MA2 - Scalar and vector fields

Weeks 3–4 of MA2 – Draft lecture slides

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Outline

Higher dimensional space

Notation $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ where $x_1 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}$.

Definition (Inner product)

$\mathbf{x} \cdot \mathbf{y} := \sum_{k=1}^{n} x_k y_k \in \mathbb{R}$

Definition (Norm)

$\|\mathbf{x}\| := \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{k=1}^{n} x_k^2}$ (E.g., in $\mathbb{R}^2$ then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$)

Recall

$|x \cdot y| \leq \|\mathbf{x}\| \ \|\mathbf{y}\|$ Cauchy-Schwarz inequality,

$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ Triangle inequality.

Scalar field

$f : \mathbb{R}^n \to \mathbb{R}$

Vector fields

$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$

- Temperature in a region,
- Wind velocity,
- Fluid flow,
- Electric field, ...

Open balls and open sets

Let $\mathbf{a} \in \mathbb{R}^n$, $r > 0$. The open n-ball of radius r and centre $\mathbf{a}$ is

$$B(\mathbf{a}, r) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < r\} .$$

Definition (Interior point)

Let $S \subset \mathbb{R}^n$. A point $\mathbf{a} \in S$ is said to be an *interior point* if there is $r > 0$ such that $B(\mathbf{a}, r) \subset S$. The set of all interior points of S is denoted $\operatorname{int} S$.

Definition (Open set)

A set $S \subset \mathbb{R}^n$ is said to be *open* if all of its points are interior points, i.e., if $\operatorname{int} S = S$.

Examples: Open intervals, disks, balls, union of open intervals, etc., are all open sets.

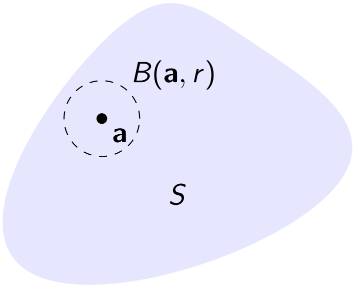

Figure: Interior points are the centre of a ball contained within the set.

Cartesian product

If $A_1, A_2 \subset \mathbb{R}$ then the Cartesian product is defined as

$$A_1 \times A_2 := \{(x_1, x_2) : x_1 \in A_1, x_2 \in A_2\} \subset \mathbb{R}^2.$$

Fact: If A_1, A_2 are open subsets of $\mathbb{R}$ then $A_1 \times A_2$ is an open subset of $\mathbb{R}^2$.

- Let $\mathbf{a} = (a_1, a_2) \in A_1 \times A_2 \subset \mathbb{R}^2$
- A_1 is open $\implies$ exists $r_1 > 0$ such that $B(a_1, r_1) \subset A_1$
- Similarly for A_2
- Let $r = \min\{r_1, r_2\}$
- $B(\mathbf{a}, r) \subset B(a_1, r_1) \times B(a_2, r_2) \subset A_1 \times A_2$

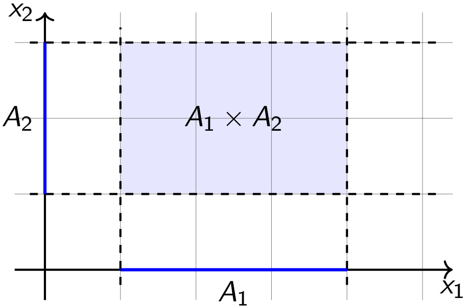

Figure: If A_1, A_2 are intervals then $A_1 \times A_2$ is a rectangle.

Exterior points and boundary

Definition (Exterior points)

Let $S \subset \mathbb{R}^n$. A point $\mathbf{a} \notin S$ is said to be an *exterior point* if there exists $r > 0$ such that $B(\mathbf{a}, r) \cap S = \emptyset$. The set of all exterior points of S is denoted $\operatorname{ext} S$.

- Observe $\operatorname{ext} S$ is an open set.
- Let $S^c := \mathbb{R}^n \setminus S$.

Definition (Boundary)

The set $\mathbb{R}^n \setminus (\operatorname{int} S \cup \operatorname{ext} S)$ is called the boundary of $S \subset \mathbb{R}^n$ and is denoted ∂S.

Definition (Closed)

A set $S \subset \mathbb{R}^n$ is said to be *closed* if $\partial S \subset S$.

Fact: S is open $\iff$ S^c is closed.

- $\mathbb{R}^n = \operatorname{int} S \cup \partial S \cup \operatorname{ext} S$ (disjointly);
- If $\mathbf{x} \in \partial S$ then, for every $r > 0$, $B(\mathbf{x}, r) \cap S \neq \emptyset$ and so $\mathbf{x} \in \partial(S^c)$;
- Similarly with S and S^c swapped and so $\partial S = \partial(S^c)$;
- If S is open then $\operatorname{int} S = S$ and $S^c = \operatorname{ext} S \cup \partial S = \operatorname{ext} S \cup \partial(S^c)$ and so S^c is closed;
- If S is not open then there exists $\mathbf{a} \in \partial S \cap S$. Additionally $\mathbf{a} \in \partial(S^c) \cap S$ hence S^c is not closed.

Limits and continuity

Notation: Let $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$. If $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ we write $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ to mean that $\lim_{\|\mathbf{x}-\mathbf{a}\|\to 0} \|\mathbf{f}(\mathbf{x}) - \mathbf{b}\| = 0$.

Definition (Continuous)

A function $\mathbf{f}$ is said to be *continuous* at $\mathbf{a}$ if $\mathbf{f}$ is defined at $\mathbf{a}$ and $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$.
We say $\mathbf{f}$ is continuous on S if $\mathbf{f}$ is continuous at each point of S.

Theorem

Suppose that $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ *and* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{c}$. *Then*

(a) $\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})) = \mathbf{b} + \mathbf{c}$,

(b) $\lim_{\mathbf{x}\to\mathbf{a}} \lambda\mathbf{f}(\mathbf{x}) = \lambda\mathbf{b}$ *for every* $\lambda \in \mathbb{R}$,

(c) $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \mathbf{b} \cdot \mathbf{c}$,

(d) $\lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{f}(\mathbf{x})\| = \|\mathbf{b}\|$.

Proof of (c)

1. $\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{b} \cdot \mathbf{c} = (\mathbf{f}(\mathbf{x}) - \mathbf{b}) \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) + \mathbf{b} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) + \mathbf{c} \cdot (\mathbf{f}(\mathbf{x}) - \mathbf{b})$
2. By the triangle inequality and Cauchy-Schwarz,

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{b} \cdot \mathbf{c}\| \leq{} & \|\mathbf{f}(\mathbf{x}) - \mathbf{b}\| \, \|\mathbf{g}(\mathbf{x}) - \mathbf{c}\| \\ & + \|\mathbf{b}\| \, \|\mathbf{g}(\mathbf{x}) - \mathbf{c}\| \\ & + \|\mathbf{c}\| \, \|\mathbf{f}(\mathbf{x}) - \mathbf{b}\|\end{aligned}$$

3. Since $\|\mathbf{f}(\mathbf{x}) - \mathbf{b}\| \to 0$ and $\|\mathbf{g}(\mathbf{x}) - \mathbf{c}\| \to 0$ as $\mathbf{x} \to \mathbf{a}$ this implies that $\|\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{b} \cdot \mathbf{c}\| \to 0$.

Proof of (d)

1. Take $\mathbf{f} = \mathbf{g}$ in part (c) implies that $\lim_{\mathbf{x} \to \mathbf{a}} \|\mathbf{f}(\mathbf{x})\|^2 = \|\mathbf{b}\|^2$.

Components of a vector field

Theorem

Let $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$. Then $\mathbf{f}$ is continuous if and only if each f_k is continuous.

Proof.

($\Rightarrow$) Let $\mathbf{e}_k = (0, \dots, 0, \overset{k^{\text{th}}\text{position}}{1}, 0, \dots, 0)$ and observe that $f_k(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{e}_k$. We have already shown that the continuity of two vector fields implies the continuity of the inner product.

($\Leftarrow$) By definition of the norm $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\|^2 = \sum_{k=1}^{m} (f_k(\mathbf{x}) - f_k(\mathbf{a}))^2$ and we know $\|f_k(\mathbf{x}) - f_k(\mathbf{a})\| \to 0$ as $\|\mathbf{x} - \mathbf{a}\| \to 0$. $\square$

Example (Polynomials)

A *polynomial* in n variables is a scalar field on $\mathbb{R}^n$ of the from

$$p(x_1, \dots, x_n) = \sum_{k_1=0}^{j} \cdots \sum_{k_n=0}^{j} c_{k_1,\dots,k_n} x_1^{k_1} \cdots x_n^{k_n}.$$

E.g., $f(x_1, x_2) := x_1 + 2x_1x_2 - x_1^2$ is a polynomial in 2 variables. Polynomials are continuous everywhere in $\mathbb{R}^n$. (Finite sum of products of continuous fields.)

Example (Rational functions)

A *rational function* is a scalar field

$$f(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})}$$

where $p(\mathbf{x})$ and $q(\mathbf{x})$ are polynomials. A rational function is continuous at every point $\mathbf{x}$ such that $q(\mathbf{x}) \neq 0$.

Composition of functions

Theorem

Suppose $S \subset \mathbb{R}^l$, $T \subset \mathbb{R}^m$, $\mathbf{f} : S \to \mathbb{R}^m$, $\mathbf{g} : T \to \mathbb{R}^n$ *and that* $\mathbf{f}(S) \subset T$ *so that*

$$(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$$

makes sense. If $\mathbf{f}$ *is continuous at* $\mathbf{a} \in S$ *and* $\mathbf{g}$ *is continuous at* $\mathbf{f}(\mathbf{a})$ *then* $\mathbf{g} \circ \mathbf{f}$ *is continuous at* $\mathbf{a}$.

Proof.

- $\lim_{\mathbf{x} \to \mathbf{a}} \|\mathbf{f}(\mathbf{g}(\mathbf{x})) - \mathbf{f}(\mathbf{g}(\mathbf{a}))\| = \lim_{\mathbf{y} \to \mathbf{g}(\mathbf{a})} \|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{g}(\mathbf{a}))\| = 0$

$\square$

Example

$f(x_1, x_2) = \sin(x_1^2 + x_2) + x_1 x_2$

Example (Continuity problem in higher dimensions)

Let $f(x_1, x_2)$ be defined as $f(0,0) = 0$ and, for all $(x_1, x_2) \neq (0,0)$,

$$f(x_1, x_2) = \frac{x_1 x_2}{x_1^2 + x_2^2}.$$

What is the behaviour of the function along the following three lines?

1. $x_1 = 0$,
2. $x_2 = 0$,
3. $x_1 = x_2$.

Directional derivatives

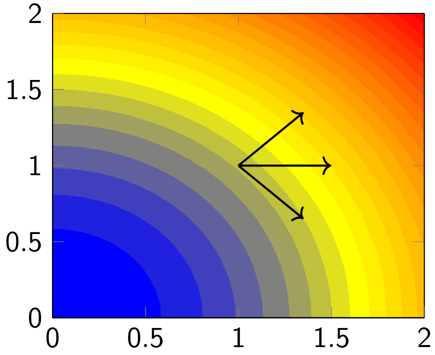

Figure: Plot where colour represents the value of $f(x_1, x_2) = x_1^2 + x_2^2$. The change in f depends on direction.

Definition (Directional derivative)

Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}$. For any $\mathbf{a} \in \operatorname{int} S$ and $\mathbf{v} \in \mathbb{R}^n$ the derivative of f with respect to $\mathbf{v}$ is defined as

$$D_{\mathbf{v}} f(\mathbf{a}) := \lim_{h \to 0} \frac{1}{h} \left(f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a}) \right).$$

- When h is small $\mathbf{a} + h\mathbf{v} \in S$ because $\mathbf{a} \in \operatorname{int} S$ so this definition makes sense.
- Many different notations in use.

Theorem

Suppose $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$, $\mathbf{a} \in \operatorname{int} S$. Let $g(t) := f(\mathbf{a} + t\mathbf{v})$. If one of the derivatives $g'(t)$ or $D_{\mathbf{v}} f(\mathbf{a})$ exists then the other also exists and

$$g'(t) = D_{\mathbf{v}} f(\mathbf{a} + t\mathbf{v}).$$

In particular $g'(0) = D_{\mathbf{v}} f(\mathbf{a})$.

Proof.

By definition $\frac{g(t+h)-g(h)}{h} = \frac{f(\mathbf{a}+h\mathbf{v})-f(\mathbf{a})}{h}$. $\square$

Theorem (Mean value)

Assume that $D_{\mathbf{v}}(\mathbf{a} + t\mathbf{v})$ exists for each $t \in [0, 1]$. Then for some $\theta \in (0, 1)$,

$$f(\mathbf{a} + \mathbf{v}) - f(\mathbf{a}) = D_{\mathbf{v}} f(\mathbf{z}), \quad \textit{where } z = \mathbf{a} + \theta\mathbf{v}.$$

Proof.

Apply mean value theorem to $g(t) = f(\mathbf{a} + t\mathbf{v})$. $\square$

Partial derivatives

For any $k \in \{1, 2, \ldots, n\}$, let $\mathbf{e}_k = (0, \ldots, 0, \overset{k^{\text{th}}\text{position}}{1}, 0, \ldots, 0)$.

Definition (Partial derivatives)

We define the *partial derivative* in x_k of $f(x_1, x_2, \ldots, x_n)$ at $\mathbf{a}$ as

$$\frac{\partial f}{\partial x_k}(\mathbf{a}) := D_{\mathbf{e}_k} f(\mathbf{a}).$$

Notation

Various symbols used for partial derivatives: $\frac{\partial f}{\partial x_k}(\mathbf{a}) = D_k f(\mathbf{a}) = \partial_k f(\mathbf{a})$. If a function is written $f(x, y)$ we write $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for the partial derivatives. Similarly for higher dimension.

In practice: To compute the partial derivative $\frac{\partial f}{\partial x_k}$, one should consider all other x_j for $j \neq k$ as constants and take the derivative with respect to x_k.

Total derivatives

Recall that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then

$$f(a+h) = f(a) + hf'(a) + hE(a,h)$$

where $E(a,h) \to 0$ as $h \to 0$. I.e., $f(a) + hf'(a)$ approximates $f(x)$ close to a.

Definition (Differentiable)

Let $S \subset \mathbb{R}^n$ be open, $f : S \to \mathbb{R}$. We say that f is *differentiable* at $\mathbf{a} \in S$ if there is $T_{\mathbf{a}} \in \mathbb{R}^n$ and $E(\mathbf{a}, \mathbf{v})$ such that, for $\mathbf{v} \in B(\mathbf{a}, r)$,

$$f(\mathbf{a}+\mathbf{v}) = f(\mathbf{a}) + T_{\mathbf{a}} \cdot \mathbf{v} + \|\mathbf{v}\| \, E(\mathbf{a}, \mathbf{v})$$

and $E(\mathbf{a}, \mathbf{v}) \to 0$ as $\mathbf{v} \to 0$.

Theorem

If f is differentiable at $\mathbf{a}$ then $T_{\mathbf{a}} = (\partial_1 f(\mathbf{a}), \ldots, \partial_n f(\mathbf{a}))$ and $D_{\mathbf{v}} f(\mathbf{a}) = T_{\mathbf{a}} \cdot \mathbf{v}$.

Total derivatives (cont.)

Proof.

Since f is differentiable $f(\mathbf{a}+h\mathbf{v}) = f(\mathbf{a}) + hT_{\mathbf{a}}\cdot\mathbf{v} + h\|\mathbf{v}\|\,E(\mathbf{a},h\mathbf{v})$ and hence

$$D_{\mathbf{v}}f(\mathbf{a}) := \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{v})-f(\mathbf{a})}{h} = \lim_{h\to 0}\frac{hT_{\mathbf{a}}\cdot\mathbf{v}+h\|\mathbf{v}\|\,E(\mathbf{a},h\mathbf{v})}{h} = T_{\mathbf{a}}\cdot\mathbf{v}.$$

In particular $T_{\mathbf{a}}\cdot\mathbf{e}_k = D_{\mathbf{e}_k}f(\mathbf{a})$. □

Definition (Gradient)

The *gradient* of f is the vector-valued function

$$\nabla f(\mathbf{a}) := \begin{pmatrix}\partial_1 f(\mathbf{a})\\ \partial_2 f(\mathbf{a})\\ \vdots\\ \partial_n f(\mathbf{a})\end{pmatrix}.$$

Theorem

If f is differentiable at $\mathbf{a}$, then it is continuous at $\mathbf{a}$.

Proof.

$$\begin{aligned}|f(\mathbf{a}+\mathbf{v})-f(\mathbf{a})| &= |T_{\mathbf{a}}\cdot\mathbf{v}+\|\mathbf{v}\|\,E(\mathbf{a},\mathbf{v})|\\ &\le \|T_{\mathbf{a}}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|\,|E(\mathbf{a},\mathbf{v})| \to 0.\end{aligned}$$ □

Theorem

Suppose that the partial derivatives $\partial_1 f(\mathbf{x}), \partial_2 f(\mathbf{x}), \ldots, \partial_n f(\mathbf{x})$ exist for all $\mathbf{x} \in B(\mathbf{a}, r)$ and are continuous at $\mathbf{a}$. Then f is differentiable at $\mathbf{a}$.

Proof.

1. Write $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{u}_k = (v_1, v_2, \ldots, v_k, 0, \ldots, 0)$;
2. Observe that $\mathbf{u}_k - \mathbf{u}_{k-1} = v_k \mathbf{e}_k$, $\mathbf{u}_0 = (0, 0, \ldots, 0)$ and $\mathbf{u}_n = \mathbf{v}$;
3. Using the mean value theorem (exists $\mathbf{z}_k = \mathbf{u}_{k-1} + \theta_k \mathbf{e}_k$)

$$\begin{aligned} f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a}) &= \sum_{k=1}^{n} f(\mathbf{a}+\mathbf{u}_k) - f(\mathbf{a}+\mathbf{u}_{k-1}) = \sum_{k=1}^{n} v_k D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{z}_k) \\ &= \sum_{k=1}^{n} v_k D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{u}_{k-1}) \\ &\qquad + \sum_{k=1}^{n} v_k \left(D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{z}_k) - D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{u}_{k-1})\right) \end{aligned}$$

4. $\sum_{k=1}^{n} v_k D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{u}_{k-1}) \to \mathbf{v} \cdot \nabla f(\mathbf{a})$. $\square$

Chain rule

If $f(t) = g \circ h(t)$ then $f'(t) = g'(h(t))\ h'(t)$. Does this extend to higher dimension?

Example

Suppose that

- $\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$ describes the position $\mathbf{x}(t)$ at time t,
- $f : \mathbb{R}^3 \to \mathbb{R}$ describes the temperature $f(\mathbf{x})$ at a point $\mathbf{x}$

The temperature at time t is equal to $g(t) = f(\mathbf{x}(t))$. We want to calculate $g'(t)$.

Derivative of $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^n$

Let $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^n$ and suppose it has the form $\boldsymbol{\alpha}(t) = (\alpha_1(t), \ldots, \alpha_n(t))$. We define the derivative as

$$\boldsymbol{\alpha}'(t) := \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.$$

- $\boldsymbol{\alpha}'$ is a vector-valued function,
- It represents the "direction of movement".

Note: We sometimes consider horizontal and sometimes vertical vectors. It can be convenient to distinguish between "position" and "direction".

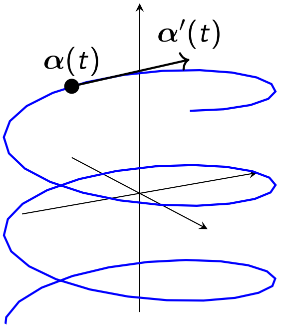

Figure: $\boldsymbol{\alpha}(t) := (\cos t, \sin t, t),\ t \in \mathbb{R}$.

Chain rule (cont.)

Theorem

Let $S \subset \mathbb{R}^n$ be open and $I \subset \mathbb{R}$ an interval. Let $\mathbf{x} : I \to S$ and $f : S \to \mathbb{R}$ and define, for $t \in I$,

$$g(t) = f(\mathbf{x}(t)).$$

Suppose that $t \in I$ is such that $\mathbf{x}'(t)$ exists and f is differentiable at $\mathbf{x}(t)$. Then $g'(t)$ exists and

$$g'(t) = \nabla f\,(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Proof.

Let $h > 0$ be small,

$$\begin{aligned}\tfrac{1}{h}\left[g(t+h) - g(t)\right] &= \tfrac{1}{h}\left[f(\mathbf{x}(t+h) - f(\mathbf{x}(t)))\right] \\ &= \tfrac{1}{h}\nabla f(\mathbf{x}(t)) \cdot (\mathbf{x}(t+h) - \mathbf{x}(t)) \\ &\quad + \tfrac{1}{h}\left\|\mathbf{x}(t+h) - \mathbf{x}(t)\right\| E(\mathbf{x}(t), \mathbf{x}(t+h) - \mathbf{x}(t)).\end{aligned}$$

Observe that $\frac{1}{h}(\mathbf{x}(t+h) - \mathbf{x}(t)) \to \mathbf{x}'(t)$ as $h \to 0$. $\square$

Chain rule example

- A particle moves in a circle and its position at time $t \in [0, 2\pi]$ is given by
$$\mathbf{x}(t) = (\cos t, \sin t).$$
- The temperature at a point $\mathbf{y} = (y_1, y_2)$ is given by the function $f(\mathbf{y}) := y_1 + y_2$,
- The temperature the particle experiences at time t is given by $g(t) = f(\mathbf{x}(t))$.

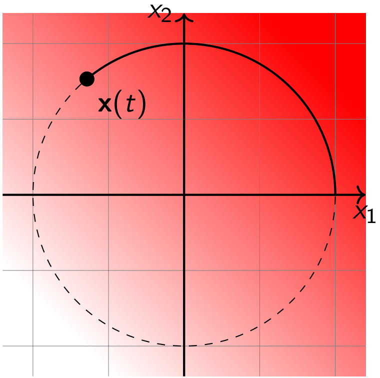

Figure: $\mathbf{x}(t)$ is the position of a particle. Shading represents temperature f.

Temperature change: $g'(t) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \cos t - \sin t.$

Level sets & tangent planes (2D)

Let $S \subset \mathbb{R}^2$, $f : S \to \mathbb{R}$. Suppose $c \in \mathbb{R}$ and

$$L(c) := \{\mathbf{x} \in S : f(\mathbf{x}) = c\}$$

is a curve at $\mathbf{a} \in S$ in the sense that $\mathbf{x} : I \to S$ is the parametric form of the curve. I.e., $\mathbf{x}(t_a) = \mathbf{a}$ for some $t_a \in I$ and

$$f(\mathbf{x}(t)) = c$$

for all $t \in I$. Then

- $\nabla f(\mathbf{a})$ is normal to the curve at $\mathbf{a}$
- Tangent line at $\mathbf{a}$ is $\{\mathbf{x} \in \mathbb{R}^2 : \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0\}$

This is because the chain rule implies that $\nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = 0$.

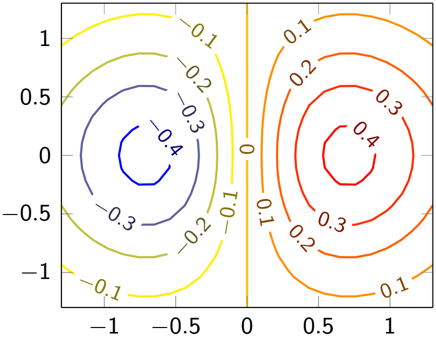

Figure: Contour plot of $x_1 \exp(-x_1^2 - x_2^2)$

- Isotherms $\leftrightarrow$ temperature;
- Contours $\leftrightarrow$ altitude.

Terminology: $L(c)$ is called the *level set*.

Level set examples

Example

Let $f(x_1, x_2, x_3) := x_1^2 + x_2^2 + x_3^2$.

- If $c > 0$ then $L(c)$ is a sphere,
- $L(0)$ is a single point $(0, 0, 0)$,
- If $c < 0$ then $L(c)$ is empty.

Example

Let $f(x_1, x_2, x_3) := x_1^2 + x_2^2 - x_3^2$.

- If $c > 0$ then $L(c)$ is a one-sheeted hyperboloid,
- $L(0)$ is an infinite cone,
- If $c < 0$ then $L(c)$ a two-sheeted hyperboloid.

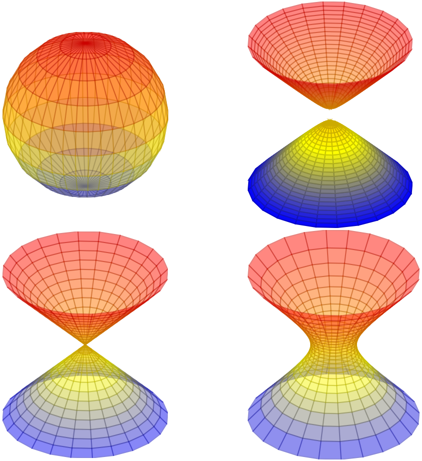

Figure: Sphere, two-sheeted hyperboloid, infinite cone, one-sheeted hyperboloid.

Level set & tangent planes (3D)

Let f be a differentiable scalar field on $S \subset \mathbb{R}^3$ and suppose that $L(c) = \{\mathbf{x} \in S : f(\mathbf{x}) = c\}$ is a surface.

- $\nabla f(\mathbf{a})$ is normal to every curve $\alpha(t)$ in the surface which passes through $\mathbf{a}$,
- Tangent plane at $\mathbf{a}$ is $\{\mathbf{x} \in \mathbb{R}^3 : \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0\}$.

Same argument as in $\mathbb{R}^2$ works in $\mathbb{R}^n$.

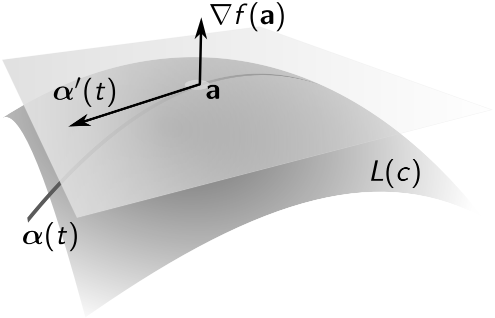

Figure: Tangent plane and normal vector

Derivatives of vector fields

Definition (Directional derivative)

Let $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$. For any $\mathbf{a} \in \operatorname{int} S$ and $\mathbf{v} \in \mathbb{R}^n$ the derivative of f with respect to $\mathbf{v}$ is defined as

$$D_{\mathbf{v}}\mathbf{f}(\mathbf{a}) := \lim_{h \to 0} \frac{\mathbf{f}(\mathbf{a} + h\mathbf{v}) - \mathbf{f}(\mathbf{a})}{h}.$$

Note: If we write $\mathbf{f} = (f_1, \ldots, f_m)$ then $D_{\mathbf{v}}\mathbf{f} = (D_{\mathbf{v}}f_1, \ldots, D_{\mathbf{v}}f_m)$.

Definition (Differentiable)

We say that f is *differentiable* at $\mathbf{a}$ if there is a linear transformation $\mathbf{T}_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{E}(\mathbf{a}, \mathbf{v})$ such that, for $\mathbf{v} \in B(\mathbf{a}, r)$,

$$\mathbf{f}(\mathbf{a} + \mathbf{v}) = \mathbf{f}(\mathbf{a}) + \mathbf{T}_{\mathbf{a}}(\mathbf{v}) + \|\mathbf{v}\| \, \mathbf{E}(\mathbf{a}, \mathbf{v})$$

and $\mathbf{E}(\mathbf{a}, \mathbf{v}) \to 0$ as $\mathbf{v} \to 0$.

Derivatives of vector fields (cont.)

Theorem

If $\mathbf{f}$ *is differentiable at* $\mathbf{a}$ *then* $\mathbf{f}$ *is continuous at* $\mathbf{a}$ *and* $\mathbf{T}_{\mathbf{a}}(\mathbf{v}) = D_{\mathbf{v}}\mathbf{f}(\mathbf{a})$*.*

Proof.

Same as for the case when $f : \mathbb{R}^n \to \mathbb{R}$. $\square$

Definition (Jacobian matrix)

The *Jacobian matrix* of $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ at $\mathbf{a}$ is defined as

$$D\mathbf{f}(\mathbf{a}) = \begin{pmatrix} \partial_1 f_1(\mathbf{a}) & \partial_2 f_1(\mathbf{a}) & \cdots & \partial_n f_1(\mathbf{a}) \\ \partial_1 f_2(\mathbf{a}) & \partial_2 f_2(\mathbf{a}) & \cdots & \partial_n f_2(\mathbf{a}) \\ \vdots & \vdots & & \vdots \\ \partial_1 f_m(\mathbf{a}) & \partial_2 f_m(\mathbf{a}) & \cdots & \partial_n f_m(\mathbf{a}) \end{pmatrix}$$

- Choosing a basis any linear transformation can be written as a $m \times n$ matrix.
- $\mathbf{T}_{\mathbf{a}}(\mathbf{v}) = D\mathbf{f}(\mathbf{a})\mathbf{v}$.

Derivatives of $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ (recap.)

Let $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$. If f is differentiable at $\mathbf{a} \in S$ then, for all $\mathbf{v} \in B(\mathbf{a}, r) \subset S$,

$$\mathbf{f}(\mathbf{a} + \mathbf{v}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})\mathbf{v} + \|\mathbf{v}\| \, \mathbf{E}(\mathbf{a}, \mathbf{v}).$$

This is like a Taylor expansion in higher dimensions.

Matrix form of the chain rule

Theorem

Let $S \subset \mathbb{R}^l$, $T \subset \mathbb{R}^m$ be open. Let $\mathbf{f} : S \to T$ and $\mathbf{g} : T \to \mathbb{R}^n$ and define

$$\mathbf{h} := \mathbf{g} \circ \mathbf{f} : S \to \mathbb{R}^n.$$

Let $\mathbf{a} \in S$. Suppose that $\mathbf{f}$ is differentiable at $\mathbf{a}$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{a})$. Then $\mathbf{h}$ is differentiable at $\mathbf{a}$ and

$$D\mathbf{h}(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a}))\ D\mathbf{f}(\mathbf{a}).$$

Proof.

Since $\mathbf{f}$ and $\mathbf{g}$ are differentiable there exists $\mathbf{E}_\mathbf{f}$ and $\mathbf{E}_\mathbf{g}$. Let $\mathbf{u} := \mathbf{f}(\mathbf{a}+\mathbf{v}) - \mathbf{f}(\mathbf{a})$.

$$\begin{aligned}
\mathbf{h}(\mathbf{a}+\mathbf{v}) - \mathbf{h}(\mathbf{a}) &= \mathbf{g}(\mathbf{f}(\mathbf{a}+\mathbf{v})) - \mathbf{f}(\mathbf{h}(\mathbf{a})) \\
&= D\mathbf{g}(\mathbf{f}(\mathbf{a}))(\mathbf{f}(\mathbf{a}+\mathbf{v}) - \mathbf{f}(\mathbf{a})) + \|\mathbf{u}\|\, \mathbf{E}_\mathbf{g}(\mathbf{f}(\mathbf{a}), \mathbf{u}) \\
&= D\mathbf{g}(\mathbf{f}(\mathbf{a}))D\mathbf{f}(\mathbf{a})\mathbf{v} + \|\mathbf{v}\|\, D\mathbf{g}(\mathbf{f}(\mathbf{a}))\mathbf{E}_\mathbf{f}(\mathbf{a}, \mathbf{v}) + \|\mathbf{u}\|\, \mathbf{E}_\mathbf{g}(\mathbf{f}(\mathbf{a}), \mathbf{u}).
\end{aligned}$$

$\square$

Polar coordinates (derivatives example)

- We can write the change of coordinates $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ as the function $\mathbf{f}(r,\theta) = (x(r,\theta), y(r,\theta))$ where $\mathbf{f} : (0,\infty) \times [0,2\pi) \to \mathbb{R}^2$.
- We calculate the Jacobian matrix

$$D\mathbf{f}(r,\theta) = \begin{pmatrix} \partial_r x(r,\theta) & \partial_\theta x(r,\theta) \\ \partial_r y(r,\theta) & \partial_\theta y(r,\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

- We wish to calculate derivatives of $h := g \circ \mathbf{f}$ for some $g : \mathbb{R}^2 \to \mathbb{R}$.

$$Dh(r,\theta) = Dg(\mathbf{f}(r,\theta))\ D\mathbf{f}(r,\theta)$$

$$\begin{pmatrix} \partial_r h(r,\theta) & \partial_\theta h(r,\theta) \end{pmatrix} = \begin{pmatrix} \partial_x g(\mathbf{f}(r,\theta)) & \partial_y g(\mathbf{f}(r,\theta)) \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

- Consequently

$$\begin{cases} \partial_r h(r,\theta) = \partial_x g(r\cos\theta, r\sin\theta)\cos\theta + \partial_y g(r\cos\theta, r\sin\theta)\sin\theta \\ \partial_\theta h(r,\theta) = -r\partial_x g(r\cos\theta, r\sin\theta)\sin\theta + r\partial_y g(r\cos\theta, r\sin\theta)\cos\theta \end{cases}.$$

Equality of partial derivatives

Does $\partial_1\partial_2 f = \partial_2\partial_1 f$, etc.?

Example (partial derivative problem)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(0,0) = 0$ and, for $(x_1, x_2) \neq (0,0)$,

$$f(x_1, x_2) := \frac{x_1 x_2 (x_1^2 - x_2^2)}{x_1^2 + x_2^2}.$$

We can calculate $\partial_2\partial_1 f(0,0) = -1$ but $\partial_1\partial_2 f(0,0) = 1$.

Theorem

Let $f : S \to \mathbb{R}$ be a scalar field such that the partial derivatives $\partial_1 f$, $\partial_2 f$ and $\partial_2\partial_1 f$ exist on an open set $S \subset \mathbb{R}^2$ containing $\mathbf{x}$. Further assume that $\partial_2\partial_1 f$ is continuous on S. Then the derivative $\partial_1\partial_2 f(\mathbf{x})$ exists and $\partial_1\partial_2 f(\mathbf{x}) = \partial_2\partial_1 f(\mathbf{x})$.

Implicit functions and partial derivatives

Implicit

- $x^2 - y = 0$
- $x^2 + y^2 - 1 = 0$
- $x^2 - y^2 - 1 = 0$
- $x^2 + y^2 - e^y - 4 = 0$

Explicit

- $f(x) = x^2$
- $f(x) = \pm\sqrt{1 - x^2}$, $|x| \leq 1$
- $f(x) = \pm\sqrt{x^2 - 1}$, $|x| \geq 1$
- ?

- We know $F : \mathbb{R}^2 \to \mathbb{R}$ and we suppose there exists some $f : \mathbb{R} \to \mathbb{R}$ such that $F(x, f(x)) = 0$ for all x.
- Let $g(x) := F(x, f(x))$ and note that $g'(x) = 0$.
- By the chain rule $g'(x)$ is equal to

$$\begin{pmatrix} \partial_1 F(x, f(x)) & \partial_2 F(x, f(x)) \end{pmatrix} \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$$

- Consequently

$$f'(x) = -\frac{\partial_1 F(x, f(x))}{\partial_2 F(x, f(x))}.$$

- A similar argument holds in higher dimension.