MA2 – Call 2 - 31/01/2022

Name:

Q1

Consider the differential equation $y'(x) = x^2 + y(x)^2$ with initial condition $y(0) = 1$. We observe that $y'' = \boxed{\text{a}}x + \boxed{\text{b}}yy'$. The differential equation has power series solution

$$y(x) = \boxed{\text{c}} + \boxed{\text{d}}x + \frac{\boxed{\text{e}}}{2!}x^2 + \frac{\boxed{\text{f}}}{3!}x^3 + \frac{28}{4!}x^4 + \cdots$$

The missing numbers are $\boxed{\text{a}} = \boxed{}$, $\boxed{\text{b}} = \boxed{}$, $\boxed{\text{c}} = \boxed{}$, $\boxed{\text{d}} = \boxed{}$, $\boxed{\text{e}} = \boxed{}$, $\boxed{\text{f}} = \boxed{}$. *Fill in each of the blanks with the correct **integer**, possibly zero or negative (1 point each).*

Q2

Let $g(x, y) = x^2 + xy + y^2$. We will find the points in the set $\{g(x, y) = 21\} \subset \mathbb{R}^2$ which are closest / furthest from the origin. Introduce a suitable function $f(x, y)$ and apply the Lagrange multiplier method with the constraint $g(x, y) = 21$ in order to find the extrema points. In total there are $\boxed{}$ extrema points. Two of these extrema points are $(-\sqrt{\boxed{\text{a}}}, \sqrt{\boxed{\text{b}}})$ and $(\sqrt{\boxed{\text{c}}}, \sqrt{\boxed{\text{d}}})$. The point(s) on the curve $g(x, y) = 21$ closest to the origin are at a distance $\sqrt{\boxed{\text{e}}}$ from the origin. The missing numbers are $\boxed{\text{a}} = \boxed{}$, $\boxed{\text{b}} = \boxed{}$, $\boxed{\text{c}} = \boxed{}$, $\boxed{\text{d}} = \boxed{}$, $\boxed{\text{e}} = \boxed{}$. *Fill in each of the blanks with the correct **integer**, possibly zero or negative (1 point each).*

Q3

Determine which of the following is a parametrization of the path

$$C = \{(x, y) : x^2 + 4y^2 = 4, x \geq 0\} \subset \mathbb{R}^2,$$

starting at $(0, -1)$ and finishing at $(0, 1)$:

- $\boldsymbol{\alpha}(t) = (-2\sin t, \cos t), t \in [-\pi, 0]$ ☐ is / ☐ is not
- $\boldsymbol{\alpha}(t) = (-2\cos t, \sin t), t \in [-\pi, 0]$ ☐ is / ☐ is not
- $\boldsymbol{\alpha}(t) = (2\sqrt{1 - t^2}, t), t \in [-1, 1]$ ☐ is / ☐ is not
- $\boldsymbol{\alpha}(t) = (2 - 2t^2, t), t \in [-1, 1]$ ☐ is / ☐ is not
- $\boldsymbol{\alpha}(t) = (\frac{2t^2-2}{t^2+1}, \frac{2t}{t^2+1}), t \in [-1, 1]$ ☐ is / ☐ is not
- $\boldsymbol{\alpha}(t) = (\frac{2-2t^2}{t^2+1}, \frac{2t}{t^2+1}), t \in [-1, 1]$ ☐ is / ☐ is not

Hint: 3 are the correct path and 3 are not. *Select the correct option for each (+1 point for each correct answer, -1 point for each incorrect, minimum score for the question is zero).*

Q4

Part A: Let $S \subset \mathbb{R}^2$ be the triangular region with vertices $(0,0)$, $(0,\pi)$, (π,π). Let $f(x,y) = 7\cos(x+y)$. Evaluate $\iint_S f(x,y)\,dxdy = \boxed{}$.
Part B: Let $T \subset \mathbb{R}^2$ be the region bounded by the curve $y = \sin x$ and the straight line between $(0,0)$ and $(\pi,0)$. Let $g(x,y) = 18y^2$. Evaluate $\iint_T g(x,y)\,dxdy = \boxed{}$. Hint: $\int_0^\pi \sin^3 x\,dx = \frac{4}{3}$.
*Fill in each of the blanks with the correct **integer**, possibly zero or negative (3 points each).*

Q5

Consider the surface $S = \{(x,y,z) : x^2 + y^2 = z, z \leq 4\}$. A possible choice for the parametric form of the surface S is to let

$$T = \{(r,\theta) : r \in [0, \boxed{\text{a}}], \theta \in [0, 2\pi]\}$$

and let

$$\mathbf{r} : (r,\theta) \mapsto \left(r\cos\theta, r\sin\theta, r^2\right)$$

such that $S = \mathbf{r}(T)$. For this parametric representation calculate the fundamental vector product $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}$. Just to check, we calculate

$$\left(\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right)(2,\pi) = \begin{pmatrix} \boxed{\text{b}} \\ \boxed{\text{c}} \\ 2 \end{pmatrix}.$$

The missing numbers are $\boxed{\text{a}} = \boxed{}$, $\boxed{\text{b}} = \boxed{}$, $\boxed{\text{c}} = \boxed{}$ (1 point each). Consider the vector field

$$\mathbf{f}(x,y,z) = \begin{pmatrix} 2 \\ 0 \\ 3z \end{pmatrix}$$

and let $\mathbf{n}$ be the unit normal to S which has **positive** z-component. Evaluate the surface integral $\iint_S \mathbf{f} \cdot \mathbf{n}\,dS = \boxed{}\pi$ (3 points). Hint: the final answer is a 2 digit number and the difference of the digits is 2. *Fill in each blank with the correct **integer**, possibly zero or negative.*

Q1 Solution:

We start with the given equation and differentiate repeatedly.

$$\begin{aligned} y' &= x^2 + y^2 \\ y'' &= 2x + 2yy' \\ y''' &= 2 + 2(y')^2 + 2yy'' \\ y'''' &= 4y'y'' + 2y'y'' + 2yy'''. \end{aligned}$$

Using the initial condition $y(0) = 1$ and the above equations we calculate that

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 + 1^2 = 1 \\ y''(0) &= 0 + 2 \cdot 1 \cdot 1 = 2 \\ y'''(0) &= 2 + 2 \cdot 1^2 + 2 \cdot 1 \cdot 2 = 8 \\ y''''(0) &= 4 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot 8 = 8 + 4 + 16 = 28. \end{aligned}$$

We then apply Taylor's formula.

Q2 Solution:

Let $g(x,y) = x^2 + xy + y^2$. One suitable choice of function for finding points closest / furthest from the origin is $f(x,y) = x^2 + y^2$. We calculate

$$\nabla g(x,y) = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}, \quad \nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$

According to the Lagrange multiplier method we introduce $\lambda \in \mathbb{R}$ and write

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}.$$

Multiplying the first line by y and the second line by x we obtain that $2xy = 2\lambda xy + \lambda y^2$ and $2xy = \lambda x^2 + 2\lambda xy$. Equating these implies that $2\lambda xy + \lambda y^2 = \lambda x^2 + 2\lambda xy$. We can discard the possibility that $\lambda = 0$ because this would contradict the constraint $g(x,y) = 21$. This means that $y^2 = x^2$. We treat the case $y = x$ and $y = -x$ independently.

Case $y = x$**:** Substituting into $x^2 + xy + y^2 - 21 = 0$ we obtain $3x^2 = 21$. Consequently $x = \pm\sqrt{7}$. This gives two solutions: $(\sqrt{7}, \sqrt{7})$ and $(-\sqrt{7}, -\sqrt{7})$.

Case $y = -x$**:** Substituting into $x^2 + xy + y^2 - 21 = 0$ we obtain $(2-1)x^2 = 21$. Consequently $x = \pm\sqrt{21}$. This gives two solutions: $(\sqrt{21}, -\sqrt{21})$ and $(-\sqrt{21}, \sqrt{21})$.

This set is an ellipse. Two extrema are the two points equally close as each other to the origin, the other two extrema are the two points equally far as each other from the origin.

The distance of the point $(\sqrt{7}, \sqrt{7})$ from the origin is $\sqrt{14}$.

Q3 Solution:

1. $\boldsymbol{\alpha}(t) = (-2\sin t, \cos t), t \in [-\pi, 0]$ **is good**.
    - End points $\boldsymbol{\alpha}(-\pi) = (0, -1), \boldsymbol{\alpha}(0) = (0, 1)$,
    - For all $t \in [-\pi, 0]$, $\sin t \leq 0$ and so $-2\sin t \geq 0$,
    - $(-2\sin t)^2 + 4(\cos t)^2 = 4$.
2. $\boldsymbol{\alpha}(t) = (-2\cos t, \sin t), t \in [-\pi, 0]$ **is not good**.
    - E.g., end point incorrect: $\boldsymbol{\alpha}(0) = (-2, 0)$.
3. $\boldsymbol{\alpha}(t) = (2\sqrt{1-t^2}, t), t \in [-1, 1]$ **is good**.
    - End points $\boldsymbol{\alpha}(-1) = (0, -1), \boldsymbol{\alpha}(1) = (0, 1)$,
    - For all $t \in [-1, 1], 2\sqrt{1-t^2} \geq 0$,
    - $(2\sqrt{1-t^2})^2 + 4(t)^2 = 4(1-t^2) + 4t^2 = 4$.
4. $\boldsymbol{\alpha}(t) = (2 - 2t^2, t), t \in [-1, 1]$ **is not good**.
    - E.g., $(2-2t^2)^2 + 4(t)^2 = 4t^4 - 4t^2 + 4 = 4(t^4 - t^2 + 1)$ but this should be equal to 4 for all $t \in [-1, 1]$.
5. $\boldsymbol{\alpha}(t) = (\frac{2t^2-2}{t^2+1}, \frac{2t}{t^2+1}), t \in [-1, 1]$ **is not good**.
    - E.g., $\boldsymbol{\alpha}(0) = (-1, 0)$ but this isn't a point in C.
6. $\boldsymbol{\alpha}(t) = (\frac{2-2t^2}{t^2+1}, \frac{2t}{t^2+1}), t \in [-1, 1]$ **is good**.
    - End points $\boldsymbol{\alpha}(-1) = (0, -1), \boldsymbol{\alpha}(1) = (0, 1)$,
    - For all $t \in [-1, 1]$, $\frac{2-2t^2}{t^2+1} \geq 0$,
    - $(\frac{2-2t^2}{t^2+1})^2 + 4(\frac{2t}{t^2+1})^2 = \frac{4-8t^2+4t^4+16t^2}{(t^2+1)^2} = \frac{4(1+2t^2+t^4)}{(t^2+1)^2} = 4$.

Q4 Solution:

Let $S \subset \mathbb{R}^2$ be the triangular region with vertices $(0,0)$, $(0,\pi)$, (π,π). Let $f(x,y) = 7\cos(x+y)$. It is convenient to write (a sketch might help to clarify this)

$$S = \{(x,y) : 0 \le x \le \pi, x \le y \le \pi\}.$$

This means that

$$\iint_S f(x,y)\,dxdy = 7\int_0^\pi \left[\int_x^\pi \cos(x+y)\,dy\right]\,dx.$$

For the inner integral,

$$\int_x^\pi \cos(x+y)\,dy = [\sin(x+y)]_{y=x}^{y=\pi} = \sin(x+\pi) - \sin(2x).$$

Now

$$\begin{aligned}\int_0^\pi \sin(x+\pi) - \sin(2x)\,dx &= \left[-\cos(x+\pi) + \frac{1}{2}\cos(2x)\right]_{x=0}^{x=\pi} \\ &= -\cos(2\pi) + \frac{1}{2}\cos(2\pi) + \cos(\pi) - \frac{1}{2}\cos(0) = -2.\end{aligned}$$

Consequently $\iint_S f(x,y)\,dxdy = -2 \cdot 7 = -14$.

Let $T \subset \mathbb{R}^2$ be the region bounded by the curve $y = \sin x$ and the straight line between $(0,0)$ and $(\pi,0)$. Let $g(x,y) = 18y^2$. It is convenient to write (a sketch might help to clarify this)

$$T = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}.$$

This means that

$$\iint_T g(x,y)\,dxdy = 18\int_0^\pi \left[\int_0^{\sin x} y^2\,dy\right]\,dx.$$

For the inner integral,

$$\int_0^{\sin x} y^2\,dy = \left[\frac{y^3}{3}\right]_{y=0}^{y=\sin x} = \frac{1}{3}\sin^3 x.$$

Now, since $\int_0^\pi \sin^3 x\,dx = \frac{4}{3}$,

$$\int_0^\pi \frac{1}{3}\sin^3 x\,dx = \frac{4}{9}.$$

This means that $\iint_T g(x,y)\,dxdy = 18 \cdot \frac{4}{9} = 8$.

Q5 Solution:

Consider the surface $S = \{(x, y, z) : x^2 + y^2 = z, z \leq 2^2\}$. We choose the parametric form of the surface S by letting $T = \{(r, \theta) : r \in [0, 2], \theta \in [0, 2\pi]\}$ and

$$\mathbf{r} : (r, \theta) \mapsto (r\cos\theta, r\sin\theta, r^2).$$

We calculate

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}.$$

We observe that this corresponds to the required normal with positive z-component. We calculate that $-2 \cdot 2^2 \cos\pi = 8$ whereas $-2 \cdot 2^2 \sin\pi = 0$. Since $\mathbf{f}(x, y, z) = \left(\begin{smallmatrix} 2 \\ 0 \\ 3z \end{smallmatrix}\right)$,

$$\mathbf{f}(\mathbf{r}(r, \theta)) = \begin{pmatrix} 2 \\ 0 \\ 3r^2 \end{pmatrix}.$$

Consequently

$$\begin{aligned} \iint_S \mathbf{f} \cdot \mathbf{n}\, dS &= \iint_T \begin{pmatrix} 2 \\ 0 \\ 3r^2 \end{pmatrix} \cdot \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix} dr d\theta \\ &= \int_0^2 \left[\int_0^{2\pi} (-4r^2\cos\theta + 3r^3)\, d\theta \right] dr. \end{aligned}$$

We calculate that

$$\int_0^2 \int_0^{2\pi} 3r^3\, d\theta dr = 6\pi \left[\frac{r^4}{4} \right]_0^2 = 3 \cdot 2^3 \pi.$$

On the other hand, the other part of the integral disappears because $\int_0^{2\pi} \cos\theta\, d\theta = 0$. This means that

$$\iint_S \mathbf{f} \cdot \mathbf{n}\, dS = 3 \cdot 2^3 \pi = 24\pi.$$