MA2 - Part 1 - Sequences and series of functions

Week 1 & 2 of MA2 – Draft lecture slides

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Outline

Sequences of functions

Here we consider sequences of functions $f_1(x), f_2(x), f_3(x), \ldots$ or $\{f_n(x)\}_{n\in\mathbb{N}}$.
For example

- $f_1(x) = x^2, f_2(x) = x^4, f_3(x) = x^6, \ldots$ or $f_n(x) = x^{2n}$
- $f_1(x) = e^x, f_2(x) = e^{2x}, f_3(x) = e^{3x}, \ldots$ or $f_n(x) = e^{nx}$
- $f_n(x) = n \exp\left(-\frac{n^2x^2}{2}\right)$

Primary reference

Apostol, "Calculus Volume 1" – Chapter 11

Definition

A sequence of numbers $a_1, a_2, a_3, \dots$ is said to be *convergent* to $a \in \mathbb{R}$ if, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for any $n \geq N$.

If a sequence $a_1, a_2, a_3, \dots$ is convergent to a then we write $a_n \to a$.
For a sequence of functions we will consider two different notions of convergence. First let us consider another example.

Example

Consider the sequence $f_n(x) = x^n$ for $x \in (0, 1)$. For each $x \in (0, 1)$ we see that $f_n(x) \to 0$. On the other hand, for each n, $2^{\frac{1}{n}} \in (0, 1)$ and $f_n(2^{\frac{1}{n}}) = \frac{1}{2}$.

Definition

Let $D \subset \mathbb{R}$, let $f_n(x)$ be a sequence of functions on D and let $f(x)$ be a function on D. If $f_n(x) \to f(x)$ for each $x \in D$ we say that f_n is *pointwise convergent* to f.

Definition

Let $f_n(x)$ be a sequence of functions on $D \subset \mathbb{R}$ and let $f(x)$ be a function on D. If, for each $\epsilon > 0$, there exists N such that for every $n \geq N$ and every $x \in D$, $|f_n(x) - f(x)| < \epsilon$ then we say that f_n is *uniformly convergent* to f.

Exercise

Show that the sequence $f_n(x) = x^n$ converges uniformly on $(0, \frac{1}{2})$.

Solution

- We expect that it converges to the constant function $f(x) = 0$.
- $|f_n(x) - f(x)| \leq \frac{1}{2^n}$ for all $x \in (0, \frac{1}{2})$.
- I.e., for every ϵ we can choose $N = -\log_2 \epsilon$.

Continuity

Definition

Let $f_n(x)$ be a sequence of functions on $D \subset \mathbb{R}$. We say that f is *continuous* at $p \in D$ if, for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ whenever $x \in D$, $|x - p| < \delta$. We say that f is *continuous on D* if f is continuous at every $p \in D$.

It is natural to consider a sequence of continuous functions which converge and ask if the function they converge to is continuous. What about the sequence of functions $f_n(x) = \arctan(nx)$?

Theorem

Suppose that $f_n \to f$ uniformly on D and that the f_n are continuous on D. Then f is continuous on D.

Proof.

Let $p \in D$.

- Uniform convergence means that, for each $\epsilon > 0$, there exists N such that for every $n \geq N$ and every $x \in D$, $|f_n(x) - f(x)| < \frac{\epsilon}{3}$.
- By continuity of $f_N(x)$ at $x = p$, there is a $\delta > 0$ such that $|f_N(x) - f_N(p)| < \frac{\epsilon}{3}$ whenever $x \in D$, $|x - p| < \delta$.
- Since

$$|f(x) - f(p)| = |f(x) - f_N(x) + f_N(x) - f_N(p) + f_N(p) - f(p)|$$

this means that, for all $|x - p| < \delta$,

$$|f(x) - f(p)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(p)| + |f_N(p) - f(p)| < 3\frac{\epsilon}{3} = \epsilon.$$

This proves the continuity of f at p. Since $p \in D$ is arbitrary this shows the continuity of f on D. $\square$

Integration

Recall that integrals are defined rigorously using the notion of a step functions.

Theorem

Suppose that f_n are continuous functions on $[a, b] \subset \mathbb{R}$, uniformly convergent to f. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx.$$

Proof.

The uniform convergence implies that for each $\epsilon > 0$, there exists N such that for every $n \geq N$ and every $x \in D$, $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$. This means that

$$\Big| \int_a^b f_n(x)\ dx - \int_a^b f(x)\ dx \Big| \leq \int_a^b |f_n(x) - f(x)|\ dx \leq (b-a)\frac{\epsilon}{b-a} = \epsilon.$$

This shows that $\int_a^b f_n(x)\ dx \to \int_a^b f(x)\ dx$. $\square$

Series of functions

Recall that for a sequence $\{a_n\}_n$ of numbers, the series $\sum_n a_n$ is the sequence $\{\sum_{k=1}^n a_k\}_n$ of numbers (the partial sums).
We say that the series $\sum_n a_n$ is convergent if $\{\sum_{k=1}^n a_k\}_n$ is convergent.

Definition

If $\{f_n\}$ is a sequence of functions we consider the series of functions $\sum_n f_n$.

- The series is said to be *pointwise convergent* if $\sum_{k=1}^n f_k(x)$ is pointwise convergent.
- The series is said to be *uniformly convergent* if $\sum_{k=1}^n f_k(x)$ is uniformly convergent.

Theorem

Suppose that the series $\sum_n f_n$ is uniformly convergent to g on D and the f_n are continuous on D. Then g is continuous on D.

Proof.

If the f_k are continuous then the $\sum_{k=1}^n f_k$ are continuous. Therefore the previous proof applies. □

Theorem

Suppose that the series $\sum_n f_n$ is uniformly convergent to g and the f_n are continuous. Then

$$\lim_{n\to\infty} \int_a^b \sum_{k=1}^n f_k(x)\; dx = \int_a^b g(x)\; dx.$$

Proof.

Again, that the f_k are continuous means that the $\sum_{k=1}^n f_k$ are continuous. Therefore the previous proof applies. □

Tests for convergence

Theorem (Ratio test)

Suppose that $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to L$. If $L < 1$ then $\sum_{n=0}^{\infty} a_n$ converges. If $L > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem (Root test)

Suppose that $a_n > 0$ and $(a_n)^{\frac{1}{n}} \to R$. If $R < 1$ then $\sum_{n=0}^{\infty} a_n$ converges. If $R > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem (Comparison test)

Suppose that $a_n > 0$, $b_n > 0$ and $a_n < b_n$. If $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges.

Weierstrass M-test

Theorem (Weierstrass M-test)

Suppose that $\{f_n\}_n$ is a sequence of functions on D, that $\{M_n\}_n$ is a sequence of positive numbers and that $|f_n(x)| \leq M_n$. If the series $\sum_{n=0}^{\infty} M_n$ is convergent then the series $\sum_{n=0}^{\infty} f_n$ is uniformly convergent.

Proof.

By the comparison test $\sum |f_n(x)|$ is convergent for all $x \in D$. I.e., for each x the series $\sum f_n(x)$ is absolutely convergent and so we let $f(x)$ be the limit. We compute

$$\left| f(x) - \sum_{k=1}^{n} f_k(x) \right| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k.$$

As $\sum_n M_n$ is convergent this last expression tends to 0 as $k \to \infty$. This estimate is independent of x. $\square$

Power series

Definition

Let $\{a_n\}_n$ be a series of complex numbers. The series $\sum_n a_n z^n$ is called a *power series*.

Typically this will converge for some z and diverge for other z.

Example

Let $a_n = 2^{-n}$. The power series $\sum_n \frac{z^n}{2^n}$ is convergent when $|z| < 2$ and divergent when $|z| > 2$.

Root test: $\left(\frac{|z|^n}{2^n}\right)^{\frac{1}{n}} = \frac{|z|}{2}$

Example

Let $a_n = \frac{1}{n!}$. The power series $\sum_n \frac{z^n}{n!}$ is convergent for all z.

Ratio test: $\left|\frac{z^{n+1}}{(n+1)!}\right| / \left|\frac{z^n}{n!}\right| = \frac{|z|}{n+1}$.

Radius of convergence

Theorem (Uniformly convergent power series)

Suppose that $\sum_n a_n z^n$ converges for some $z = z_0 \neq 0$. Let $R < |z_0|$. Then the series is uniformly and absolutely convergent for all z such that $|z| \leq R$.

Proof.

- $\sum_n a_n z_0^n$ is convergent $\implies$ for all n, $|a_n z_0^n| \leq M$ for some $M > 0$;
- $|a_n z^n| = |a_n z_0^n| \cdot \left|\frac{z}{z_0}\right|^n \leq M \frac{R^n}{|z_0|^n}$;
- $\sum_n M \frac{R^n}{|z_0|^n}$ is a geometric sum and so convergent;
- M-test $\implies$ series is uniformly and absolutely convergent when $|z| \leq R$.

$\square$

Theorem (Radius of convergence)

Suppose exists $z_1, z_2 \neq 0$ such that $\sum_n a_n z_1^n$ is convergent and $\sum_n a_n z_2^n$ is divergent. Then exists $r > 0$ such that $\sum_n a_n z^n$ is convergent for $|z| < r$ and divergent for $|z| > r$.

Proof.

- Let A be the set of real numbers for which $\sum_n a_n z^n$ is convergent;
- Let r be the least upper bound of A;
- Series $\sum_n a_n z^n$ is convergent whenever $|z| < r$;
- If $|z| > r$ and $\sum_n a_n z^n$ is convergent then this contradicts the definition of A and so $\sum_n a_n z^n$ is divergent for $|z| > r$. $\square$

Definition

This r is called the *radius of convergence* for the power series $\sum_n a_n z^n$.

Convention

If $\sum_n a_n z^n$ converges for all $z \in \mathbb{C}$ we say the radius of convergence is ∞. If $\sum_n a_n z^n$ doesn't converge except $z = 0$ we say the radius of convergence is 0.

Integration of power series

Let $a_n \in \mathbb{R}$, $x \in \mathbb{R}$. If $\sum_n a_n x^n$converges we define the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Theorem (Integrating power series)

Suppose that, for $x \in (-r, r)$, the series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent. Then $f(x)$ is continuous and $\int_0^x f(y)\, dy = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

Proof.

- Let $|x| < R < r$;
- Series is uniformly convergent for $y \in [-R, R]$;
- $f(x)$ is continuous and so we can interchange limit and integral:

$$\int_0^x f(y)\, dy = \sum_{n=0}^{\infty} \int_0^x a_n x^n = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

$\square$

Derivative of power series

Theorem (Differentiating power series)

Suppose that, for $x \in (-r, r)$, the series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent. Then $f(x)$ is differentiable and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, convergent for $x \in (-r, r)$.

Proof.

- Let $|x| < R < r$;
- $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} a_n R^n \cdot \frac{n}{R} \cdot \frac{x^{n-1}}{R^{n-1}}$;
- $\sum_{n=1}^{\infty} a_n R^n$ is absolutely convergent and $\frac{n}{R} \cdot (\frac{|x|}{R})^{n-1}$ is bounded;
- $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is absolutely convergent;
- $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ is a power series;
- $\int_0^x g(y)\, dy = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$.

$\square$

Example ($\log(x+1)$)

- $\frac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$;
- $(\log(x+1))' = \frac{1}{x+1}$;
- $\log(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ for $|x| < 1$.

Example ($\arctan x$)

- $\frac{1}{x^2+1} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $|x| < 1$;
- $(\arctan x)' = \frac{1}{x^2+1}$;
- $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $|x| < 1$.

Exercises

1. Determine the radius of convergence r of the following power series. If r is finite, test for convergence at the boundary points.
   1.1 $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{3^n}$,
   1.2 $\displaystyle\sum_{n=0}^{\infty} \frac{(z+3)^n}{(n+1)2^n}$,
   1.3 $\displaystyle\sum_{n=1}^{\infty} \frac{n!z^n}{n^n}$;
2. Let $f_n(x) = \frac{\sin(nx)}{n}$, $n \in \mathbb{N}$. For each fixed x let $f(x) = \lim_{n\to\infty} f_n(x)$. Calculate
   2.1 $\lim_{n\to\infty} f_n'(0)$,
   2.2 $f'(0)$.

Exercises

1. Calculate the Taylor's series for $f(x) = e^x$ at $x = 0$.
2. Consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

   2.1 Is $f(x)$ infinitely differentiable for every $x \in \mathbb{R}$?
   2.2 What is the Taylor's series for $f(x)$ at 0?
   2.3 What is the difficulty?

Representing functions

Let a, x and the coefficients a_n be real numbers. The series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

defines a function on the interval $(a-r, a+r)$, where r is the *radius of convergence*.
The series is said to *represent* the function f. It is called the *power series expansion* of f about a.

Question

Given the series, what are the properties of f?

Question

Given a function f, can it be represented by a power series?

Uniqueness of power series

The theorems on differentiating and integrating hold as before. In particular:

Theorem

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is convergent for $x \in (a-r, a+r)$. Then $f(x)$ has derivatives of every order and, for $k \in \mathbb{N}$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-a)^{n-k}.$$

Theorem (Uniqueness of power series)

If two power series $\sum_n a_n(x-a)^n$ and $\sum_n b_n(x-a)^n$ are equal in a neighbourhood of a then the two series are equal term-by-term: $a_n = b_n = \frac{f^{(n)}(a)}{n!}$ for every $n \in \mathbb{N}$.

Proof.

$$f^{(k)}(x) = k!a_k + \sum_{n=k+1}^{\infty} n\cdots(n-k+1)a_n(x-a)^{n-k} \text{ so } f^{(k)}(a) = k!a_k. \qquad \square$$

Taylor's series

Suppose that a function $f(x)$ is infinitely differentiable on an open interval about a. We consider the *Taylor's series generated by* f *at* a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Question

Does the series converge on the entire interval? In general, no.

Question

If it converges, is it equal to $f(x)$ on the interval? In general, no.

Error term

Let (error term in n^{th} approximation)

$$E_n(x) := f(x) - \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-a)^k.$$

Integration by parts implies that

$$E_n(x) = \frac{1}{n!}\int_a^x (x-y)^n f^{(n+1)}(y)\, dy.$$

Convergence of the Taylor's series to $f(x)$ is implied by $E_n(x) \to 0$ as $n \to \infty$.

A sufficient condition for convergence of a Taylor's series

Theorem

Assume f is infinitely differentiable on $I = (a - r, a + r)$ and exists $A > 0$ such that

$$\left| f^{(n)}(x) \right| \leq A^n, \quad \textit{for all } n \in \mathbb{N}, x \in I.$$

Then Taylor's series generated by f at a converges to $f(x)$ for each $x \in I$.

Proof.

$$|E_n(x)| \leq \frac{1}{n!} \int_a^x |x - y|^n A^{n+1} \, dy \leq \frac{1}{n!} r r^n A^{n+1} = rA \frac{(rA)^n}{n!}.$$

$\frac{(rA)^n}{n!} \to 0$ as $n \to \infty$ and so $|E_n(x)| \to 0$ as $n \to \infty$. □

Examples of Taylor's series

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots,$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$
- $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

Valid on what interval?

Power series and differential equations

Problem

Find function $y(x)$ such that $(1-x^2)y''(x) = -2y(x)$ with $y(0)=1$, $y'(0)=1$.

1. Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
   Then $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$;
2. And so

$$\begin{aligned}
-2\sum_{n=0}^{\infty} a_n x^n = (1-x^2)y''(x) &= (1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n
\end{aligned}$$

3. Consequently $0 = 2a_n + (n+2)(n+1)a_{n+2} - n(n-1)a_n$ for each $n \in \mathbb{N}_0$;

4. Equivalently $a_{n+2} = \frac{n-2}{n+2} a_n$;
5. $a_0 = y(0) = 1$, $a_1 = y'(0) = 1$;
6.
   - $a_2 = \frac{0-2}{0+2} a_0 = -1$,
   - $a_4 = \frac{2-2}{2+2} a_2 = 0$,
   - $a_6 = \frac{4-2}{4+2} a_4 = 0, \ldots$;
7.
   - $a_3 = \frac{1-2}{1+2} a_1 = -\frac{1}{3}$,
   - $a_5 = \frac{3-2}{3+2} a_3 = \frac{1}{5}(-\frac{1}{3}), \ldots$
   - $a_{2n+1} = \frac{1}{(2n+1)(2n-1)}$;
8. Convergent for $|x| < 1$,

$$y(x) = 1 - x^2 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)(2n-1)}.$$

Terminology

We call this the "Method of undetermined coefficients".

Binomial series

For any $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, let $\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$.

Theorem

Let $\alpha \in \mathbb{R}$. Then $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ *whenever* $|x| < 1$.

Proof.

1. By the ratio test the right hand side converges;
2. Let $f(x) = (1+x)^\alpha$;
3. $f'(x) = \alpha(1+x)^{\alpha-1}$ and so $f(x)$ is a solution to the differential equation

$$y'(x) = \frac{\alpha}{x+1} y(x)$$

   and satisfies the initial condition $f(0) = 1$.
4. $g(x) := \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ satisfies the same differential equation and initial condition. $\square$

Does $g(x)$ satisfy this differential equation?

First observe that

$$(n+1)\binom{\alpha}{n+1} = (\alpha - n)\binom{\alpha}{n} \iff (n+1)\binom{\alpha}{n+1} + n\binom{\alpha}{n} = \alpha\binom{\alpha}{n}.$$

We defined $g(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ and so

$$g'(x) = \sum_{n=1}^{\infty} n\binom{\alpha}{n} x^{n-1} = \sum_{n=0}^{\infty} (n+1)\binom{\alpha}{n+1} x^n.$$

Consequently

$$\begin{aligned}(1+x)g'(x) &= \sum_{n=0}^{\infty}\left((n+1)\binom{\alpha}{n+1} + n\binom{\alpha}{n}\right)x^n \\ &= \alpha\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n = \alpha g(x).\end{aligned}$$

Additionally $g(0) = 1$.

Finding Taylor's series

Problem

Find the Taylor expansion around the point 0 of the function

$$f(x) := \frac{1}{x^3 - 2x^2 + x - 2}.$$

Find the radius of convergence r and study the convergence at $x = \pm r$.

1. Observe that $x^3 - 2x^2 + x - 2 = (x^2 + 1)(x - 2)$;
2. Writing $f(x) = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$ we obtain
$$f(x) = -\frac{1}{10} \cdot \frac{1}{1 - (\frac{x}{2})} - \frac{1}{5} \cdot \frac{2 + x}{1 - (-x^2)};$$
3. We know that, for $|y| < 1$,
$$\frac{1}{1 - y} = \sum_{n=0}^{\infty} y^n.$$

4. Apply this expansion for $y = \frac{x}{2}, -x^2$ and obtain

$$f(x) = -\frac{1}{10} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n - \frac{1}{5} \cdot \left(x \sum_{n=0}^{\infty} (-x^2)^n + 2 \sum_{n=0}^{\infty} (-x^2)^n \right)$$
$$= \sum_{n=0}^{\infty} -\frac{2^{-n}}{10} x^n + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{5} x^{2n+1} + \sum_{n=0}^{\infty} \frac{2}{5} (-1)^{n+1} x^{2n};$$

5. This is a powers series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where

$$\begin{cases} a_{2m} = -\frac{1}{5} \left(2^{-(2m+1)} + 2(-1)^m \right) \\ a_{2m+1} = -\frac{1}{5} \left(2^{-2(m+1)} + (-1)^{m+1} \right) \end{cases} ;$$

6. Radius of convergence for the three sums are $2, 1, 1$ respectively. This means that the radius of convergence is at least 1. However if $1 \leq x$, the latter sums diverge.

Exercises

1. Prove the expansion $\frac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$.
2. Solve $(1-x)^2 y'' - 2xy' + 6y = 0$ with initial conditions $y(0) = 1$, $y'(0) = 0$.