MA2 Part 0 Introduction

Week 1 of MA2 – Draft lecture slides

Oliver Butterley

University of Rome Tor Vergata

2020/21

Outline

Why Analysis?

MA1 versus MA2

Why Analysis?

Example (Divide by zero)

The cancellation law $ac = bc \implies a = b$ does not work when $c = 0$. For instance, the identity $1 \times 0 = 2 \times 0$ is true but if one blindly cancels the 0 then one obtains $1 = 2$ which is false.

In this case it was obvious but in other cases it can be more hidden.

Suggested reference

Tao's "Analysis 1", §1.2.

Example (Series)

The geometric series $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ can be summed by a simple trick. Multiplying by 2 we obtain that
$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2 + S$ and so $S = 2$. If we try to do the same to the sum $T = 1 + 2 + 4 + 8 + 16 + \cdots$ we get the nonsensical answer $2T = 2 + 4 + 8 + 16 + \cdots = T - 1$ and so $T = -1$.

Why should we trust the argument in the first case and not in the second?

Example (Interchanging sums)

If we consider any matrix of numbers, e.g.,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

we can sum first the rows $6 + 15 + 24 = 45$ or first the columns $12 + 15 + 18 = 45$ to obtain the total sum of all numbers. This is the rule $\sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} = \sum_{k=1}^{n} \sum_{j=1}^{m} a_{jk}$. We would like to believe that also $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{jk}$. However this doesn't work for the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We often want to swap the order of summing (or integrating) and often need to consider infinite sums (or integrals). When can we do this and can't we?

Example (Interchanging integrals)

Let's try to integrate $e^{-xy} - xye^{xy}$ with respect to both x and y. We would like to believe that

$$\int_0^\infty \int_0^1 (e^{-xy} - xye^{xy})\ dy\ dx = \int_0^1 \int_0^\infty (e^{-xy} - xye^{xy})\ dx\ dy.$$

Since

$$\int_0^1 (e^{-xy} - xye^{xy})\ dy = \left[ye^{-xy}\right]_{y=0}^1 = e^{-x},$$

the left-hand side is $\int_0^\infty e^{-x}\ dx = [-e^{-x}]_0^\infty = 1$. However, since

$$\int_0^\infty (e^{-xy} - xye^{xy})\ dx = \left[xe^{-xy}\right]_{x=0}^\infty = 0,$$

the right-hand side is $\int_0^1 0\ dx = 0$.

So how do we know when to trust the interchange of intervals?

Example (Iinterchanging limits)

We could easily believe that

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2}{x^2+y^2}=\lim_{y\to 0}\lim_{x\to 0}\frac{x^2}{x^2+y^2}.$$

However $\lim_{y\to 0}\frac{x^2}{x^2+y^2}=\frac{x^2}{x^2+0}=1$ and so the left-hand side is 1 whereas $\lim_{x\to 0}\frac{x^2}{x^2+y^2}=\frac{0}{0+y^2}=0$ so the right-hand side is 0.
This example shows that the interchange of limits is untrustworthy. Under what circumstances is it legitimate? In general we must be rigorous in our logic.

Further reading

"The basics of mathematical logic" [Tao's "Analysis", Appendix A].

MA1 versus MA2

Mathematical Analysis 1

- Sequences of numbers $a_1, a_2, a_3, \ldots$
- Series of numbers $\sum_j a_j$
- *Functions* $f(x)$ on $\mathbb{R}$
  - Limits $\lim_{x \to a} f(x)$
  - Derivative $f'(x) = \frac{df}{dx}(x)$
  - Integral $\int_a^b f(x)\, dx$

Mathematical Analysis 2

- Sequences of functions $f_1(x), f_2(x), f_3(x), \ldots$
- Series of functions $\sum_j f_j(x)$
- *Functions* $f(x_1, \ldots, x_n)$ on[1] $\mathbb{R}^n$ and *vector fields* $\mathbf{F}(x_1, \ldots, x_n)$
  - Partial derivatives $\frac{\partial f}{\partial x_j}(x_1, \ldots, x_n)$
  - Find extrema in $\mathbb{R}^n$
  - Multiple integral
  - Line integral
  - Surface integral
  - Applications to mechanics, electrodynamics, etc.

[1] $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}$