MA2 – Part 3 – Applications of the differential calculus

Weeks 5–6 of MA2 – Draft lecture slides

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Partial differential equations

First order linear PDEs

1D wave equation

Extrema

Stationary points

Second order Taylor formula and Hessian matrix

Classifying stationary points

Extreme value theorem for continuous scalar fields

Extrema with constraints

Lagrange multipliers

Outline

Partial differential equations
- First order linear PDEs
- 1D wave equation

Extrema
- Stationary points
- Second order Taylor formula and Hessian matrix
- Classifying stationary points
- Extreme value theorem for continuous scalar fields

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First order linear PDE

Huge number of different partial differential equations – we consider a few types.

Example

Find all solutions of the partial differential equation $3\frac{\partial f}{\partial x}(x,y) + 2\frac{\partial f}{\partial y}(x,y) = 0$.

Solution:

1. Equivalent to $\binom{3}{2} \cdot \nabla f(x,y) = 0$;
2. Directional derivative $D_v f(x,y) = 0$ where $v = \binom{3}{2}$;
3. This means that f is constant in the direction $\binom{3}{2}$;
4. All solutions have the form $f(x,y) = g(2x - 3y)$ for some $g : \mathbb{R} \to \mathbb{R}$.

Theorem

Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable, $a, b \in \mathbb{R}$, $(a,b) \neq (0,0)$. If $f(x,y) := g(bx - ay)$ then

$$a\frac{\partial f}{\partial x}(x,y) + b\frac{\partial f}{\partial y}(x,y) = 0.$$

Conversely, every f which satisfies this equation is of the form $g(bx - ay)$.

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Partial differential equations
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1D wave equation
Extrema
Stationary points
Second order Taylor formula and Hessian matrix
Classifying stationary points
Extreme value theorem for continuous scalar fields
Extrema with constraints
Lagrange multipliers

PDE (cont.)

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Proof.

($\Rightarrow$)

1. If $f(x,y) = g(bx - ay)$ then, by the chain rule, $\partial_x f(x,y) = bg'(bx - ay)$ and $\partial_y f(x,y) = -ag'(bx - ay)$.
2. Consequently $a\partial_x f(x,y) + b\partial_y f(x,y) = abg'(bx - ay) - abg'(bx - ay) = 0$.

($\Leftarrow$)

1. Let $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ and so $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{a^2+b^2} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$.
2. Let $h(u,v) = f\left(\frac{au+bv}{a^2+b^2}, \frac{bu-av}{a^2+b^2}\right)$.
3. Calculate

$$\partial_u h(u,v) = \frac{1}{a^2+b^2} \left(a\partial_x f + b\partial_y f\right)(au + bv, bu - av) = 0.$$

4. Namely, $h(u,v)$ is a function of v only so take $g(v) = h(u,v)$ and so $f(x,y) = g(bx - ay)$.

$\square$

1D wave equation

The *1D wave equation* is

$$\frac{\partial^2 f}{\partial x^2}(x,t) = c^2 \frac{\partial^2 f}{\partial t^2}(x,t).$$

- x – position along string
- t – time
- $f(x,t)$ – displacement
- c – constant depending on the string

Derived from the equation of motion $F = ma$ where F is the tension in the string, a is the acceleration from horizontal and m is the mass of a little piece of the string. Good for small displacement.

Boundary conditions: Are the ends fixed? Does it start moving?

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1D wave equation (cont.)

Theorem

1. *Let F be a twice differentiable function and G a differentiable function. Then*

$$f(x,t) := \frac{1}{2}(F(x+ct) + F(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\, ds \qquad (1)$$

   satisfies $\frac{\partial^2 f}{\partial x^2}(x,t) = c^2 \frac{\partial^2 f}{\partial t^2}(x,t)$, $f(x,0) = F(x)$ *and* $\frac{\partial f}{\partial t}(x,0) = G(x)$.

2. *Conversely, if a solution of* $\frac{\partial^2 f}{\partial x^2}(x,t) = c^2 \frac{\partial^2 f}{\partial t^2}(x,t)$ *satisfies*

$$\frac{\partial^2 f}{\partial x \partial t}(x,t) = \frac{\partial^2 f}{\partial t \partial x}(x,t),$$

   then it has the above form (1).

1D wave equation (cont.)

Partial differential equations
First order linear PDEs
1D wave equation
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Proof of part 1.

1. Let $f(x,t)$ be as defined (1) and calculate

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,t) &= \tfrac{1}{2}\left(F'(x+ct)+F'(x-ct)\right)+\tfrac{1}{2c}\left(G(x+ct)-G(x-ct)\right)\\
\frac{\partial^2 f}{\partial x^2}(x,t) &= \tfrac{1}{2}\left(F''(x+ct)+F''(x-ct)\right)+\tfrac{1}{2c}\left(G'(x+ct)-G'(x-ct)\right)\\
\frac{\partial f}{\partial t}(x,t) &= \tfrac{1}{2}\left(cF'(x+ct)-cF'(x-ct)\right)+\tfrac{1}{2}\left(G(x+ct)+G(x-ct)\right)\\
\frac{\partial^2 f}{\partial t^2}f(x,t) &= \tfrac{1}{2}\left(c^2F''(x+ct)+c^2F''(x-ct)\right)+\tfrac{c}{2}\left(G'(x+ct)+G'(x-ct)\right)
\end{aligned}$$

2. Observe that $\frac{\partial^2 f}{\partial x^2}(x,t) = c^2\frac{\partial^2 f}{\partial t^2}(x,t)$.
3. Observe that $f(x,0)=F(x)$ and $\frac{\partial f}{\partial t}(x,0)=G(x)$.

$\square$

1D wave equation (cont.)

Proof of part 2.

1. Introduce $u = x + ct$, $v = x - ct$ and observe that $x = \frac{u+v}{2}$, $t = \frac{u-v}{2c}$;
2. Define $g(u, v) = f(x, t) = f(\frac{u+v}{2}, \frac{u-v}{2c})$;
3. By the chain rule

$$\frac{\partial g}{\partial u}(u, v) = \frac{1}{2}\frac{\partial f}{\partial x}\Big(\frac{u+v}{2}, \frac{u-v}{2c}\Big) + \frac{1}{2c}\frac{\partial f}{\partial t}\Big(\frac{u+v}{2}, \frac{u-v}{2c}\Big)$$

$$\begin{aligned}\frac{\partial^2 g}{\partial v \partial u}(u, v) &= \frac{1}{4}\frac{\partial^2 f}{\partial x^2}\Big(\frac{u+v}{2}, \frac{u-v}{2c}\Big) - \frac{1}{4c}\frac{\partial^2 f}{\partial x \partial t}\Big(\frac{u+v}{2}, \frac{u-v}{2c}\Big) \\ &\quad + \frac{1}{4c}\frac{\partial^2 f}{\partial x \partial t}\Big(\frac{u+v}{2}, \frac{u-v}{2c}\Big) - \frac{1}{4c^2}\frac{\partial^2 f}{\partial t^2}\Big(\frac{u+v}{2}, \frac{u-v}{2c}\Big) = 0;\end{aligned}$$

4. So $\frac{\partial g}{\partial u}(u, v) = \varphi_0(u)$ and $g(u, v) = \varphi_1(u) + \varphi_2(v)$. I.e., $f(x, t) = \varphi_1(x + ct) + \varphi_2(x - ct)$;
5. Let $F(x) := \varphi_1(x) + \varphi_2(x)$;
6. $F'(x) = \varphi_1'(x) + \varphi_2'(x)$ and $\frac{\partial f}{\partial t}(x, t) = c\varphi_1(x + ct) - c\varphi_2(x - ct)$;
7. Let $G(x) := \frac{\partial f}{\partial t}(x, 0) = c\varphi_1(x) - c\varphi_2(x)$.

Partial differential equations
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Minima, maxima and saddle points

Let $S \subset \mathbb{R}^n$ be open, $f : S \to \mathbb{R}$ be a scalar field and $\mathbf{a} \in S$.

Definition (absolute min/max)

If $f(\mathbf{a}) \leq f(\mathbf{x})$ (resp. $f(\mathbf{a}) \geq f(\mathbf{x})$) for all $\mathbf{x} \in S$, then $f(\mathbf{a})$ is said to be the *absolute* minimum (resp. maximum) of f.

Definition (relative min/max)

If $f(\mathbf{a}) \leq f(\mathbf{x})$ (resp. $f(\mathbf{a}) \geq f(\mathbf{x})$) for all $\mathbf{x} \in B(\mathbf{a}, r)$ for some $r > 0$, then $f(\mathbf{a})$ is said to be a *relative* minimum (resp. maximum) of f.

Terminology: "absolute" = "global"; "relative" = "local".

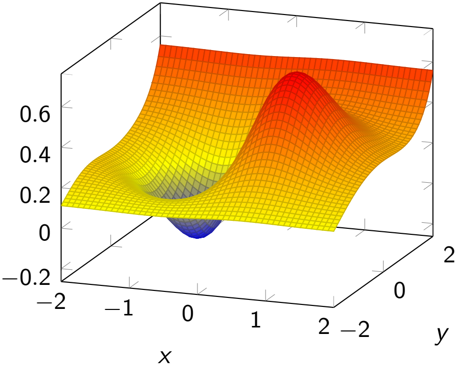

Figure: $f(x, y) := xe^{-(x^2y^2)} + \frac{1}{4}e^{y^{\frac{3}{10}}}$

Partial differential equations
First order linear PDEs
1D wave equation
Extrema
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Lagrange multipliers

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Theorem

If $f : S \to \mathbb{R}$ is differentiable and has a relative minimum or maximum at $\mathbf{a}$, then $\nabla f(\mathbf{a}) = \mathbf{0}$.

Proof.

1. Suppose f has a relative minimum at $\mathbf{a}$ (or consider $-f$);
2. For any unit vector $\mathbf{v}$ let $g(u) = f(\mathbf{a} + u\mathbf{v})$;
3. g has relative minimum at $u = 0$ so $u'(0) = 0$;
4. This means that $D_{\mathbf{v}} f(\mathbf{a}) = 0$ for every $\mathbf{v}$ and so $\nabla f(\mathbf{a}) = \mathbf{0}$. $\square$

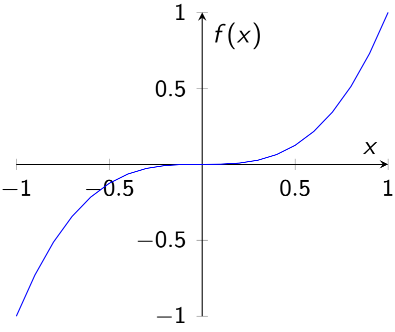

Figure: $\nabla f(\mathbf{a}) = \mathbf{0}$ doesn't imply a minimum or maximum at $\mathbf{a}$ as seen for the function $f(x) := x^3$.

Stationary points (cont.)

Definition (stationary point)

If $\nabla f(\mathbf{a}) = 0$ then $\mathbf{a}$ is called a *stationary point*.

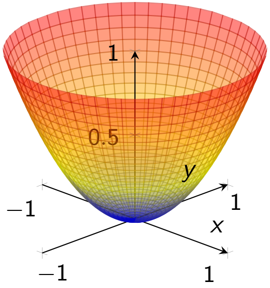

Figure: If $f(x,y) = x^2 + y^2$ then $\nabla f(x,y) = \left(\begin{smallmatrix} 2x \\ 2y \end{smallmatrix}\right)$ and $\nabla f(0,0) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. The point $(0,0)$ is an absolute minimum for f.

Definition (saddle point)

If $\nabla f(\mathbf{a}) = 0$ and $\mathbf{a}$ is neither a minimum nor a maximum then $\mathbf{a}$ is said to be a *saddle point*.

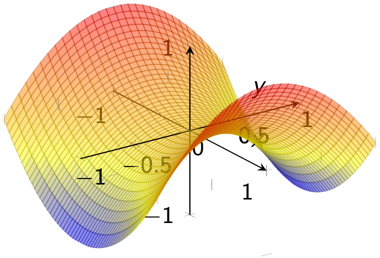

Figure: If $f(x,y) = x^2 - y^2$ then $\nabla f(x,y) = \left(\begin{smallmatrix} 2x \\ -2y \end{smallmatrix}\right)$ and $\nabla f(0,0) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. The point $(0,0)$ is a saddle point for f.

Hessian matrix

Definition (Hessian matrix)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. The *Hessian matrix* at $\mathbf{a} \in \mathbb{R}^n$ is defined

$$\mathbf{H}f(\mathbf{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_1\,\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_n\,\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{pmatrix}.$$

- The Hessian matrix $\mathbf{H}f(\mathbf{a})$ is symmetric;
- If $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ then $\mathbf{v}^{\mathbf{t}}\, \mathbf{H}f(\mathbf{a})\, \mathbf{v} \in \mathbb{R}$.

$\mathbf{v}^{\mathrm{t}}\ \mathbf{H}f(\mathbf{a})\ \mathbf{v}$

Let $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. We use the notation $\partial_j \partial_k f(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j\, \partial x_k}(\mathbf{a})$.

Then

$$\mathbf{v}^{\mathrm{t}}\ \mathbf{H}f(\mathbf{a})\ \mathbf{v} = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \partial_1\partial_1 f(\mathbf{a}) & \cdots & \partial_1\partial_n f(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \partial_n\partial_1 f(\mathbf{a}) & \cdots & \partial_n\partial_n f(\mathbf{a}) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= \sum_{j,k=0}^{n} \partial_j \partial_k f(\mathbf{a}) v_j v_k.$$

Hessian matrix (cont.)

Example

Let $f(x,y) = x^2 - y^2$. The gradient is

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2x \\ -2y \end{pmatrix}.$$

The Hessian is

$$\mathbf{H}f(x,y) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y) & \dfrac{\partial^2 f}{\partial x\,\partial y}(x,y) \\ \dfrac{\partial^2 f}{\partial y\,\partial x}(x,y) & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The point $(0,0)$ is a stationary point since $\nabla f(0,0) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$.

Second order Taylor formula for scalar fields

Recall first order Taylor approximation: If f is differentiable at $\mathbf{a}$ then $f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v} + \|\mathbf{v}\| \, E(\mathbf{a},\mathbf{v})$. If $\mathbf{a}$ is a stationary point then this only tells us that $f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a}) = \|\mathbf{v}\| \, E(\mathbf{a},\mathbf{v})$ but we want better information.

Theorem (second order Taylor)

Let f be a scalar field twice differentiable on $B(\mathbf{a}, r)$. Then, if $\|\mathbf{v}\| \leq r$,

$$f(\mathbf{a}+\mathbf{v}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{v} + \frac{1}{2}\mathbf{v}^{\mathbf{t}} \; \mathbf{H}f(\mathbf{a}) \; \mathbf{v} + \|\mathbf{v}\|^2 \, E_2(\mathbf{a},\mathbf{v})$$

and $E_2(\mathbf{a},\mathbf{v}) \to 0$ as $\|\mathbf{v}\| \to 0$.

Partial differential equations
First order linear PDEs
1D wave equation
Extrema
Stationary points
Second order Taylor formula and Hessian matrix
Classifying stationary points
Extreme value theorem for continuous scalar fields
Extrema with constraints
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Proof of second order Taylor formula.

1. Let $g(u) = f(\mathbf{a} + u\mathbf{v})$;
2. Taylor's expansion $g(1) = g(0) + g'(0) + \frac{1}{2}g''(c)$ for some $c \in (0,1)$;
3. Since $g(u) = f(a_1 + uv_1, \ldots, a_n + uv_n)$, by the chain rule,
$$g'(u) = \sum_{j=1}^{n} \partial_j f(a_1 + uv_1, \ldots, a_n + uv_n) v_j = \nabla f(\mathbf{a} + u\mathbf{v}) \cdot \mathbf{v};$$
4. Similarly
$$g''(u) = \sum_{j,k=1}^{n} \partial_j \partial_k f(a_1 + uv_1, \ldots, a_n + uv_n) v_j v_k = \mathbf{v}^{\mathbf{t}}\ \mathbf{H} f(\mathbf{a} + u\mathbf{v})\ \mathbf{v};$$
5. Consequently $f(\mathbf{a} + \mathbf{v}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{v} + \frac{1}{2}\mathbf{v}^{\mathbf{t}}\ \mathbf{H} f(\mathbf{a} + c\mathbf{v})\ \mathbf{v}$;
6. We define $E_2(\mathbf{a}, \mathbf{v}) = \frac{1}{2}\frac{1}{\|\mathbf{v}\|^2}\mathbf{v}^{\mathbf{t}}(\mathbf{H} f(\mathbf{a} + c\mathbf{v}) - \mathbf{H} f(\mathbf{a}))\mathbf{v}$.
7. $|E_2(\mathbf{a}, \mathbf{v})| \leq \sum_{j,k=0}^{n} \frac{v_j v_k}{\|\mathbf{v}\|^2} \left(\partial_j \partial_k f(\mathbf{a} + c\mathbf{v}) - \partial_j \partial_k f(\mathbf{a})\right).$ $\square$

Classifying stationary points

Theorem

Let A be a real symmetric matrix and let $Q(\mathbf{v}) = \mathbf{v}^{\mathbf{t}} A\mathbf{v}$. Then

- $Q(\mathbf{v}) > 0$ *for all* $\mathbf{v} \neq \mathbf{0}$ *if and only if all eigenvalues of* A *are positive;*
- $Q(\mathbf{v}) < 0$ *for all* $\mathbf{v} \neq \mathbf{0}$ *if and only if all eigenvalues of* A *are negative.*

Proof.

1. A can be diagonalised by matrix B which is orthogonal ($B^{\mathbf{t}} = B^{-1}$)

$$D = B^{\mathbf{t}} AB = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix};$$

2. $Q(\mathbf{v}) = \mathbf{v}^{\mathbf{t}} B^{\mathbf{t}} BAB^{\mathbf{t}} B\mathbf{v} = \mathbf{w}^{\mathbf{t}} D\mathbf{w} = \sum_j \lambda_j w_j^2$ where $\mathbf{w} = B\mathbf{v}$;
3. If all $\lambda_j > 0$ then $\sum_j \lambda_j w_j^2 > 0$;
4. $Q(B\mathbf{u}_k) = \lambda_k$ so, if $Q(\mathbf{v}) > 0$ for all $\mathbf{v} \neq \mathbf{0}$ then $\lambda_k > 0$ for all k. $\square$

Partial differential equations
First order linear PDEs
1D wave equation
Extrema
Stationary points
Second order Taylor formula and Hessian matrix
Classifying stationary points
Extreme value theorem for continuous scalar fields
Extrema with constraints
Lagrange multipliers

Theorem (classification of stationary points)

Let f be a scalar field twice differentiable on $B(\mathbf{a}, r)$. Suppose $\nabla f(\mathbf{a}) = \mathbf{0}$. Then

- *All eigenvalues of $\mathbf{H}f(\mathbf{a})$ are positive then f has a relative minimum at $\mathbf{a}$;*
- *All eigenvalues of $\mathbf{H}f(\mathbf{a})$ are negative then f has a relative maximum at $\mathbf{a}$;*
- *Some eigenvalues positive and some negative then $\mathbf{a}$ is a saddle point.*

Proof.

1. Let $Q(\mathbf{v}) = \mathbf{v}^{\mathbf{t}}\mathbf{H}f(\mathbf{a})\mathbf{v}$, $\mathbf{w} = B\mathbf{v}$ and let $\Lambda := \min_j \lambda_j$;
2. Observe that $\|\mathbf{w}\| = \|\mathbf{v}\|$ and that $Q(\mathbf{v}) = \sum_j \lambda_j w_j^2 \geq \Lambda \sum_j w_j^2 = \Lambda \|\mathbf{v}\|^2$;
3. 2nd-order Taylor
$$f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a}) = \frac{1}{2}\mathbf{v}^{\mathbf{t}}\ \mathbf{H}f(\mathbf{a})\ \mathbf{v} + \|\mathbf{v}\|^2 E_2(\mathbf{a}, \mathbf{v}) \geq \left(\tfrac{\Lambda}{2} - E_2(\mathbf{a}, \mathbf{v})\right) \|\mathbf{v}\|^2\,;$$
4. Since $E_2(\mathbf{a}, \mathbf{v}) \to 0$ as $\|\mathbf{v}\| \to 0$, $|E_2(\mathbf{a}, \mathbf{v})| < \frac{\Lambda}{2}$ when $\|\mathbf{v}\|$ is small.

Analogous argument for the second part. For final part consider $\mathbf{v}_j$ which is eigenvector for λ_j and apply the argument of first or second part. $\square$

Extreme value theorem for continuous scalar fields

Partial differential equations
First order linear PDEs
1D wave equation
Extrema
Stationary points
Second order Taylor formula and Hessian matrix
Classifying stationary points
Extreme value theorem for continuous scalar fields
Extrema with constraints
Lagrange multipliers

The argument will be in two parts:

1. Continuity implies boundedness;
2. Boundedness implies that the maximum and minimum are attained.

Notation: Intervals / rectangles / etc. . . If $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ then we consider the n-dimensional closed cartesian product

$$[\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n].$$

We call this set a *rectangle*.

Theorem (Bolzano–Weierstrass)

If $\{\mathbf{x}_n\}_n$ is a sequence in $[\mathbf{a}, \mathbf{b}]$ there exists a convergent subsequence $\{\mathbf{x}_{n_j}\}_j$.

Proof.

1. Divide $[\mathbf{a}, \mathbf{b}]$ into sub-rectangles of size half the original;
2. Choose a sub-rectangle which contains infinite elements of the sequence and choose the first of these elements to be part of the sub-sequence;
3. Now divide again this sub-rectangle by half and repeat to give the subsequence.

[Insert: illustration of proof]

$\square$

Boundedness

Partial differential equations
First order linear PDEs
1D wave equation
Extrema
Stationary points
Second order Taylor formula and Hessian matrix
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Extreme value theorem for continuous scalar fields
Extrema with constraints
Lagrange multipliers

Theorem (boundedness of continuous scalar fields)

Suppose that f is a scalar field continuous at every point in the closed rectangle $[\mathbf{a}, \mathbf{b}]$. Then f is bounded on $[\mathbf{a}, \mathbf{b}]$ in the sense that there exists $C > 0$ such that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$.

Proof.

1. Suppose the contrary: for all $n \in \mathbb{N}$ there exists $\mathbf{x}_n \in [\mathbf{a}, \mathbf{b}]$ such that $|f(\mathbf{x}_n)| > n$;
2. Bolzano–Weierstrass theorem means that there exists a subsequence $\{\mathbf{x}_{n_j}\}_j$ converges to $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$;
3. Continuity of f means that $f(\mathbf{x}_{n_j})$ converges to $f(\mathbf{x})$. This is a contradiction.

$\square$

Attaining extreme values

Theorem (extreme value theorem)

Suppose that f is a scalar field continuous at every point in the closed rectangle $[\mathbf{a},\mathbf{b}]$. There there exist points $\mathbf{x},\mathbf{y}\in[\mathbf{a},\mathbf{b}]$ such that

$$f(\mathbf{x}) = \inf f \quad \textit{and} \quad f(\mathbf{y}) = \sup f.$$

Proof.

- By the boundedness theorem $\sup f$ is finite and so there exists a sequence $\{\mathbf{x}_n\}_n$ such that $f(\mathbf{x}_n)$ converges to $\sup f$;
- Bolzano–Weierstrass theorem implies that there exists a subsequence $\{\mathbf{x}_{n_j}\}_j$ which converges to $\mathbf{x}\in[\mathbf{a},\mathbf{b}]$;
- By continuity $f(\mathbf{x}_n)\to f(\mathbf{x}) = \sup f$.

$\square$

The geometric idea of Lagrange multipliers

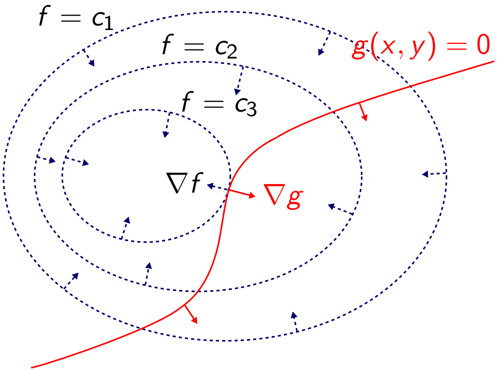

Figure: Extrema of f under constraint g

Problem: Minimise (or maximise) $f(x, y)$ under the constraint $g(x, y) = 0$.

- At "touching point" the gradient vectors are parallel;
- I.e., $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$.

Lagrange multipliers

Method of Lagrange's multipliers

If a scalar field $f(x_1, \ldots, x_n)$ has a relative extremum when it is subject to m constraints

$$g_1(x_1, \ldots, x_n) = 0, \ldots, g_m(x_1, \ldots, x_n) = 0,$$

where $m < n$, then there exist m scalars $\lambda_1, \ldots, \lambda_m$ such that

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_m \nabla g_m$$

at the extremum point.

MA2 – Part 3 – Applications of the differential calculus

Partial differential equations

First order linear PDEs

1D wave equation

Extrema

Stationary points

Second order Taylor formula and Hessian matrix

Classifying stationary points

Extreme value theorem for continuous scalar fields

Extrema with constraints

Lagrange multipliers

Lagrange multipliers (cont.)

Example

Find the extrema of $f(x, y) = xy$ subject to the constraint $g(x, y) = x + y - 1 = 0$.

- $\nabla f(x, y) = \left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right)$ and $\nabla g(x, y) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$;
- According to the Lagrange multiplier method there is $\lambda \in \mathbb{R}$ such that $\nabla f(x, y) = \lambda \nabla g(x, y)$ at the extremum point (x, y);
- We must solve the simultaneous equations

$$\left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right) = \lambda \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \quad g(x, y) = 0;$$

- I.e., $x = \lambda, \quad y = \lambda, \quad x + y = 1$;
- This has the solution $(x, y) = (\frac{1}{2}, \frac{1}{2})$, $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.

Lagrange multipliers (cont.)

Example

Find the points closest and furthest from the origin on the curve defined by the intersection of the two surfaces

$$x^2 - xy + y^2 - z^2 = 1 \quad \text{and} \quad x^2 + y^2 = 1.$$

- Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$;
- Let $g_1(x, y, z) = x^2 - xy + y^2 - z^2 - 1$, $g_2(x, y, z) = x^2 + y^2 - 1$;
- Calculate ∇f, ∇g_1 and ∇g_2;
- Solve the system of 5 equations (and 5 unknowns):

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z),$$

$$g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0;$$

- Check which are closest to and which are furthest from the origin.