1. Questions - Call 6 - 16/09/2021

Solutions to each question are included at the end of this document.

Call 6.

(1) **Q1**

CLOZE | 1 point | 0.10 penalty

The Taylor Series for $f(x) = \frac{7}{x^4}$ about $x = -3$ is

$$f(x) = \sum_{n=\boxed{\text{a}}}^{\infty} \frac{7(n+\boxed{\text{b}})(n+2)(n+1)}{6(\boxed{\text{c}})^{n+4}}(x+3)^n$$

where $\boxed{\text{a}}$ =

NUMERICAL | 1 point

0 ✓	

; $\boxed{\text{b}}$ =

NUMERICAL | 1 point

3 ✓	

; $\boxed{\text{c}}$ =

NUMERICAL | 1 point

3 ✓	

. The radius of convergence of this series is $r =$

NUMERICAL | 1 point

3 ✓	

. Fill in the above blanks with the correct **integers**, possibly zero or negative (1 point each). At $x = -3 - r$ the series

MULTI | 1 point | Single | Shuffle

converges	
diverges ✓	

and at $x = -3 + r$ the series

MULTI | 1 point | Single | Shuffle

converges	
diverges ✓	

(1 point each).

(2) **Q2**

CLOZE | 1 point | 0.10 penalty

(Fill in each of the following blanks with the correct **integer**, possibly zero or negative.) Let $g(x, y) := x^2 + 3xy + y^2 - 35$. We will find the points in the set $\{g(x, y) = 0\} \subset \mathbb{R}^2$ which are closest / furthest from the origin. Introduce a suitable function $f(x, y)$ and apply the Lagrange multiplier method with the constraint $g(x, y) = 0$ in order to find the extrema points. There are

NUMERICAL | 2 points

2 ✓	

extrema points (2 points). There is a single extrema point in the upper right quadrant and it is equal to $(\sqrt{\boxed{\text{a}}}, \sqrt{\boxed{\text{b}}})$ where $\boxed{\text{a}}$ =

NUMERICAL | 1 point

7 ✓	

; $\boxed{\text{b}}$ =

NUMERICAL | 1 point

7 ✓	

(1 point each). The extrema points are:

MULTI | 2 points | Single

- all equally the closest points to the origin ✓
- some are the closest and some are the furthest
- all equally the furthest points to the origin
- something else

(2 point). Hint: draw a sketch of the set.

(3) **Q3**

CLOZE | 1 point | 0.10 penalty

Determine which of the following is a parametrization of the path

$$C = \{(x, y) : x^2 + y^2 = 4, x \geq 0, y \leq 0\} \subset \mathbb{R}^2,$$

starting at $(-2, 0)$ and finishing at $(0, 2)$ ($\frac{1}{2}$ point each):

- $(2\sin t, -2\cos t)$, $t \in [0, \frac{\pi}{2}]$

  MULTI | 1 point | Single

is ✓	
is not	

- $(2\cos t, 2\sin t)$, $t \in [0, \frac{\pi}{2}]$

  MULTI | 1 point | Single

is	
is not ✓	

- $(t, \sqrt{4 - t^2})$, $t \in [0, 2]$

  MULTI | 1 point | Single

is	
is not ✓	

- $(2\sin t, 2\cos t)$, $t \in [\frac{\pi}{2}, \pi]$

  MULTI | 1 point | Single

is	
is not ✓	

- $(2\cos t, 2\sin t)$, $t \in [-\frac{\pi}{2}, 0]$

  MULTI | 1 point | Single

is ✓	
is not	

- $(\sqrt{4 - t^2}, t)$, $t \in [-2, 0]$

  MULTI | 1 point | Single

is ✓	
is not	

- $(t, -\sqrt{4-t^2})$, $t \in [0,2]$

MULTI | 1 point | Single

is ✓	
is not	

- $(t+2, t)$, $t \in [-2,0]$

MULTI | 1 point | Single

is	
is not ✓	

If C is the path above and

$$\mathbf{f}(x,y) = \begin{pmatrix} 2 \\ x^2 \end{pmatrix}$$

is a vector field on $\mathbf{R}^2$ then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} =$

NUMERICAL | 4 points

28 ✓	
-28 (50%)	

/3. Fill in the blank with the correct integer, possibly zero or negative (2 points). Hint: $\int \sin^3 t\ dt = (\cos 3t - 9\cos t)/12$.

(4) **Q4**

CLOZE | 1 point | 0.10 penalty

Let V denote the three dimensional region which is contained within the cylinder $x^2+y^2 = 4$, lies below $z = 8-x^2-y^2$ and above $z = -\sqrt{4x^2+4y^2}$. Let $D = \{(x,y) : x^2+y^2 \leq 4\}$ and show that the volume of V is equal to

$$\iint_D \boxed{\text{a}} - (x^2+y^2) + 2\sqrt{(x^2+y^2)}\ dxdy$$

where $\boxed{\text{a}}$ =

NUMERICAL | 3 points

8 ✓	

. *Hint: write $V = \{(x,y,z) : (x,y) \in D, f(x,y) \leq z \leq g(x,y)\}$ for a suitable choice of functions f and g, then integrate with respect to z.* Now evaluate this integral show that the volume of V is equal to

NUMERICAL | 3 points

104 ✓	

$\frac{\pi}{3}$. *Hint: use polar coordinates.* Fill in the blanks with the correct integers, possibly zero or negative (3 points each).

(5) **Q5**

CLOZE | 1 point | 0.10 penalty

Consider the parametric surface $\mathbf{r}(S)$ where

$$\mathbf{r}(u,v) = (3u\cos v, 3u\sin v, -u^2),$$

and $S = \left\{(u,v) : 0 \leq u \leq \frac{\sqrt{19}}{6}, 0 \leq v \leq 2\pi\right\}$. This surface has the Cartesian equation $x^2+y^2 =$

NUMERICAL | 1 point

-9 ✓	
9 (50%)	

z (1 point). The fundamental vector product is

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 6u^2 \cos v \\ \boxed{\text{a}}\, u^2 \sin v \\ \boxed{\text{b}}\, u \end{pmatrix}$$

where $\boxed{\text{a}}$:

NUMERICAL 1 point

6 ✓	
-6 (50%)	

, $\boxed{\text{b}}$:

NUMERICAL 1 point

9 ✓	

(1 point each). Now calculate $\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|$ and then integrate to calculate the area of the surface Area $(\mathbf{r}(S)) =$

NUMERICAL 3 points

271 ✓	

$\frac{\pi}{54}$ (3 points). Fill in the blanks with the correct **integers**, possibly zero or negative.

Q1 Solution: For the Taylor expansion see `https://tutorial.math.lamar.edu/Solutions/CalcII/TaylorSeries/Prob5.aspx`. We obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}} (x+3)^n.$$

Fixing x we let

$$a_n = \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}} |x+3|^n$$

and consider $\sum_{n=0}^{\infty} a_n$. Using the ratio test we see that this converges when $|x+3| < 3$ and diverges when $|x+3| > 3$ and so the radius of convergence is r. Now we investigate convergence for $x = -3 - r = -6$:

$$f(-6) = \sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}} (-3)^n = \frac{7}{6(3)^4} \sum_{n=0}^{\infty} (n+3)(n+2)(n+1)(-1)^n.$$

Now we investigate convergence for $x = -3 + r = 0$:

$$f(0) = \sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}} (3)^n = \frac{7}{6(3)^4} \sum_{n=0}^{\infty} (n+3)(n+2)(n+1).$$

In both cases the series diverge.

Q2 Solution: Let $g(x,y) := x^2 + 3xy + y^2 - 45$. One suitable choice of function for finding points closest / furthest from the origin is $f(x,y) = x^2 + y^2$. We calculate

$$\nabla g(x,y) = \begin{pmatrix} 2x + 3y \\ 3x + 2y \end{pmatrix}, \quad \nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$

According to the Lagrange multiplier method we introduce $\lambda \in \mathbb{R}$ and write

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 2x + 3y \\ 3x + 2y \end{pmatrix}.$$

Multiplying the first line by y and the second line by x we obtain that $2xy = 2\lambda xy + 3\lambda y^2$ and $2xy = \lambda 3x^2 + 2\lambda xy$. Equating these implies that $2\lambda xy + 3\lambda y^2 = \lambda 3x^2 + 2\lambda xy$ and so $y^2 = x^2$. We treat the case $y = x$ and $y = -x$ independently.

Case $y = x$**:** Substituting into $x^2 + 3xy + y^2 - 35 = 0$ we obtain $(2+3)x^2 = 35$. Consequently $x = \pm\sqrt{\frac{35}{5}}$. This gives two solutions: $(\sqrt{7}, \sqrt{7})$ and $(-\sqrt{7}, -\sqrt{7})$.

Case $y = -x$**:** Substituting into $x^2 + 3xy + y^2 - 35 = 0$ we obtain $(2-3)x^2 = 35$. However $(2-3)$ is negative and so there are no solutions in this case.

This set consists of two curves and has mirror symmetry along the line $y = x$ and the line $y = -x$. The set is unbounded, it contains points infinitely far from the origin. The two extrema are the two points equally close to the origin.

Q3 Solution: Consider the vector field

$$\mathbf{f}(x,y) = \begin{pmatrix} 2 \\ x^2 \end{pmatrix}$$

Picking the parametrization $\boldsymbol{\alpha}(t) = (2\sin t, -2\cos t)$, $t \in [0, \frac{\pi}{2}]$ we calculate that

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix}$$

and also

$$\begin{pmatrix} 2 \\ 4\sin^2 t \end{pmatrix} \cdot \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix} = 4\cos t + 8\sin^3 t.$$

Consequently

$$\int_C \mathbf{f}\ d\boldsymbol{\alpha} = 8\int_0^{\frac{\pi}{2}} \sin^3 t\ dt + 4\int_0^{\frac{\pi}{2}} \cos t\ dt.$$

Since $\int \sin^3 t\ dt = (\cos 3t - 9\cos t)/12$,

$$\begin{aligned} \int_C \mathbf{f}\ d\boldsymbol{\alpha} &= \frac{2}{3}\left[\cos 3t - 9\cos t\right]_0^{\frac{\pi}{2}} + 4\left[-\sin t\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{3}(-1+9) + 4 = \frac{16}{3} + 4 = \frac{28}{3}. \end{aligned}$$

Q4 Solution: `https://tutorial.math.lamar.edu/Solutions/CalcIII/TripleIntegrals/Prob9.aspx`

Q5 Solution: Consider the parametric surface $\mathbf{r}(S)$ where

$$\mathbf{r}(u,v) = (3u\cos v, 3u\sin v, -u^2),$$

and $S = \left\{(u,v) : 0 \le u \le \frac{\sqrt{19}}{6}, 0 \le v \le 2\pi\right\}$.

(1) We observe that $(3u\cos v)^2 + (3u\sin v)^2 = 9u^2$ and so this surface has the Cartesian equation $x^2 + y^2 = -9z$.

(2) We calculate that

$$\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 3\cos v \\ 3\sin v \\ -2u \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -3u\sin v \\ 3u\cos v \\ 0 \end{pmatrix}$$

and so the fundamental vector product is

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -6u^2\cos v \\ 6u^2\sin v \\ 9u \end{pmatrix}.$$

(3) Hence

$$\left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| = \left(36u^4 + 81u^2\right)^{\frac{1}{2}} = u\left(36u^2 + 81\right)^{\frac{1}{2}}.$$

(4) Using this we calculate the surface area

$$\begin{aligned} \text{Area}\,(\mathbf{r}(S)) &= \iint_S \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\|\ dudv \\ &= \int_0^{2\pi}\left[\int_0^{\frac{\sqrt{19}}{6}} \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\|\ du\right]\ dv \\ &= 2\pi\int_0^{\frac{\sqrt{19}}{6}} \left(u\left(36u^2+81\right)^{\frac{1}{2}}\right)\ du \\ &= 2\pi\left[\frac{2}{3\cdot 36\cdot 2}\left(36u^2+81\right)^{\frac{3}{2}}\right]_0^{\frac{\sqrt{19}}{6}} = \frac{\pi}{3\cdot 18}\left(\left(36\left(\frac{\sqrt{19}}{6}\right)^2 + 81\right)^{\frac{3}{2}} - (0+81)^{\frac{3}{2}}\right) \\ &= \frac{\pi}{3\cdot 18}\left(100^{\frac{3}{2}} - 81^{\frac{3}{2}}\right) = \frac{\pi}{3\cdot 18}(1000-729) = \frac{271\pi}{54}. \end{aligned}$$