MA2 – Part 6 – Surface integrals

Weeks 11–12 of MA2 – Draft lecture slides

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Outline

Parametric representation of a surface
Fundamental vector product
Area of a parametric surface
Surface integrals
Change of parametric representation
Surface integral of a vector field
Curl and divergence
Theorem of Stokes
Theorem of Gauss

Example: Parametric representation of a hemisphere

Recall: Half circle $C = \{(x,y) : x^2 + y^2 = 1, y \geq 1\}$ can be parametrized

- $\boldsymbol{\alpha}(x) := (x, \sqrt{1 - x^2})$, $x \in [-1, 1]$, or
- $\boldsymbol{\alpha}(t) := (\cos t, \sin t)$, $t \in [0, \pi]$.

Example (hemisphere)

The hemisphere $S = \{(x,y,z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ can be parametrized

- $\mathbf{r}(x,y) := (x, y, \sqrt{1 - x^2 - y^2})$, $(x,y) \in \{x^2 + y^2 \leq 1\}$, or
- $\mathbf{r}(u,v) := (\cos u \cos v, \sin u \cos v, \sin v)$, $(u,v) \in [0, 2\pi] \times [0, \pi/2]$.

Remark

Second form can be deduced from spherical coordinates (fixed distance from origin).

Example: Parametric representation of a cone

Example (cone)

The cone $S = \{(x, y, z) : z^2 = x^2 + y^2, z \in [0, 1]\}$ can be parametrized

- $\mathbf{r}(x, y) := (x, y, \sqrt{x^2 + y^2})$, $(x, y) \in \{x^2 + y^2 \leq 1\}$, or
- $\mathbf{r}(u, v) := (v \cos u, v \sin u, v)$, $(u, v) \in [0, 2\pi] \times [0, 1]$.

Remark

Second form can be deduced from spherical coordinates (fixed angle from z-axis).

Fundamental vector product

Consider the parametric surface, denoted $\mathbf{r}(T)$,

$$\mathbf{r}(u,v) := (X(u,v), Y(u,v), Z(u,v)), \quad (u,v) \in T.$$

Definition (fundamental vector product)

The vector-valued function

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} \partial_u X \\ \partial_u Y \\ \partial_u Z \end{pmatrix} \times \begin{pmatrix} \partial_v X \\ \partial_v Y \\ \partial_v Z \end{pmatrix}$$

is called the *fundamental vector product* of the representation $\mathbf{r}$.

Remarks

- The vector-valued functions $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are tangent to the surface,
- The fundamental vector product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is normal to the surface,
- Represents local scaling of area (small parallelograms).

Further details on parametric surface representations

Definition (regular point)

If (u, v) is a point in T at which $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are continuous and the fundamental vector product is non-zero then $\mathbf{r}(u, v)$ is said to be a *regular point* for that representation.

Definition

A surface $\mathbf{r}(T)$ is said to be smooth if all its points are regular points.

Parametric representation of surface with explicit form

Suppose that a surface S has the form $z = f(x,y)$, i.e., it is described explicitly.

- We can use x, y as the parameters and show

$$\mathbf{r}(x,y) := (x, y, f(x,y)),$$

- The region T is called the projection of S onto the xy-plane,
- We compute

$$\frac{\partial \mathbf{r}}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix},$$

- Consequently

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix} = \begin{pmatrix} -\partial_x f \\ -\partial_y f \\ 1 \end{pmatrix}.$$

Example: Hemisphere representation 1

- Let $T := \{x^2 + y^2 \leq 1\}$,
- Let $\mathbf{r}(x, y) := (x, y, \sqrt{1 - x^2 - y^2})$.
- The surface $\mathbf{r}(T)$ is the unit hemisphere.
- The fundamental vector product of this representation is

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}(x, y) = \begin{pmatrix} x(1-x^2-y^2)^{-1/2} \\ y(1-x^2-y^2)^{-1/2} \\ 1 \end{pmatrix} = z^{-1}\, \mathbf{r}(x, y).$$

- All points are regular except the equator.

Example: Hemisphere representation 2

- Let $T := [0, 2\pi] \times [0, \pi/2]$,
- Let $\mathbf{r}(u, v) := (\cos u \cos v, \sin u \cos v, \sin v)$.
- The surface $\mathbf{r}(T)$ is the unit hemisphere.
- $$\frac{\partial \mathbf{r}}{\partial u}(u, v) = \begin{pmatrix} -\sin u \cos v \\ \cos u \cos v \\ 0 \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial v}(u, v) = \begin{pmatrix} -\cos u \sin v \\ -\sin u \sin v \\ \cos v \end{pmatrix}.$$
- The fundamental vector product of this representation is
$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u, v) = \cos v \; \mathbf{r}(u, v).$$
- Many points map to the north pole, north pole is not a regular point, many points map to a line between equator and north pole.

Area of a parametric surface

Definition (area of a parametric surface)

The area of the parametric surface $S = \mathbf{r}(T)$ is defined as the double integral

$$\text{Area}(S) := \iint_T \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, dudv.$$

Remarks

- Similar to the definition of the length of a curve,
- We can show that Area(S) is *independent* of the choice of representation.

Example: Area of a hemisphere

- Let, as before, $T := [0, 2\pi] \times [0, \pi/2]$,
- Let $\mathbf{r}(u, v) := (\cos u \cos v, \sin u \cos v, \sin v)$.
- Norm of fundamental vector product:

$$\left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}(u, v) \right\| = \|\cos v \ \mathbf{r}(u, v)\| = \cos v.$$

- Hence:

$$\text{Area}(S) := \iint_T \cos v \ dudv = \int_0^{2\pi} \left[\int_0^{\pi/2} \cos v \ dv \right] \ du = 2\pi.$$

Surface integrals

Similar to line integrals, defined using a parametrization.

Definition (surface integral)

Let $S = \mathbf{r}(T)$ be a parametric surface and let f be a scalar field defined on S. The surface integral of f over S is defined as

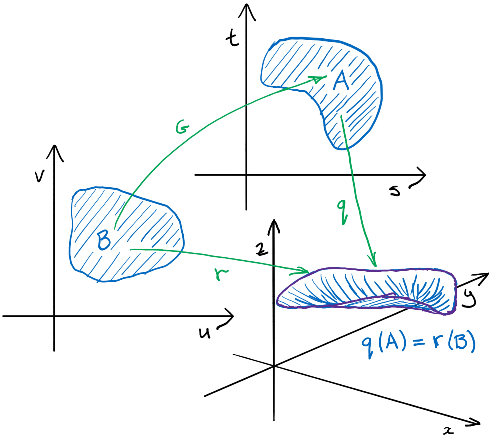
$$\iint_{\mathbf{r}(T)} f \; dS = \iint_T f(\mathbf{r}(u,v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u,v) \right\| \; dudv$$

whenever the double integral on the right exists.

Remarks:

- When $f \equiv 1$ the surface integral reduces to surface area.
- If f is the density of thin material in the form S then $\iint_S f \; dS$ is the mass.
- We can calculate the centre of mass of material in the form S.

Change of parametric representation



Suppose that

- $\mathbf{q}(A)$ and $\mathbf{r}(B)$ are both representations of the same surface,
- $\mathbf{r} = \mathbf{q} \circ G$ for some differentiable $G : B \to A$.

Then

$$\iint_A f \circ \mathbf{q} \left\| \frac{\partial \mathbf{q}}{\partial s} \times \frac{\partial \mathbf{q}}{\partial t} \right\| \, dsdt = \iint_B f \circ \mathbf{r} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, dudv$$

Proof of change of parametric representation

1. Since $\mathbf{r}(u, v) = \mathbf{q}(S(u, v), T(u, v))$ we calculate (chain rule and vector product) that

$$\left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right](u, v) = \left[\left(\frac{\partial \mathbf{q}}{\partial s} \times \frac{\partial \mathbf{q}}{\partial t}\right)\left(\frac{\partial S}{\partial u}\frac{\partial T}{\partial v} - \frac{\partial S}{\partial v}\frac{\partial T}{\partial u}\right)\right](S(u, v), T(u, v)),$$

2. Observe that $\frac{\partial S}{\partial u}\frac{\partial T}{\partial v} - \frac{\partial S}{\partial v}\frac{\partial T}{\partial u}$ is the Jacobian determinant associated to change of variables $(u, v) \mapsto (S(u, v), T(u, v))$,
3. Consequently, by the change of variables theorem,

$$\iint_A f \circ \mathbf{q} \left\|\frac{\partial \mathbf{q}}{\partial s} \times \frac{\partial \mathbf{q}}{\partial t}\right\| dsdt = \iint_B f \circ \mathbf{r} \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| dudv.$$

Normal vector of a surface

Definition (normal vector)

Let $S = \mathbf{r}(T)$ be a parametric surface and let $\mathbf{N} := \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$. At each regular point there are two unit normals

$$\mathbf{n}_1 := \frac{\mathbf{N}}{\|\mathbf{N}\|} \quad \text{and} \quad \mathbf{n}_2 := -\mathbf{n}_1.$$

Remark:

- If $\mathbf{f}$ is a vector field then $\mathbf{f} \cdot \mathbf{n}$ is the component of the flow in direction of $\mathbf{n}$.

Normal vector and surface integral of a vector field

Definition (surface integral of a vector field)

Let $S = \mathbf{r}(T)$ be a parametric surface and $\mathbf{f}$ a vector field. The integral

$$\iint_S \mathbf{f} \cdot \mathbf{n} \; dS$$

is said to be the *surface integral of* $\mathbf{f}$ *with respect to the normal* $\mathbf{n}$.

Remarks:

- $\iint_S \mathbf{f} \cdot \mathbf{n} \; dS = \iint_T (\mathbf{f} \circ \mathbf{r}) \cdot \mathbf{n} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \; dudv = \pm \iint_T (\mathbf{f} \circ \mathbf{r}) \cdot \mathbf{N} \; dudv.$
- $\iint_S \mathbf{f} \cdot \mathbf{n}_1 \; dS = -\iint_S \mathbf{f} \cdot \mathbf{n}_2 \; dS$ because $\mathbf{n}_1 = -\mathbf{n}_2$.

Curl and divergence

Suppose that $\mathbf{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$ is a vector field.

Definition (curl)

The *curl* of $\mathbf{f}$ is defined as

$$\nabla \times \mathbf{f} = \begin{pmatrix} \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \\ \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \end{pmatrix}.$$

Definition (divergence)

The *divergence* of $\mathbf{f}$ is

$$\nabla \cdot \mathbf{f} := \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}.$$

Alternative notation:

- $\operatorname{curl} \mathbf{f} = \nabla \times \mathbf{f}$
- $\operatorname{div} \mathbf{f} = \nabla \cdot \mathbf{f}$

Properties:

- If $\mathbf{f} = \nabla\varphi$ then $\nabla \times \mathbf{f} = \mathbf{0}$,
- $\nabla^2\varphi := \nabla \cdot (\nabla\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$ is called the Laplacian,
- $\nabla \cdot (\nabla \times \mathbf{f}) = 0$,
- $\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2\mathbf{f}$.

Theorem

Let $S \subset \mathbb{R}^3$ be convex. Then $\nabla \times \mathbf{f} = \mathbf{0}$ if and only if $\mathbf{f}$ is a gradient.

Examples (curl and divergence)

Example

If $\mathbf{f}(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\nabla \times \mathbf{f} = \mathbf{0}$, $\nabla \cdot \mathbf{f} = 3$.

Example

If $\mathbf{f}(x,y,z) = \nabla\varphi$ then $\nabla \times \mathbf{f} = \mathbf{0}$.

Example

If $\mathbf{f}(x,y,z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ then $\nabla \times \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $\nabla \cdot \mathbf{f} = 0$.

Theorem of Stokes

Theorem (Stokes)

Let $S = \mathbf{r}(T)$ be a parametric surface. Suppose that T is simply connected and that the boundary of T is mapped to C, the boundary of S. Let β be a counter clockwise parametrization of the boundary of T and let $\boldsymbol{\alpha}(t) = \mathbf{r}(\beta(t))$. Then

$$\iint_S (\nabla \times \mathbf{f}) \cdot \mathbf{n} \, dS = \int \mathbf{f} \cdot d\boldsymbol{\alpha}.$$

Sketch of proof.

1. Write $\mathbf{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$ and suppose that $f_y = f_z = 0$;
2. Use Green's theorem;
3. Conclude for general $\mathbf{f}$ by linearity.

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$\square$

Extension of Stokes' theorem

Linearity of integrals allows to extend Stokes' theorem to other parametric surfaces:

- Surfaces with holes
- Cylinder
- Möbius band (does **not** hold)
- Sphere

Theorem of Gauss

Theorem (Gauss)

Let $V \subset \mathbb{R}^3$ be a solid with boundary the parametric surface S and let $\mathbf{n}$ be the outward normal unit vector. If $\mathbf{f}$ is a vector field then

$$\iiint_V \nabla \cdot \mathbf{f} \; dxdydz = \iint_S \mathbf{f} \cdot \mathbf{n} \; dS.$$

Sketch of proof.

1. Write $\iiint_V \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \right) \; dxdydz = \iint_S \left(f_x n_x + f_y n_y + f_z n_z \right) \; dS$,
2. Suffices to show that $\iiint_V \left(\frac{\partial f_x}{\partial x} \right) \; dxdydz = \iint_S \left(f_x n_x \right) \; dS$,
3. Suppose solid is xy-projectable,
4. Basic calculus to express f_x as the integral of the derivative.

□

Remark: Also called the "Divergence Theorem".

Interpretation of divergence as a limit

Theorem

Let V_t be the ball of radius $t > 0$ centred at $\mathbf{a} \in \mathbb{R}^3$ and let S_t be its boundary with outgoing unit normal vector $\mathbf{n}$. Then

$$\nabla \cdot \mathbf{f} = \lim_{t \to 0} \frac{1}{\mathrm{Vol}(V_t)} \iint_{S_t} \mathbf{f} \cdot \mathbf{n} \, dS.$$

Proof.

Using Gauss' theorem. $\square$

Remark:

Curl can also be written as a similar limit.

Relation of curl and divergence to the Jacobian matrix

$$\mathrm{Jac}(\mathbf{f}) = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{pmatrix}$$

- The divergence is the trace of the Jacobian matrix
- Every real matrix A can be written as the sum of a symmetric matrix $\frac{1}{2}(A + A^T)$ and a skew-symmetric matrix $\frac{1}{2}(A - A^T)$.

$$\frac{1}{2}(\mathrm{Jac}(\mathbf{f}) - \mathrm{Jac}(\mathbf{f})^T) = \begin{pmatrix} 0 & \frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x} & \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} & 0 & \frac{\partial f_y}{\partial z} - \frac{\partial f_z}{\partial y} \\ \frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z} & \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} & 0 \end{pmatrix}$$