MA2 – Part 5 – Multiple integrals

Weeks 9–10 of MA2 – Draft lecture slides

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Outline

Plan of action

Recall: One-dimensional case

1. define the integral for step functions;
2. define integral for "integrable functions";
3. show that continuous functions are integrable.

For higher dimensions we follow the same logic. We will then show that we can evaluate higher dimensional integrals by repeated one-dimensional integration.

Partitions of rectangles

Observe:

- Partition divides R into nm sub-rectangles.
- If $P \subseteq Q$ then we say that Q is a finer partition than P.

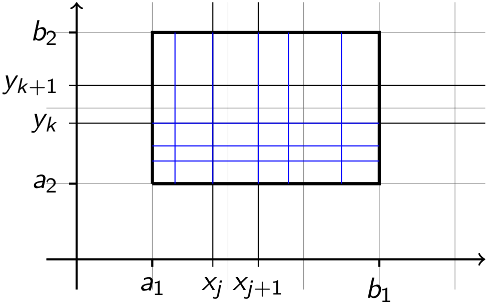

Figure: A partition of a rectangle R.

Definition (partition)

Let $R = [a_1, b_1] \times [a_2, b_2]$ be a rectangle. Suppose that $P_1 = \{x_0, \ldots, x_m\}$ and $P_2 = \{y_0, \ldots, y_n\}$ such that $a_1 = x_0 < x_2 < \cdots < x_m = b_1$ and $a_2 = y_0 < y_2 < \cdots < y_n = b_2$. $P = P_1 \times P_2$ is said to be a partition of R.

Step functions

Definition (step function)

A function $f : R \to \mathbb{R}$ is said to be a *step function* if there is a partition P of R such that f is constant on each sub-rectangle of the partition.

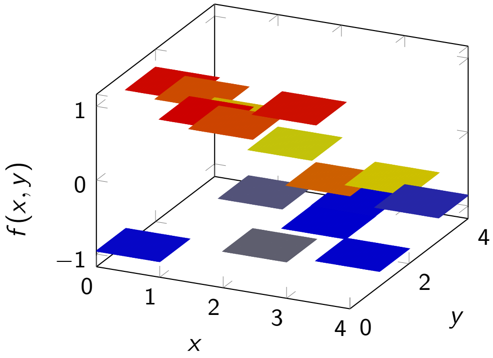

Figure: Graph of a step function.

- If f and q are step functions and $c, d \in \mathbb{R}$, then $cf + dg$ is also a step function;
- The area of the sub-rectangle $Q_{jk} := [x_j, x_{j+1}] \times [y_k, y_{k+1}]$ is equal to $(x_{j+1} - x_j)(y_{k+1} - y_k)$.

Integral of a step function

Definition (integral of a step function)

Suppose that f is a step function with value c_{jk} on the sub-rectangle $(x_j, x_{j+1}) \times (y_k, y_{k+1})$. Then we define the integral as

$$\iint_R f \; dxdy = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} c_{jk}(x_{j+1} - x_j)(y_{k+1} - y_k).$$

- The value of the integral is independent of the partition, as long as the function is constant on each sub-rectangle,
- If $Q_{jk} = (x_j, x_{j+1}) \times (y_k, y_{k+1})$ then

$$\iint_{Q_{jk}} f \; dxdy = c_{jk}(x_{j+1} - x_j)(y_{k+1} - y_k)$$

$$= \int_{x_j}^{x_{j+1}} \left[\int_{y_k}^{y_{k+1}} f(x,y) \; dy \right] \; dx \;\; = \int_{y_k}^{y_{k+1}} \left[\int_{x_j}^{x_{j+1}} f(x,y) \; dx \right] \; dy.$$

Properties of the integral of step functions

Theorem (basic properties of the integral)

Let f, g be step functions.

1. $\displaystyle\iint_R (af+bg)\ dxdy = a\iint_R f\ dxdy + b\iint_R g\ dxdy$ *for all* $a, b \in \mathbb{R}$;
2. $\displaystyle\iint_R f\ dxdy = \iint_{R_1} f\ dxdy + \iint_{R_2} f\ dxdy$ *if* R *is divided into* R_1 *and* R_2;
3. $\displaystyle\iint_R f\ dxdy \leq \iint_R g\ dxdy$ *if* $f(x,y) \leq g(x,y)$.

Proof.

Straightforward from the definition. □

Upper / lower integrals and integrability

Definition (integrability)

Let R be a rectangle and let $f : R \to \mathbb{R}$ be a bounded function. If there is one and only one number $I \in \mathbb{R}$ such that

$$\iint_R g(x,y)\ dxdy \leq I \leq \iint_R h(x,y)\ dxdy$$

for every pair of step functions g, h such that, for all $(x,y) \in R$,

$$g(x,y) \leq f(x,y) \leq h(x,y).$$

This number I is called the integral of f on R and is denoted $\iint_R f(x,y)\ dxdy$.

Theorem (evaluating by repeated integration)

Let f be a bounded integrable function on $R = [a_1, b_1] \times [a_2, b_2]$. Suppose that, for every $y \in [a_2, b_2]$, the integral $\int_{a_1}^{b_1} f(x,y)\; dx =: A(y)$ exists. Then $\int_{a_2}^{b_2} A(y)\; dy$ exists and

$$\iint_R f\; dxdy = \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x,y)\; dx \right] dy.$$

Proof.

1. Choose step functions g, h such that $g \le f \le h$,
2. By assumption $\int_{a_1}^{b_1} g(x,y)\; dx \le A(y) \le \int_{a_1}^{b_1} h(x,y)\; dx$,
3. Observe that $\int_{a_1}^{b_1} g(x,y)\; dx$ and $\int_{a_1}^{b_1} h(x,y)\; dx$ are step functions (in y) and so $A(y)$ is integrable and, moreover,

$$\int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} g(x,y)\; dx \right] dy \le \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x,y)\; dx \right] dy \le \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} h(x,y)\; dx \right] dy.$$

□

Integrability of continuous functions

Theorem (integral of continuous functions)

Suppose that f is a continuous function defined on the rectangle R. Then f is integrable and

$$\iint_R f(x,y)\; dxdy = \int_{a_2}^{b_2}\left[\int_{a_1}^{b_1} f(x,y)\; dx\right]\; dy = \int_{a_1}^{b_1}\left[\int_{a_2}^{b_2} f(x,y)\; dy\right]\; dx.$$

Proof.

1. Continuity implies boundedness and so upper and lower integrals exist,
2. Let $\epsilon > 0$. Exists $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| \leq \epsilon$ whenever $\|\mathbf{x} - \mathbf{y}\| \leq \delta$,
3. Partition such $\|\mathbf{x} - \mathbf{y}\| \leq \delta$ whenever $\mathbf{x}, \mathbf{y}$ are in the same sub-rectangle Q_{jk},
4. Step functions g, h s.t. $g(\mathbf{x}) = \inf_{Q_{jk}} f$, $h(\mathbf{x}) = \sup_{Q_{jk}} f$ when $\mathbf{x} \in Q_{jk}$,
5. $\left|\inf_{Q_{jk}} f - \sup_{Q_{jk}} f\right| \leq \epsilon$ and $\epsilon > 0$ can be made arbitrarily small.

$\square$

Volume of a solid

Let $f(x,y)$ be positive on the rectangle $R \subset \mathbb{R}^2$ and consider the 3D set

$$V := \{(x,y,z) : (x,y) \in R, 0 \le z \le f(x,y)\}.$$

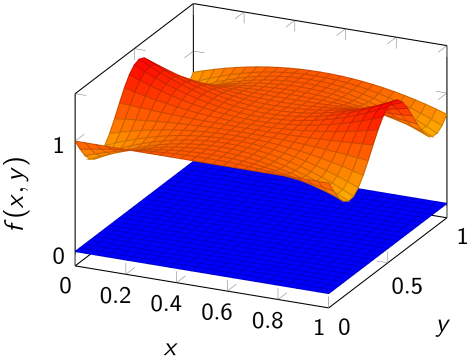

Figure: Set enclosed by xy-plane & $f(x,y)$.

- The volume of the set is equal to

$$\mathrm{Vol}(V) = \iint_R f(x,y)\,dxdy.$$

- Also possible: $f(x,y) \le z \le g(x,y)$.

Functions with discontinuities

Definition (content zero set)

A bounded subset $A \subset \mathbb{R}^2$ is said to have content zero if, for every $\epsilon > 0$, there exists a finite set of rectangles whose union includes A and the sum of the areas of the rectangles is not greater than ϵ.

Example

- Finite set of points,
- Bounded segment,
- Continuous path.

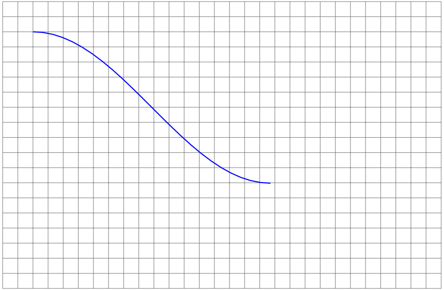

Figure: The graph of a continuous function has content zero.

Integrating functions with discontinuities

Theorem

Let f be a bounded function on R and suppose that the set of discontinuities $A \subset R$ had content zero. Then the double integral $\iint_R f(x,y)\,dxdy$ exists.

Proof.

1. Take a cover of A by rectangles with total area not greater than $\delta > 0$,
2. Let P be a partition of R which is finer than the cover of A,
3. We may assume that $\left|\inf_{Q_{jk}} f - \sup_{Q_{jk}} f\right| \leq \epsilon$ on each sub-rectangle of the partition which doesn't contain a discontinuity of f,
4. The contribution to the integral of bounding step functions from the cover of A is bounded by $\delta \sup |f|$.

$\square$

Integrals over regions bounded by continuous functions

Definition (integral on general regions)

Suppose $S \subset R$ and f is a bounded function on S. We extend f to R by defining

$$f_R(x, y) := \begin{cases} f(x, y) & \text{if } (x, y) \in S \\ 0 & \text{otherwise.} \end{cases}$$

We say that f is integrable if f_R is integrable and define

$$\iint_S f(x, y)\ dxdy = \iint_R f_R(x, y)\ dxdy.$$

Regions bounded by continuous functions

Definition (type 1)

$S \subset \mathbb{R}^2$ is *type 1* if there are continuous functions φ_1, φ_2 such that

$$S = \{(x, y) : x \in [a, b], \varphi_1(x) \leq y \leq \varphi_2(x)\} .$$

Definition (type 2)

$S \subset \mathbb{R}^2$ is *type 2* if there are continuous functions φ_1, φ_2 such that

$$S = \{(x, y) : y \in [a, b], \varphi_1(y) \leq x \leq \varphi_2(y)\} .$$

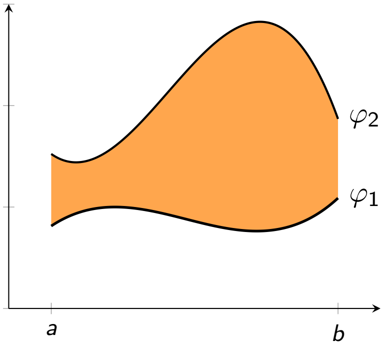

Figure: A region of type 1.

Theorem

Let φ be a continuous function on $[a, b]$. Then the graph $\{(x, y) : x \in [a, b], y = \varphi(x)\}$ has zero content.

Proof.

1. Continuity means that, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\varphi(x) - \varphi(y)| \leq \epsilon$ whenever $|x - y| \leq \delta$,
2. Take partition of $[a, b]$ into subintervals of length less than δ,
3. Using this partition we generate a cover of the graph which has area not greater than $2\epsilon\,|b - a|$.

□

Integrals over regions bounded by continuous functions

Theorem

Let $S = \{(x, y) : x \in [a, b], \varphi_1(x) \leq y \leq \varphi_2(x)\}$ be a region of type 1 and let f be a bounded continuous function of S. Then f is integrable on S and

$$\iint_S f(x, y)\ dxdy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\ dy \right]\ dx.$$

Proof.

1. The set of discontinuity of f_R is the boundary of S in $R = [a, b] \times [\tilde{a}, \tilde{b}]$ which consists of the graphs of φ_1, φ_2,
2. These graphs have zero content as we proved before,
3. For each x, $f(x, y)$ is integrable since it has only two discontinuity points,
4. Additionally $\int_{\tilde{a}}^{\tilde{b}} f_R(x, y)\ dy = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\ dy$. $\square$

Remark: A similar result holds for type 2 regions.

Area of S

Let $S \subset \mathbb{R}^2$ be a *type 1* region, i.e., φ_1, φ_2 are continuous functions,

$$S := \{(x, y) : x \in [a, b], \varphi_1(x) \le y \le \varphi_2(x)\}.$$

Integrating:

$$\iint_S dxdy = \int_a^b (\varphi_2(x) - \varphi_1(x))\, dx.$$

Volume of V

Let $f(x, y) \ge g(x, y)$ be continuous functions on S and

$$V := \{(x, y, z) : x \in [a, b], \varphi_1(x) \le y \le \varphi_2(x), f(x, y) \le z \le g(x, y)\}.$$

Integrating:

$$\iiint_V dxdydz = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} (f(x, y) - g(x, y))dy\right] dx.$$

Mass, centre of mass, centroid

- Suppose we have several particles[1] each with mass m_k at point (x_k, y_k).
  - Total mass is $M := \sum_k m_k$,
  - Centre of mass is the point (p, q) such that

$$pM = \sum_k m_k x_k, \quad qM = \sum_k m_k y_k.$$

- Suppose a disk has the shape of a region S and the density of the material is $f(x, y)$ at point (x, y).
  - Total mass is $M := \iint_S f(x, y)\, dxdy$,
  - Centre of mass is the point (p, q) such that

$$pM = \iint_S x\ f(x, y)\ dxdy, \quad qM = \iint_S y\ f(x, y)\ dxdy.$$

- If the density is constant the centre of mass is called the centroid.

[1] In general, mass m_k at point $\mathbf{x}_k$, the centre of mass is point $\mathbf{X}$ such that $M\mathbf{X} = \sum_k m_k \mathbf{x}_k$.

Green's theorem

Theorem (Green's theorem)

Let $C \subset \mathbb{R}^2$ be a piecewise-smooth simple (no intersections) curve and $\boldsymbol{\alpha}$ a path that parametrizes C in the counter-clockwise direction. Let S be the region enclosed by C. Suppose that $\mathbf{f}(x, y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$ is a vector field continuously differentiable on an open set containing S. Then

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy = \int_C \mathbf{f} \cdot d\boldsymbol{\alpha}.$$

Proof of Green's theorem.

1. Assume that S is a type 1 region and that $Q = 0$,
2. Since $S = \{(x, y) : x \in [a, b], \varphi_1(x) \le y \le \varphi_2(x)\}$,

$$\iint_S \left(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\right)\, dxdy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} (-\tfrac{\partial P}{\partial y})\, dy\right] dx = \int_a^b (P(x, \varphi_1(x)) - P(x, \varphi_2(x)))dx,$$

3. Choose paths $\boldsymbol{\alpha}_1(t) = (t, \varphi_1(t))$, $\boldsymbol{\alpha}_2(t) = (a, t)$, $\boldsymbol{\alpha}_3(t) = (t, \varphi_2(t))$, $\boldsymbol{\alpha}_4(t) = (b, t)$,
4. $\int_C \mathbf{f} \cdot d\boldsymbol{\alpha} = \int \mathbf{f} \cdot d\boldsymbol{\alpha}_1 - \int \mathbf{f} \cdot d\boldsymbol{\alpha}_3 = \int_a^b P(t, \varphi_1(t))\, dt - \int_a^b P(t, \varphi_2(t))\, dt$,
5. If S is also type 2 then this works for $P = 0$ and linearity means it works for $\mathbf{f} = \left(\begin{smallmatrix} P \\ 0 \end{smallmatrix}\right) + \begin{pmatrix} 0 \\ Q \end{pmatrix}$,
6. More general regions can be formed by "glueing" together simpler regions.

$\square$

Simply connected regions

Definition (simply connected)

The set $S \subset \mathbb{R}^n$ is said to be *simply connected* if every closed path $\alpha(t)$ can be continuously shrunk to a point.

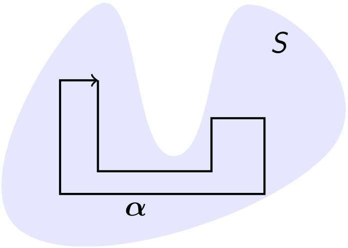

Figure: Simply connected.

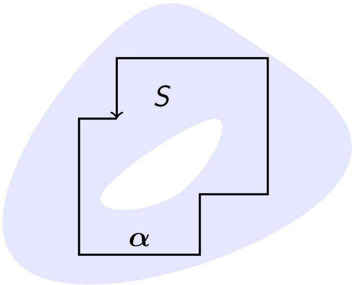

Figure: Not simply connected.

Theorem (conservative vector fields on simply connected regions)

Let S be a simply connected region and suppose that $\mathbf{f} = \begin{pmatrix} P \\ Q \end{pmatrix}$ is a vector field, continuously differentiable on S. Then $\mathbf{f}$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Proof.

1. We already proved that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ whenever $\mathbf{f}$ is conservative;
2. Now suppose that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ and consider any closed path $\boldsymbol{\alpha}$ in S,
3. By Green's theorem $\int_C \mathbf{f} \cdot d\boldsymbol{\alpha} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy = 0$,
4. By the conservative vector field theorem this implies that $\mathbf{f}$ is conservative.

$\square$

Remarks:

- Invariance of a line integral under deformation of a path.
- Multiply connected regions.

Jacobian determinant

Recall 1D case: If $g : [a, b] \to [g(a), g(b)]$ is onto with continuous derivative and f is continuous then

$$\int_{g(a)}^{g(b)} f(x)\ dx = \int_a^b f(g(u))\ g'(u)\ du.$$

Two dimensional change of coordinates

We consider maps $(u, v) \mapsto (X(u, v), Y(u, v))$ mapping $T \subset \mathbb{R}^2$ to $S \subset \mathbb{R}^2$.

Change of variables

Change of variables formula

Suppose that $(u, v) \mapsto (X(u, v), Y(u, v))$ which maps T to S is one-to-one with X, Y continuously differentiable. Then

$$\iint_S f(x, y)\ dxdy = \iint_T f(X(u, v), Y(u, v))\ |J(u, v)|\ dudv.$$

where $J(u, v) := \begin{pmatrix} \partial_u X & \partial_u Y \\ \partial_v X & \partial_v Y \end{pmatrix}$ is the Jacobian matrix.

Remark

The Jacobian is scaling of area: $\iint_S\ dxdy = \iint_T |J(u, v)|\ dudv$.

Polar coordinates

Coordinate mapping:

- $x = r\cos\theta$
- $y = r\sin\theta$

Jacobian determinant:

$$|J(r,\theta)| = \left|\begin{pmatrix} \partial_u X & \partial_u Y \\ \partial_v X & \partial_v Y \end{pmatrix}\right| = \left|\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\right| = r(\cos^2\theta + \sin^2\theta) = r.$$

Change of coordinates:

$$\iint_S f(x,y)\ dxdy = \iint_T r\ f(r\cos\theta, r\sin\theta)\ drd\theta.$$

Change of variables for linear transformations

Coordinate mapping:

- $x = Au + Bv$
- $y = Cu + Dv$

Jacobian determinant:

$$|J(u,v)| = \left| \begin{pmatrix} \partial_u X & \partial_u Y \\ \partial_v X & \partial_v Y \end{pmatrix} \right| = \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| = |AD - BC| \, .$$

Change of coordinates:

$$\iint_S f(x,y)\; dxdy = |AD - BC| \iint_T f(Au + Bv, Cu + Dv)\; dudv.$$

Extension to higher dimensions

- Integrability defined using step functions for $\iiint_S f(x, y, z)\ dxdydz$
- Repeated one dimensional integration: If $V = \{(x, y, z) : (x, y) \in S, \psi_1(x, y) \leq z \leq \psi_2(x, y)\}$,

$$\iiint_S f(x, y, z)\ dxdydz = \iint_S \left[\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\ dz\right]\ dxdy$$

- Change of variables $(u, v, w) \mapsto (X(u, v, w), Y(u, v, w), Z(u, v, w))$

$$\iiint_S f(x, y, z)\, dxdydz = \iiint_T f(X(u, v, w), Y(u, v, w), Z(u, v, w))\ |J(u, v, w)|\ dudvdw.$$

where $J(u, v) := \begin{pmatrix} \partial_u X & \partial_u Y & \partial_u Z \\ \partial_v X & \partial_v Y & \partial_v Z \\ \partial_w X & \partial_w Y & \partial_w Z \end{pmatrix}$

Cylindrical coordinates

Coordinate mapping (require $r > 0$, $0 \leq \theta \leq 2\pi$):

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

Jacobian determinant:

$$|J(r,\theta,z)| = \left|\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = \left|r(\cos^2\theta + \sin^2\theta)\right| = r.$$

Change of coordinates:

$$\iiint_S f(x,y,z)\ dxdydz = \iiint_T r\ F(r,\theta,z)\ drd\theta dz.$$

where $F(r,\theta,z) = f(r\cos\theta, r\sin\theta, z)$.

Spherical coordinates

Coordinate mapping (require $\rho > 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi < \pi$):

- $x = \rho \cos\theta \sin\varphi$
- $y = \rho \sin\theta \sin\varphi$
- $z = \rho \cos\varphi$

Jacobian determinant:

$$|J(\rho,\theta,\varphi)| = \left|\begin{pmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\rho\sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & 0 \\ \rho\cos\theta\cos\varphi & \rho\sin\theta\cos\varphi & -\rho\sin\varphi \end{pmatrix}\right| = \left|-\rho^2\sin\varphi\right| = \rho^2\sin\varphi.$$

Change of coordinates:

$$\iiint_S f(x,y,z)\ dxdydz = \iiint_T F(\rho,\theta,\varphi)\rho^2\sin\varphi\ d\rho d\theta d\varphi.$$

where $F(\rho,\theta,\varphi) = f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$.