxed{c}$ *are prime.*

Solution. See MA2 2018-19 Call2 Q4.

Question 5. Let $a > 0$ be fixed and let $S = \left\{(x, y, z) : x^2 + y^2 + z^2 = a^2\right\}$. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$,

$$\mathbf{F}(x, y, z) = \begin{pmatrix} x(x^2 + y^2 + z^2) \\ y(x^2 + y^2 + z^2) \\ z(x^2 + y^2 + z^2) \end{pmatrix}.$$

Compute the divergence of $\mathbf{F}$. In particular, $(\nabla \cdot \mathbf{F})(1, 2, 3) = \boxed{}$ (2 pts). Let $\mathbf{n}$ denote the outgoing normal vector on S and recall the divergence theorem ($\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dxdydz$). Compute the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \boxed{\text{a}}\pi a^{\boxed{\text{b}}},$$

where $\boxed{\text{a}} = \boxed{}$, $\boxed{\text{b}} = \boxed{}$ (2 pts each). *Hint:* $\boxed{a} + \boxed{b} = 9$.

Solution. See MA2 2018-19 Call1 Q5.