1. MA2 – CALL 1 – 17/01/2022

Q1. Fill in the following blanks with the correct **integer**, possibly zero or negative. In this problem we will find a power series solution of the differential equation

$$(1 + 9x^2)y''(x) = 18y(x)$$

subject to the initial conditions $y(0) = 3$, $y'(0) = 1$.

We suppose that $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substitute to obtain the formula

$$\sum_{n=0}^{\infty} \left[(n + \boxed{\text{a}})(n+1)a_{n+2} + \boxed{\text{b}}(n-2)(n+1)a_n \right] x^n = 0$$

where $\boxed{\text{a}} = \boxed{}$, $\boxed{\text{b}} = \boxed{}$(1 point each). We also note that $a_0 = \boxed{}$and $a_1 = 1$ because of the given initial conditions ($\frac{1}{2}$ point).

Our previous calculation, by the uniqueness of power series, gives us a recursion formula for the coefficients. We calculate the first few with even index: $a_2 = \boxed{}$, $a_4 = \boxed{}$, $a_6 = \boxed{}$($\frac{1}{2}$ point each).

We also consider the coefficients with odd index. Using the recursion formula you should now see that $a_3 = 3$. (If not, now is a good time to check the recursion formula.) Instead of explicitly writing a formula for each of the coefficients with odd index we can continue working with just the recursion formula. The radius of convergence of the power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{\boxed{\text{c}}}$ where $\boxed{\text{c}} = \boxed{}$(2 points).

Q2. In this question we will find and classify the extrema points of $f(x, y) = 3x^4 + 4xy + 2y^2$. The gradient of this function is

$$\nabla f(x, y) = \begin{pmatrix} \boxed{\text{a}}x^3 + \boxed{\text{b}}y \\ \boxed{\text{c}}x + \boxed{\text{d}}y \end{pmatrix},$$

where $\boxed{\text{a}} = \boxed{}$, $\boxed{\text{b}} = \boxed{}$, $\boxed{\text{c}} = \boxed{}$, $\boxed{\text{d}} = \boxed{}$.

There are three stationary points: $\left(\frac{-1}{\sqrt{\boxed{\text{e}}}}, \frac{1}{\sqrt{\boxed{\text{f}}}}\right)$, $(0, \boxed{\text{g}})$ and $\left(\frac{1}{\sqrt{\boxed{\text{e}}}}, \frac{\boxed{\text{h}}}{\sqrt{\boxed{\text{f}}}}\right)$ where $\boxed{\text{e}} = \boxed{}$, $\boxed{\text{f}} = \boxed{}$, $\boxed{\text{g}} = \boxed{}$, $\boxed{\text{h}} = \boxed{}$.

Computing the Hessian at each stationary point we deduce that there are $\boxed{}$relative minima, $\boxed{}$saddle point and $\boxed{}$relative maxima. Moreover $f(x, y)$ is $\boxed{\text{bounded}}$ or $\boxed{\text{unbounded}}$? Fill in the blanks with **integers**, possibly 0 or negative. Each part is worth $\frac{1}{2}$ point.

Q3. Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

(a) If C is the path from $(0, 1)$ to $(1, e^{\sin(1)})$ along the curve $y = e^{\sin(x)}$ and

$$\mathbf{f}(x, y) = \begin{pmatrix} x^2 \\ y \end{pmatrix}.$$

is a vector field then $\int_C \mathbf{f}\, d\boldsymbol{\alpha} = -\frac{1}{\boxed{\text{a}}} + \frac{1}{2}e^{2\sin(1)}$ where $\boxed{\text{a}} = \boxed{}$.

(b) If C be the line segment from $(0, 2)$ to $(2, 4)$ and

$$\mathbf{f}(x, y) = \begin{pmatrix} x^2 y \\ x + y \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\, d\boldsymbol{\alpha} = \frac{\boxed{\text{b}}}{3}$ where $\boxed{\text{b}} = \boxed{}$.

(c) If Let C be the path on the unit circle (i.e., $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$) along the anticlockwise direction.

$$\mathbf{f}(x,y) = \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\text{c}}\pi$ where $\boxed{\text{c}} = \boxed{}$.

Q4. Fill in the following blanks with the correct **integer**, possibly zero or negative.

The set

$$V = \left\{(x,y,z) : 0 \le z \le 12 - 4\sqrt{x^2+y^2}\right\}$$

is a cone of height 12 with base in the xy-plane. The base of this cone is a disc of radius $\boxed{}$(1 point). The set

$$W = \left\{(x,y,z) : \left(x - \frac{3}{2}\right)^2 + y^2 \le \frac{9}{4}\right\}$$

is a cylinder. Let $U \subset \mathbb{R}^3$ be the subset of the cone V which is contained within the cylinder W. We will calculate the volume of U.

Let $S = \left\{(x,y) : (x - \frac{3}{2})^2 + y^2 \le (\frac{3}{2})^2\right\}$. We can then write

$$U = \left\{(x,y,z) : (x,y) \in S, 0 \le z \le \boxed{\text{a}} - 4\sqrt{x^2+y^2}\right\}$$

where $\boxed{\text{a}} = \boxed{}$(1 point). This is convenient since the volume of U is equal to $\iint_S \boxed{\text{a}} - 4\sqrt{x^2+y^2}\ dxdy$.

To proceed we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. The region S, under this change of coordinates, corresponds to the region

$$\widetilde{S} = \left\{(r,\theta) : \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right], 0 \le r \le \boxed{\text{b}}\cos\theta\right\}$$

where $\boxed{\text{b}}$: $\boxed{}$(2 points).

Use this change of variables in order to calculate that the volume of U is equal to $27\pi + \boxed{\text{c}}$ where $\boxed{\text{c}} = \boxed{}$(2 points). **Hints:** $\int \cos^2\ d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$ and $\int \cos^3\theta\ d\theta = \sin\theta - \frac{1}{3}\sin^3\theta + C$.

Q5. Consider the solid $V \subset \mathbb{R}^3$ defined as $V = \{(x,y,z) : x^2+y^2+z^2 \le 25, z \ge 3\}$ and the vector field $\mathbf{f}$ defined as

$$\mathbf{f}(x,y,z) = \begin{pmatrix} xz \\ yz \\ 2 \end{pmatrix}.$$

The intersection of the two surfaces $x^2+y^2+z^2 = 25$ and $z = 3$ is the curve $\{(x,y,z) : z = 3, \sqrt{x^2+y^2} = \boxed{\text{a}}\}$, i.e., a circle of radius $\boxed{\text{a}}$ where $\boxed{\text{a}} = \boxed{}$(1 point). Calculate the divergence $(\nabla \cdot \mathbf{f})(x,y,z)$. In particular $(\nabla \cdot \mathbf{f})(7,8,9) = \boxed{}$(1 point). (There is nothing special about the point $(7,8,9)$, it's just to test the formula for divergence.)

Let S_1 denote the planar surface which is the base of V and let S_2 denote the curved surface, the upper surface of V. Let $\mathbf{n}$ denote the outward unit normal of the surface of V.

$$\iiint_V \nabla \cdot \mathbf{f}\ dxdydz = \boxed{\text{b}}\pi, \qquad \iint_{S_1} \mathbf{f} \cdot \mathbf{n}\ dS = \boxed{\text{c}}\pi,$$

$$\iint_{S_2} \mathbf{f} \cdot \mathbf{n}\ dS = 160\pi.$$

where $\boxed{\text{b}} = \boxed{}$and $\boxed{\text{c}} = \boxed{}$(2 points each).

Q1 Solution: In this problem we will find a power series solution of the differential equation

$$(1+9x^2)y''(x) = 18y(x)$$

subject to the initial conditions $y(0) = 3$, $y'(0) = 1$.

(1) To proceed we substitute $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ to obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 9x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 18 \sum_{n=0}^{\infty} a_n x^n = 0.$$

(2) Rearranging and shifting indexes, noting also that $n(n-1) = 0$ whenever $n \in \{0, 1\}$, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 9 \sum_{n=0}^{\infty} n(n-1)a_n x^n - 18 \sum_{n=0}^{\infty} a_n x^n = 0.$$

This is equivalent to

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 9n(n-1)a_n - 18a_n\right] x^n = 0.$$

(3) By the uniqueness of power series, this implies that, for all $n \geq 0$,

$$(n+2)(n+1)a_{n+2} = -9\left(n(n-1) - 2\right)a_n = -9(n-2)(n+1)a_n.$$

This leads to the recursion relation

$$a_{n+2} = -9\frac{(n-2)}{(n+2)}a_n.$$

(4) Using the initial conditions we have $a_0 = y(0) = 3$ and $a_1 = y'(0) = 1$. Using the recursion relation we observe that $a_2 = 9a_0 = 27$ but $a_4 = 0$, $a_6 = 0$, $a_8 = 0$, etc. I.e., there are only a finite number of coefficients with even index.

(5) Using again the recursion relation we know that $a_3 = 3a_1 = 3$. Instead of explicitly writing a formula for each of the coefficients with odd index we can continue working with just the recursion formula. The power series has the form (infinite odd terms and two even terms):

$$y(x) = 3 + x + 9x^2 + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1}.$$

(6) We use the ratio test to deduce that the radius of convergence. Fix x and, for convenience, let $A_k = a_{2k+1}x^{2k+1}$ and we consider the convergence of the series $\sum_k A_k$. Observe that

$$\frac{|A_{k+1}|}{|A_k|} = \frac{|a_{2k+3}|\,|x|^{2k+3}}{|a_{2k+1}|\,|x|^{2k+1}} = \frac{|a_{2k+3}|}{|a_{2k+1}|}|x|^2 = 9\frac{2k-1}{2k+3}|x|^2.$$

Consequently $\frac{|A_{k+1}|}{|A_k|} \to 9|x|^2$ as $k \to \infty$. The means that the series converges when $9|x|^2 < 1$ and diverges when $9|x|^2 > 1$. This means that the radius of convergence is $\frac{1}{3}$.

Q2 Solution: Let $f(x,y) = 3x^4 + 4xy + 2y^2$.

(1) We calculate the gradient of this function

$$\nabla f(x,y) = \begin{pmatrix} 12x^3 + 4y \\ 4x + 4y \end{pmatrix}.$$

(2) To find the stationary points we suppose $\nabla f(x,y) = 0$ and solve for (x,y). The second equation $(4x + 4y = 0)$ implies that $y = -x$. Substituting this into the first equation $(12x^3 + 4y = 0)$ we obtain $12x^3 - 4x = 0$. Consequently, either $x = 0$ or $x^2 - \frac{1}{3} = 0$. In the first case we obtain the solution $(0,0)$. In the second case we have $x = \pm\frac{1}{\sqrt{3}}$. Using again that $y = -x$ we obtain the solutions $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

(3) We calculate the Hessian matrix

$$\mathbf{H}f(x,y) = \begin{pmatrix} 36x^2 & 4 \\ 4 & 4 \end{pmatrix}.$$

This means that

$$\mathbf{H}f(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix},$$

and

$$\mathbf{H}f(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}) = \mathbf{H}f(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}) = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}.$$

(4) $\mathbf{H}f(0,0)$ has one positive eigenvalue and one negative eigenvalue so this is a saddle point. The eigenvalues of $\mathbf{H}f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \mathbf{H}f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ are both not negative and so these two points are relative minima.

Q3 Solution:

(a) Let C be the path from $(0,1)$ to $(1, e^{\sin(1)})$ along the curve $y = e^{\sin(x)}$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} x^2 \\ y \end{pmatrix}.$$

Choose $\boldsymbol{\alpha}(t) = (t, e^{\sin(t)})$, $t \in [0,1]$. We calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 1 \\ e^{\sin(t)} \cos(t) \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} t^2 \\ e^{\sin(t)} \end{pmatrix}.$$

Consequently $\boldsymbol{\alpha'}(t) \cdot \mathbf{f}(\boldsymbol{\alpha}(t)) = t^2 + e^{2\sin(t)} \cos(t)$. And so (using also substitution $u = \sin t$)

$$\begin{aligned} \int \mathbf{f} \, d\boldsymbol{\alpha} &= \int_0^1 (t^2 + e^{2\sin(t)} \cos(t)) \, dt = \int_0^1 t^2 \, dt + \int_0^{\sin(1)} e^{2u} \, du \\ &= \left[\frac{1}{3}t^3\right]_0^1 + \left[\frac{1}{2}e^{2t}\right]_0^{\sin(1)} = \frac{1}{3} + \frac{1}{2}e^{2\sin(1)} - \frac{1}{2} = -\frac{1}{6} + \frac{1}{2}e^{2\sin(1)}. \end{aligned}$$

(b) Let C be the line segment from $(0,2)$ to $(2,4)$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} x^2 y \\ x + y \end{pmatrix}.$$

Choose $\boldsymbol{\alpha}(t) = (2t, 2+2t)$, $t \in [0,1]$. We calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} 2^2 t^2 (2+2t) \\ 2 + 4t \end{pmatrix}.$$

Consequently $\boldsymbol{\alpha}'(t) \cdot \mathbf{f}(\boldsymbol{\alpha}(t)) = 2(2t)^2(2+2t) + 2(2+4t)$. And so

$$\begin{aligned} \int \mathbf{f} \, d\boldsymbol{\alpha} &= \int_0^1 2(2t)^2(2+2t) + 2(2+4t) \, dt \\ &= \int_0^1 16t^3 + 16t^2 + 8t + 4 \, dt = \left[4t^4 + \frac{16}{3}t^3 + 4t^2 + 4t\right]_0^1 \\ &= \frac{52}{3}. \end{aligned}$$

(c) Let C be the anticlockwise path on the circle of radius 1 (i.e., $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$) and

$$\mathbf{f}(x,y) = \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}.$$

Choose $\boldsymbol{\alpha}(t) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$. We calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$

Consequently $\boldsymbol{\alpha}'(t) \cdot \mathbf{f}(\boldsymbol{\alpha}(t)) = \sin(t)^2 + \cos(t)^2 = 1$. And so

$$\int \mathbf{f} \, d\boldsymbol{\alpha} = \int_0^{2\pi} 1 \, dt = 2\pi.$$

Q4 Solution: The set $V = \{(x,y,z) : 0 \le z \le 12 - 4\sqrt{x^2+y^2}\}$ is a cone of height 12 with base in the xy-plane. This is the volume bounded below by the surface $z = 0$ and above by the surface $z = 12 - 4\sqrt{x^2+y^2}$. These two surfaces meet where $0 = 12 - 4\sqrt{x^2+y^2}$. I.e., when $x^2 + y^2 = 9$. This means that the base of the cone in a disc of radius 3.

The set $W = \{(x,y,z) : (x - \frac{3}{2})^2 + y^2 \le \frac{9}{4}\}$ is a cylinder. Let $U \subset \mathbb{R}^3$ be the subset of the cone V which is contained within the cylinder W. We will calculate the volume of U.

(1) We define $S = \left\{(x,y) : (x - \frac{3}{2})^2 + y^2 \le \frac{9}{4}\right\}$ (the projection of the cylinder on to the xy-plane). Consequently
$$U = \left\{(x,y,z) : (x,y) \in S, 0 \le z \le 12 - 4\sqrt{x^2+y^2}\right\}.$$
In particular the volume of U is equal to $\iint_S 12 - 4\sqrt{x^2+y^2}\ dxdy$.

(2) To proceed we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ which means that the Jacobian is $J(r,\theta) = r$ and the corresponding region is (a sketch might help)
$$\widetilde{S} = \left\{(r,\theta) : \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right], 0 \le r \le 3\cos\theta\right\}.$$
The condition on r is because $(x - \frac{3}{2})^2 + y^2 \le \frac{9}{4}$ implies $(r\cos\theta - \frac{3}{2})^2 + r^2\sin^2\theta \le \frac{9}{4}$ which in turn implies that $r - 3\cos\theta \le 0$.

(3) This all means that the volume of U is equal to
$$\iint_{\widetilde{S}} r(12 - 4r)\ drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{3\cos\theta} 12r - 4r^2\ dr\right] d\theta$$

(4) For the inner integral we calculate
$$\int_0^{3\cos\theta} 12r - 4r^2\ dr = \left[6r^2 - \frac{4}{3}r^3\right]_0^{3\cos\theta} = 54\cos^2\theta - 36\cos^3\theta.$$

(5) Consequently the volume of U is equal to
$$54\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\ d\theta - 36\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\ d\theta.$$
We use the indefinite integrals $\int \cos^2\ d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$ and $\int \cos^3\theta\ d\theta = \sin\theta - \frac{1}{3}\sin^3\theta$. It is also convenient to note that both $\cos^2\theta$ and $\cos^3\theta$ are even.

(6) Putting everything together we have calculated that the volume of U is equal to
$$108\left[\tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\cos\theta\right]_0^{\frac{\pi}{2}} - 72\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{2}}.$$
We finally obtain that this is equal to $27\pi - 72(1 - \frac{1}{3}) = 27\pi - 48$.

Q5 Solution: We define the solid $V \subset \mathbb{R}^3$ as

$$V = \{(x, y, z) : x^2 + y^2 + z^2 \leq 25, z \geq 3\}.$$

We define the vector field $\mathbf{f}$ as

$$\mathbf{f}(x, y, z) = \begin{pmatrix} xz \\ yz \\ 2 \end{pmatrix}.$$

(1) We first calculate the intersection of the surfaces $x^2 + y^2 + z^2 = 25$ and $z = 3$. Substituting we obtain that $x^2 + y^2 + 3^2 = 25$ which is equivalent to $x^2 + y^2 = 25 - 9 = 16 = 4^2$. This means that the intersection of the two surfaces is the curve $\{(x, y, z) : z = 3, \sqrt{x^2 + y^2} = 4\}$, i.e., a circle of radius 4.

(2) We calculate the divergence $(\nabla \cdot \mathbf{f})(x, y, z) = z + z + 0 = 2z$. In particular $(\nabla \cdot \mathbf{f})(7, 8, 9) = 18$.

(3) Using the idea of projecting onto the xy-plane we define $D = \{(x, y) : x^2 + y^2 \leq 4^2\}$ and we can write V in the convenient form

$$V = \{(x, y, z) : (x, y) \in D, 3 \leq z \leq \sqrt{25 - (x^2 + y^2)}\}.$$

(4) The triple integral is evaluated as

$$\iiint_V \nabla \cdot \mathbf{f} \; dxdydz = \iint_D \left[\int_3^{\sqrt{25-(x^2+y^2)}} (2z) \; dz \right] \; dxdy$$

Evaluating the inside integral we obtain

$$\begin{aligned} \int_3^{\sqrt{25-(x^2+y^2)}} (2z) \; dz &= \left[z^2 \right]_3^{\sqrt{25-(x^2+y^2)}} = 25 - (x^2 + y^2) - 9 \\ &= 16 - (x^2 + y^2). \end{aligned}$$

Consequently the full integral is equal to (using that $\iint_D \; dxdy = 4^2\pi$)

$$\iint_D \left[16 - (x^2 + y^2) \right] \; dxdy = \pi 4^4 - \iint_D \left(x^2 + y^2 \right) \; dxdy.$$

To integrate the final term we use polar coordinates (remembering the Jacobian),

$$\iint_D \left(x^2 + y^2 \right) \; dxdy = \int_0^{2\pi} \left[\int_0^4 r^3 \; dr \right] \; d\theta = 2\pi \left[\frac{r^4}{4} \right]_0^4 = 2\pi 4^3.$$

Summing the two parts of the calculation we obtain a final answer of $(4 - 2)4^3\pi = 128\pi$.

(5) The planar surface, the base of V, can be written in parametric form as $S_1 = \mathbf{r}_1(D)$ where $\mathbf{r}_1(x, y) = (x, y, 3)$. The fundamental vector product is

$$\mathbf{N}(x, y) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We note that this vector points into V but we need the outward going normal. I.e., $\mathbf{n} = -\frac{\mathbf{N}}{\|\mathbf{N}\|}$. In preparation for the integral we calculate the inner product

$$-\mathbf{f} \cdot \mathbf{N} = -\begin{pmatrix} xz \\ yz \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -2$$

$$\iint_{S_1} \mathbf{f}\cdot\mathbf{n}\ dS = \iint_D (-2)\ dxdy = -2(\pi 4^2) = -32\pi.$$

(6) (This part of the calculation isn't required for this exam and is only given here for interest.) The curved upper portion of the surface of V can be written in parametric form as $S_2 = \mathbf{r}_2(D)$ where $\mathbf{r}_2(x,y) = \left(x, y, \sqrt{25-(x^2+y^2)}\right)$. The fundamental vector product is

$$\mathbf{N}(x,y) = \begin{pmatrix} 1 \\ 0 \\ -x(25-(x^2+y^2))^{-\frac{1}{2}} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -y(25-(x^2+y^2))^{-\frac{1}{2}} \end{pmatrix}$$
$$= \begin{pmatrix} x(25-(x^2+y^2))^{-\frac{1}{2}} \\ y(25-(x^2+y^2))^{-\frac{1}{2}} \\ 1 \end{pmatrix} = \begin{pmatrix} xz^{-1} \\ yz^{-1} \\ 1 \end{pmatrix}.$$

In this case the normal is already in the correct outward direction.

$$\mathbf{f}\cdot\mathbf{N} = \begin{pmatrix} xz \\ yz \\ 2 \end{pmatrix} \cdot \begin{pmatrix} xz^{-1} \\ yz^{-1} \\ 1 \end{pmatrix} = x^2 + y^2 + 2.$$

Consequently

$$\iint_{S_2} \mathbf{f}\cdot\mathbf{n}\ dS = \iint_D (x^2+y^2)\ dxdy + 2\iint_D\ dxdy = 2(4^3+4^2)\pi = 160\pi.$$