MA2 – Part 4 – Line integrals

Weeks 7–8 of MA2 – Draft lecture slides

Oliver Butterley

University of Rome Tor Vergata

2020/21

Outline

Some curves

Recall some "curves" we already saw:

- Circle $\{(x, y) : x^2 + y^2 = 4\}$
- Half a circle $\{(x, y) : x^2 + y^2 = 4, x \geq 0\}$
- Ellipse $\{(x, y) : (\frac{x}{2})^2 + (\frac{y}{3})^2 = 4\}$
- Line $\{(x, y) : y = 5x + 2\}$
- Line in 3D space $\{(x, y, z) : x + 2y + 3z = 0, x = 4y\}$
- Parabola $\{(x, y) : y = x^2\}$

Curves and paths

- Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ be continuous.
- In components $\boldsymbol{\alpha}(t) = (\alpha_1(t), \dots, \alpha_n(t))$.
- We say that $\boldsymbol{\alpha}(t)$ is *continuously differentiable* if each component $\alpha_k(t)$ is differentiable on $[a, b]$ and $\alpha_k'(t)$ is continuous.
- We say that $\boldsymbol{\alpha}(t)$ is *piecewise continuously differentiable* if $[a, b] = [a, c_1] \cup [c_1, c_2] \cup \cdots \cup [c_l, b]$ and $\boldsymbol{\alpha}(t)$ is *continuously differentiable* on each of these intervals.

Definition

If $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ is piecewise continuously differentiable then we call it a *path*.

- Different functions can trace out the same curve in different ways.
- The path has an inherent direction.
- This is a *parametric representation* of the curve.

Examples of paths

- $\boldsymbol{\alpha}(t) := (t, t),\ t \in [0, 1]$
- $\boldsymbol{\alpha}(t) := (\cos t, \sin t),\ t \in [0, 2\pi]$
- $\boldsymbol{\alpha}(t) := (\cos t, \sin t),\ t \in [-\pi/2, \pi/2]$
- $\boldsymbol{\alpha}(t) := (\cos t, -\sin t),\ t \in [0, 2\pi]$
- $\boldsymbol{\alpha}(t) := (t, t, t),\ t \in [0, 1]$
- $\boldsymbol{\alpha}(t) := (\cos t, \sin t, t),\ t \in [-10, 10]$
- etc. . .

[View graphic of the spiral and circle in part 2]

Definition of the line integral

- Let $\boldsymbol{\alpha}(t)$ be a (piecewise continuously differentiable) path on $[a, b]$,
- Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field,
- Recall that $\boldsymbol{\alpha}'(t) = \begin{pmatrix} \alpha_1'(t) \\ \vdots \\ \alpha_n'(t) \end{pmatrix}$ and $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$.

Definition (line integral)

The *line integral* of the vector field $\mathbf{f}$ along the path $\boldsymbol{\alpha}$ is

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} := \int_a^b \mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)\ dt.$$

Other possible notation:

- $\int_C \mathbf{f} \cdot d\boldsymbol{\alpha}$ (if the parametrization of the curve C is clear);
- $\int f_1\ d\alpha_1 + \cdots + f_n\ d\alpha_n$ or $\int f_1\ dx_1 + \cdots + f_n\ dx_n$.

Example of calculating a line integral

Example

Consider the vector field $\mathbf{f}(x, y) := \begin{pmatrix} \sqrt{y} \\ x^3 + y \end{pmatrix}$ and the path $\boldsymbol{\alpha}(t) := (t^2, t^3)$, $t \in (0, 1)$. Evaluate $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$.

Solution.

1. $\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$;
2. $\mathbf{f}(\boldsymbol{\alpha}(t)) := \begin{pmatrix} t^{\frac{3}{2}} \\ t^6 + t^3 \end{pmatrix}$;
3. $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = \begin{pmatrix} t^{\frac{3}{2}} \\ t^6 + t^3 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} = 2t^{\frac{5}{2}} + 3t^8 + 3t^5$;
4. $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_0^1 (2t^{\frac{5}{2}} + 3t^8 + 3t^5)\, dt = \frac{59}{42}$. $\square$

Basic properties of the line integral

Linearity: Suppose $\mathbf{f}$, $\mathbf{g}$ are vector fields and $\boldsymbol{\alpha}(t)$ is a path. For any $c, d \in \mathbb{R}$,

$$\int (c\mathbf{f} + d\mathbf{g}) \cdot d\boldsymbol{\alpha} = c \int \mathbf{f} \cdot d\boldsymbol{\alpha} + d \int \mathbf{g} \cdot d\boldsymbol{\alpha}.$$

Joining / dividing paths: Suppose $\mathbf{f}$ is a vector field and that

$$\boldsymbol{\alpha}(t) = \begin{cases} \boldsymbol{\alpha}_1(t) & t \in [a, c] \\ \boldsymbol{\alpha}_2(t) & t \in [c, b] \end{cases}$$

is a path. Then

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int \mathbf{f} \cdot d\boldsymbol{\alpha}_1 + \int \mathbf{f} \cdot d\boldsymbol{\alpha}_2.$$

Or: If we write C, C_1, C_2 for the corresponding curves, then

$$\int_C \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_{C_1} \mathbf{f} \cdot d\boldsymbol{\alpha} + \int_{C_2} \mathbf{f} \cdot d\boldsymbol{\alpha}.$$

Choices of parametrization

Consider the curve $C = \{(x, y) : x^2 + y^2 = 1, y \geq 0\}$ (Half circle).
Many possible path parametrization, e.g.,

- $\alpha(t) := (-t, \sqrt{1 - t^2})$, $t \in [-1, 1]$
- $\beta(t) := (\cos t, \sin t)$, $t \in [0, \pi]$

Definition (equivalent paths)

We say that two paths $\alpha(t)$ and $\beta(t)$ are *equivalent* if there exists a continuously differentiable function $u : [c, d] \to [a, b]$ such that $\alpha(u(t)) = \beta(t)$. Furthermore,

- if $u(c) = a$ and $u(d) = b$ we say that $\alpha(t)$ and $\beta(t)$ are in the same direction,
- if $u(c) = b$ and $u(d) = a$ we say that $\alpha(t)$ and $\beta(t)$ are in the opposite direction.

Change of parametrization

Theorem (Change of parametrization)

Let $\mathbf{f}$ be a continuous vector field and let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be equivalent paths. Then

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \begin{cases} \int \mathbf{f} \cdot d\boldsymbol{\beta} & \textit{if the paths are in the same direction,} \\ -\int \mathbf{f} \cdot d\boldsymbol{\beta} & \textit{if the paths are in the opposite direction.} \end{cases}$$

Proof.

1. Suppose continuously differentiable path (decomposing if required);
2. Since $\boldsymbol{\alpha}(u(t)) = \boldsymbol{\beta}(t)$ chain rule implies that $\boldsymbol{\beta}'(t) = \boldsymbol{\alpha}'(u(t))\ u'(t)$;
3. $\int \mathbf{f} \cdot d\boldsymbol{\beta} = \int_c^d \mathbf{f}(\boldsymbol{\beta}(t)) \cdot \boldsymbol{\beta}'(t)\ dt = \int_c^d \mathbf{f}(\boldsymbol{\alpha}(u(t))) \cdot \boldsymbol{\alpha}'(u(t))\ u'(t)\ dt$;
4. Change variables (gives minus if path is opposite direction). □

Gradients and line integrals

- Let $h(x, y)$ be a scalar field in $\mathbb{R}^2$;
- Recall that the gradient $\nabla h(x, y)$ is a vector field;
- Let $\boldsymbol{\alpha}(t)$, $t \in [0, 1]$ be a path;
- $\frac{d}{dt} h(\boldsymbol{\alpha}(t)) = \nabla h(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)$;
- $$\begin{aligned} \int \nabla h \cdot d\boldsymbol{\alpha} &= \int_0^1 \nabla h(\boldsymbol{\alpha}(t)) \cdot \alpha'(t)\ dt \\ &= \int_0^1 \frac{d}{dt} h(\boldsymbol{\alpha}(t))\ dt = h(\boldsymbol{\alpha}(1)) - h(\boldsymbol{\alpha}(0)). \end{aligned}$$

[Graphic of person walking on a map with contour lines]

Work in physics 1 (Gravity)

- Gravitational field $\mathbf{f}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix}$;
- Move particle from $\mathbf{a} = (a_1, a_2, a_3)$ to $\mathbf{b} = (b_1, b_2, b_3)$ along the path $\boldsymbol{\alpha}(t)$, $t \in [0,1]$;
- Work done is defined as $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$.

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_0^1 \mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)\ dt = \int_0^1 mg\ \alpha_3'(t)\ dt$$
$$= mg\ \left[\alpha_3(t)\right]_0^1 = mg(b_3 - a_3).$$

I.e., work done depends only on the change in height.

Work in physics 2 (force field)

- Let $\mathbf{f}$ be a force field;
- Let $\mathbf{x}(t)$ be the position at time t of a particle moving in the field;
- Let $\mathbf{v}(t) = \mathbf{x}'(t)$ be the velocity at time t of the particle;
- Define kinetic energy as $\frac{m}{2}\|\mathbf{v}(t)\|^2$.

Newton's law: $\mathbf{f}(\mathbf{x}(t)) = m\mathbf{x}''(t) = m\mathbf{v}'(t)$.

Work done:

$$\begin{aligned}\int \mathbf{f} \cdot d\mathbf{x} &= \int_0^1 \mathbf{f}(\mathbf{x}(t)) \cdot \mathbf{v}(t)\ dt = \int_0^1 m\mathbf{v}'(t) \cdot \mathbf{v}(t)\ dt \\ &= \int_0^1 \frac{d}{dt}\left(\frac{m}{2}\|\mathbf{v}(t)\|^2\right) = \left(\frac{m}{2}\|\mathbf{v}(1)\|^2 - \frac{m}{2}\|\mathbf{v}(0)\|^2\right)\end{aligned}$$

I.e., work done on the particle moving in the force field is equal to the change in kinetic energy.

Length of a curve

Let $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ be a path.

Definition (length of a curve)

The length of the piece of the curve between $\boldsymbol{\alpha}(a)$ and $\boldsymbol{\alpha}(t)$ is defined as

$$s(t) := \int_a^t \|\boldsymbol{\alpha}'(u)\| \; du.$$

- $s'(t) = \|\boldsymbol{\alpha}'(t)\|$.
- If the path represents a wire and the wire has density $\varphi(\boldsymbol{\alpha}(t))$ at the point $\boldsymbol{\alpha}(t)$ then the mass of the wire is defined as $M = \int \varphi(\boldsymbol{\alpha}(t)) \; s'(t) \; dt$.

The second fundamental theorem of calculus

Recall: If $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable then $\int_a^b \varphi'(t)\ dt = \varphi(b) - \varphi(a)$.

Theorem (2nd fundamental theorem in $\mathbb{R}^n$)

Suppose that φ is a continuously differentiable scalar field on $S \subset \mathbb{R}^n$ and suppose that $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ is a path in S. Let $\mathbf{a} := \boldsymbol{\alpha}(a)$, $\mathbf{b} := \boldsymbol{\alpha}(b)$. Then

$$\int \nabla\varphi \cdot d\boldsymbol{\alpha} = \varphi(\mathbf{b}) - \varphi(\mathbf{a}).$$

Proof.

1. Suppose that $\boldsymbol{\alpha}(t)$ is continuously differentiable;
2. By the chain rule $\frac{d}{dt}\varphi(\boldsymbol{\alpha}(t)) = \nabla\varphi(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)$;
3. Consequently $\int \nabla\varphi \cdot d\boldsymbol{\alpha} = \int_0^1 \nabla\varphi(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)\ dt = \int_0^1 \frac{d}{dt}\varphi(\boldsymbol{\alpha}(t))\ dt$
4. By 2nd fund. theorem in $\mathbb{R}$, $\int_0^1 \frac{d}{dt}\varphi(\boldsymbol{\alpha}(t))\ dt = \varphi(\boldsymbol{\alpha}(b)) - \varphi(\boldsymbol{\alpha}(a))$.

$\square$

Potential energy example

- Earth has mass M with centre at $(0,0,0)$,
- Small particle close to earth has mass m,
- Force field of gravitation is equal to $\mathbf{f}(\mathbf{x}) := \frac{-GmM}{\|\mathbf{x}\|^3}\mathbf{x}$,
- Define potential energy as $\varphi(\mathbf{x}) := \frac{GmM}{\|\mathbf{x}\|}$.

We write $\varphi(x_1, x_2, x_3) = \frac{GmM}{\sqrt{x_1^2+x_2^2+x_3^2}}$ and calculate

$$\nabla\varphi(\mathbf{x}) = \begin{pmatrix} (GmM)\ (2x_1)\ (-\frac{1}{2})\ (x_1^2+x_2^2+x_3^2)^{-\frac{3}{2}} \\ (GmM)\ (2x_2)\ (-\frac{1}{2})\ (x_1^2+x_2^2+x_3^2)^{-\frac{3}{2}} \\ (GmM)\ (2x_3)\ (-\frac{1}{2})\ (x_1^2+x_2^2+x_3^2)^{-\frac{3}{2}} \end{pmatrix} = \frac{-GmM}{\|\mathbf{x}\|^3}\mathbf{x}.$$

Connected sets

Definition (connected)

The set $S \subset \mathbb{R}^n$ is said to be *connected* if, for every pair of points $\mathbf{a}, \mathbf{b} \in S$, there exists a path $\alpha(t), t \in [a, b]$ such that

- $\alpha(t) \in S$ for every $t \in [a, b]$,
- $\alpha(a) = \mathbf{a}$ and $\alpha(b) = \mathbf{b}$.

Terminology: Sometimes this is called "path connected" to distinguish between different notions.

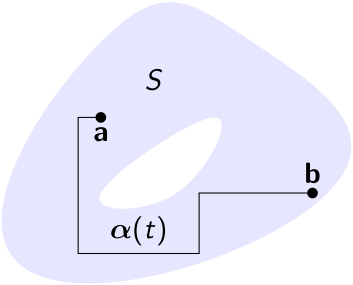

Figure: A connected set.

The first fundamental theorem of calculus

Recall: If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\varphi(x) := \int_a^x f(t)\, dt$ then $\varphi'(x) = f(x)$.

Theorem (1^{st} fundamental theorem in $\mathbb{R}^n$)

Let $\mathbf{f}$ be a continuous vector field on a connected set $S \subset \mathbb{R}^n$. Suppose that, for $\mathbf{x}, \mathbf{a} \in S$, the line integral $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ is equal for every path $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}(a) = \mathbf{a}$, $\boldsymbol{\alpha}(b) = \mathbf{x}$. Fix $\mathbf{a} \in S$ and define $\varphi(\mathbf{x}) := \int \mathbf{f} \cdot d\alpha$. Then φ is continuously differentiable and $\nabla\varphi = \mathbf{f}$.

Sketch of proof.

1. As before $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$;
2. $\varphi(\mathbf{x} + h\mathbf{e}_k) - \varphi(\mathbf{x}) = \int \mathbf{f} \cdot d\boldsymbol{\beta}_k$ where $\boldsymbol{\beta}_k(t) := \mathbf{x} + t\mathbf{e}_k$, $t \in [0, h]$;
3. Observe that $\boldsymbol{\beta}_k'(t) = \mathbf{e}_k$;
4. $\frac{\partial\varphi}{\partial x_k} = \lim_{h\to 0} \frac{1}{h}(\varphi(\mathbf{x} + h\mathbf{e}_k) - \varphi(\mathbf{x})) = \lim_{h\to 0} \frac{1}{h} \int_0^h \mathbf{f}(\boldsymbol{\beta}_k(t)) \cdot \mathbf{e}_k \, dt = f_k(\mathbf{x})$;
5. I.e., $\nabla\varphi(\mathbf{x}) = \mathbf{f}(\mathbf{x})$;

$\square$

Closed paths

Definition (closed path)

We say a path $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ is *closed* if $\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b)$.

Remarks

- If $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ is a closed path then we can divided it into two paths: Let $c \in [a, b]$ and consider the two paths $\boldsymbol{\alpha}(t)$, $t \in [a, c]$ and $\boldsymbol{\alpha}(t)$, $t \in [c, b]$.
- Suppose $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ and $\boldsymbol{\beta}(t)$, $t \in [c, d]$ are two path starting at $\mathbf{a}$ and finishing at $\mathbf{b}$. The these can be combined to define a closed path (by following one backward).

Conservative vector fields

Definition (conservative vector field)

A vector field $\mathbf{f}$, continuous on $S \subset \mathbb{R}^n$ is said to be conservative if there exists a scalar field φ such that, on S,

$$\mathbf{f} = \nabla \varphi.$$

Terminology:

- Some authors call such a vector field a *gradient* (i.e., the vector field is the gradient of some scalar).
- If $\mathbf{f} = \nabla \varphi$ the φ is called the *potential*.

Non-uniqueness:

- Observe that $\nabla \varphi = \nabla(\varphi + C)$ for any constant $C \in \mathbb{R}$.

Conservative vector fields

Theorem (conservative vector fields)

The following are equivalent for a vector field $\mathbf{f}$*:*

(a) *There exists* φ *such that* $\mathbf{f} = \nabla\varphi$,

(b) $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ *does not depend on* $\boldsymbol{\alpha}$, *as long as* $\boldsymbol{\alpha}(a) = \mathbf{a}$, $\boldsymbol{\alpha}(b) = \mathbf{b}$,

(c) $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = 0$ *for any closed path* $\boldsymbol{\alpha}$.

Proof.

(a) $\Leftrightarrow$ (b) We proved in the previous theorems (the two fundamental theorems);

(b) $\Rightarrow$ (c) Let $\boldsymbol{\alpha}(t)$ be a closed path and let $\boldsymbol{\beta}(t)$ be the same path in the opposite direction. Observe that $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = -\int \mathbf{f} \cdot d\boldsymbol{\beta}$ but that $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int \mathbf{f} \cdot d\boldsymbol{\beta}$ and so $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = 0$;

(b) $\Leftarrow$ (c) The two paths between $\mathbf{a}$ and $\mathbf{b}$ can be combined (with a minus sign) to give a closed path.

$\square$

Mixed partial derivatives

Theorem

Suppose that $\mathbf{f}$ *is a continuously differential vector field. If* $\mathbf{f} = \nabla\varphi$ *for some scalar field* φ *then, for each* l, k*,*

$$\frac{\partial f_l}{\partial x_k} = \frac{\partial f_k}{\partial x_l}.$$

Notation: Here we write, as usual, $\mathbf{f}(x_1, \ldots, x_n) = \begin{pmatrix} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_n(x_1,\ldots,x_n) \end{pmatrix}$.

Proof.

By assumption the second order partial derivatives exist and so

$$\frac{\partial f_l}{\partial x_k} = \frac{\partial^2 \varphi}{\partial x_k \partial x_l} = \frac{\partial^2 \varphi}{\partial x_l \partial x_k} = \frac{\partial f_k}{\partial x_l}. \qquad \square$$

Useful: If a pair of mixed derivatives is not equal then $\mathbf{f}$ is *not* conservative.

Constructing a potential

Question: Suppose we are given a vector field $\mathbf{f}$ and we know that $\mathbf{f} = \nabla\varphi$ for some φ. How can we calculate φ?

Two methods:

1. by line integral;
2. by indefinite integrals.

Constructing a potential by line integral

1. Suppose that $\mathbf{f}$ is a conservative vector field on the rectangle $[a_1, b_1] \times [a_2, b_2]$;
2. We will define $\varphi(\mathbf{x})$ as the line integral $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is a path between $\mathbf{a} = (a_1, a_2)$ and $\mathbf{x}$;
3. For any $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ consider the two paths:
   - H: $\boldsymbol{\alpha}_1(t) := (t, a_2),\ t \in [a_1, x_1]$;
   - V: $\boldsymbol{\alpha}_2(t) := (x_1, t),\ t \in [a_2, x_2]$;

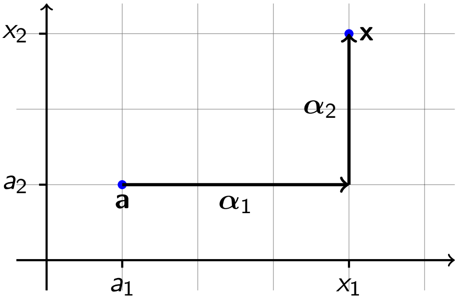

Figure: The paths $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.

4. Let $\boldsymbol{\alpha}(t)$ denote the combination of the two paths;
5. Calculate $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_{a_1}^{x_1} \mathbf{f}(\boldsymbol{\alpha}_1(t)) \cdot \boldsymbol{\alpha}_1'(t)\ dt + \int_{a_2}^{x_2} \mathbf{f}(\boldsymbol{\alpha}_2(t)) \cdot \boldsymbol{\alpha}_2'(t)\ dt$;
6. And so $\varphi(\mathbf{x}) = \int_{a_1}^{x_1} f_1(t, a_2)\ dt + \int_{a_2}^{x_2} f_2(x_1, t)\ dt$.

Constructing a potential by indefinite integrals

1. Again suppose that $\mathbf{f} = \nabla\varphi$ for some scalar field $\varphi(x,y)$ which we wish to find;
2. Observe that $\frac{\partial\varphi}{\partial x} = f_1$ and $\frac{\partial\varphi}{\partial y} = f_2$;
3. This means that ($A(y)$, $B(x)$ are constants of integration)

$$\int_a^x f_1(t,y)\ dt + A(y) = \varphi(x,y) = \int_b^y f_2(x,t)\ dt + B(x);$$

4. Calculating and comparing we can obtain a formula for $\varphi(x,y)$.

Example

Find a potential for $\mathbf{f}(x,y) = \begin{pmatrix} e^x y^2+1 \\ 2e^x y \end{pmatrix}$ on $\mathbb{R}^2$.

- $\int_a^x f_1(t,y)\ dt + A(y) = e^x y^2 + x + A(y) = \varphi(x,y)$;
- $\int_b^y f_2(x,t)\ dt + B(x) = e^x y^2 + B(x) = \varphi(x,y)$;
- we can choose $A(y) = 0$ and $B(x) = x$ to obtain equality above;
- potential is $\varphi(x,y) = e^x y^2 + x$.

Convex sets

Definition (convex set)

A set $S \subset \mathbb{R}^n$ is said to be *convex* if for any $\mathbf{x}, \mathbf{y} \in S$ the segment $\{t\mathbf{x} + (1-t)\mathbf{y}, t \in [0,1]\}$ is contained in S.

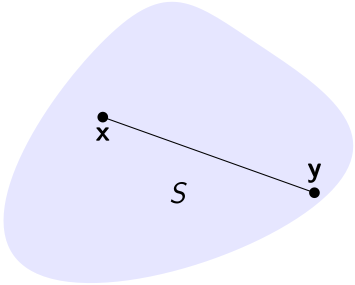

Figure: A convex set.

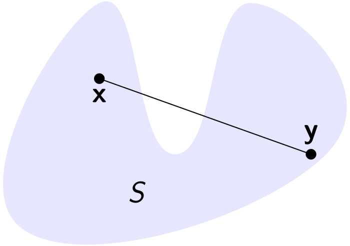

Figure: A set which is not convex.

Sufficient condition for a vector field to be conservative

Theorem

Let[1] $\mathbf{f}$ be a continuously differentiable vector field on a convex region $S \subset \mathbb{R}^n$. Then $\mathbf{f}$ is conservative if and only if

$$\frac{\partial f_l}{\partial x_k} = \frac{\partial f_k}{\partial x_l}, \quad \textit{for each } l, k.$$

Sketch of proof.

1. Need only assume $\partial_g f_l = \partial_l f_k$ and construct a potential;
2. Let $\varphi(\mathbf{x}) = \int \mathbf{f} \cdot d\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}(t) = t\mathbf{x}$, $t \in [0, 1]$;
3. Since $\boldsymbol{\alpha}'(t) = \mathbf{x}$, $\varphi(\mathbf{x}) = \int_0^1 \mathbf{f}(t\mathbf{x}) \cdot \mathbf{x}$;
4. Also (needs proving) $\partial_k \varphi(\mathbf{x}) = \int_0^1 (t\partial_k \mathbf{f}(t\mathbf{x}) \cdot \mathbf{x} + f_k(t\mathbf{x}))\ dt$;
5. This is equal to $\int_0^1 (t\nabla f_k(t\mathbf{x}) \cdot \mathbf{x} + f_k(t\mathbf{x}))\ dt$ because $\partial_g f_l = \partial_l f_k$;
6. By the chain rule (to $g(t) := t\nabla f_k(t\mathbf{x})$) this is equal to $f_k(\mathbf{x})$ as required.

$\square$

[1] As usual $f_k(x_1, \ldots, x_n)$ denotes the k^{th} component of the vector field $\mathbf{f}$.

Conservative or non-conservative vector field?

Example

Consider the vector field $\mathbf{f}(x, y) := \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}$ on $S = \mathbb{R}^2 \setminus (0,0)$.

1. Verify that $\frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial x}$;
2. Evaluate the line integral $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}(t) := (a\cos t, a\sin t)$, $t \in [0, 2\pi]$.

Remarks

- S is not convex;
- Line integral is the same for any circle, independent of the radius.

Evaluation of line integral

1. $\boldsymbol{\alpha}'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix}$ and $\mathbf{f}(\boldsymbol{\alpha}(t)) = \frac{1}{a^2}\begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix}$;
2. $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = \sin^2 t + \cos^2 t = 1$;
3. $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_0^{2\pi} (1)\ dt = 2\pi$.

Exact differential equations $p(x, y) + q(x, y)\frac{dy}{dx} = 0$

Theorem

(a) *If* $\varphi(x, y)$ *satisfies* $\nabla\varphi(x, y) = \begin{pmatrix} p(x,y) \\ q(x,y) \end{pmatrix}$ *then the solution* $y(x)$ *of the equation* $p(x, y) = q(x, y)\frac{dy}{dx}$ *satisfies* $\varphi(x, y(x)) = C$ *for some* $C \in \mathbb{R}$*.*

(b) *Conversely, if* $\varphi(x, y(x)) = C$ *defines implicitly a function* $y(x)$*, then* $y(x)$ *is a solution to the equation* $p(x, y) = q(x, y)\frac{dy}{dx}$*.*

Proof.

1. If $y(x)$ satisfies $\varphi(x, y(x)) = C$, then by the chain rule and $\nabla\varphi = \begin{pmatrix} p \\ q \end{pmatrix}$, $p(x, y(x)) + y'(x)q(x, y(x)) = 0$;
2. Conversely, if $y(x)$ is a solution, $\varphi(x, y(x))$ must be constant in x. □

Example

Solve $y^2 + 2xyy' = 0$. Let $p(x, y) = y^2$, $q(x, y) = 2xy$ and find $\varphi(x, y) = xy^2$ so $\nabla\varphi = \begin{pmatrix} p \\ q \end{pmatrix}$. Solutions satisfy $\varphi(x, y(x)) = xy(x)^2 = C$, i.e., $y(x) = \sqrt{\frac{C}{x}}$.