1. Questions - Call 5 - 13/09/2021

Solutions to each question are included at the end of this document.

Call 5.

(1) **Q1**

CLOZE | 1 point | 0.10 penalty

The Taylor series of $f(x) = \ln(3+4x)$ about $x = 0$ is

$$f(x) = \ln(\boxed{\text{a}}) + \sum_{n=\boxed{\text{b}}}^{\infty} \frac{(-1)^{n+1}\boxed{\text{c}}^n}{n3^n} x^n$$

where $\boxed{\text{a}}$ =

NUMERICAL | 1 point

3 ✓	

; $\boxed{\text{b}}$ =

NUMERICAL | 1 point

1 ✓	

; $\boxed{\text{c}}$ =

NUMERICAL | 1 point

4 ✓	

. The radius of convergence of this series is $r =$

NUMERICAL | 1 point

3 ✓	

/4. Fill in the above blanks with the correct **integers**, possibly zero or negative (1 point each). At $x = r$ the series

MULTI | 1 point | Single | Shuffle

converges ✓	
diverges	

and at $x = -r$ the series

MULTI | 1 point | Single | Shuffle

converges	
diverges ✓	

(1 point each).

(2) **Q2**

CLOZE | 1 point | 0.10 penalty

Let $g_1(x,y,z) = x^2 + y^2 - xy - z^2 + 1$ and let $g_2(x,y,z) = x^2 + y^2 - 18$. Consider the set defined as the intersection between the two-sheeted hyperboloid $g_1(x,y,z) = 0$ and the cylinder $g_2(x,y,z) = 0$. Use the Lagrange multiplier method to find the points in this set which are closest and furtherest from the origin. These 8 extremal points are $(a, a, \pm\sqrt{b})$, $(-a, -a, \pm\sqrt{b})$, $(a, -a, \pm\sqrt{c})$, $(-a, a, \pm\sqrt{c})$, where $a, b, c > 0$. Fill in the following blanks with the correct **integers**, possibly zero or negative (2 points each): $a =$

NUMERICAL | 2 points

3 ✓	

; $b =$

NUMERICAL | 2 points

10 ✓	

; $c =$

NUMERICAL | 2 points

28 ✓	

.

(3) **Q3**

CLOZE | 1 point | 0.10 penalty

Consider the curve $C = \left\{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} = 1, x \geq 0\right\}$. A parametrization of C, starting at $(0,3)$ and finishing at $(0,-3)$ is $\boldsymbol{\alpha}(t) := ($

NUMERICAL | 2 points

2 ✓	

$\cos t, -3\sin t)$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Consider the vector field $\mathbf{f}(x,y) = \begin{pmatrix} x^2 + y \\ y^2 \end{pmatrix}$ and compute $\int \mathbf{f}\ d\boldsymbol{\alpha} =$

NUMERICAL | 1 point

-18 ✓	
18 (50%)	

$+$

NUMERICAL | 1 point

3 ✓	
-3 (50%)	

π. Let $\boldsymbol{\beta}(t) := (0,t)$, $t \in (-3,3)$ and compute $\int \mathbf{f}\ d\boldsymbol{\beta} =$

NUMERICAL | 2 points

-18 ✓	
18 (50%)	

. Fill in the above blanks with the correct **integers**, possibly zero or negative (2 points each).

(4) **Q4**

CLOZE | 1 point | 0.10 penalty

Let S denote the region determined by $y \leq 4 - x^2$, $x \geq 0$, $y \geq 0$. The area of S is $\iint_S\ dxdy =$

NUMERICAL | 2 points

16 ✓	

/3. The two moments are $\iint_S y\ dxdy =$

NUMERICAL | 2 points

128 ✓	

/15 and $\iint_S x\ dxdy = 4$. Using this we conclude that the centre of mass of S is $(\frac{3}{4}, \frac{\boxed{\text{a}}}{5})$ where $\boxed{\text{a}} =$

NUMERICAL | 2 points

8 ✓	

. Fill in the above blanks with the correct **integers**, possibly zero or negative (2 points each).

(5) **Q5**

CLOZE | 1 point | 0.10 penalty

The surface area of the portion of $z = 2x^2 + 2y^2 - 7$ that is inside the cylinder $x^2 + y^2 = 4$ is equal to

$$\frac{\pi}{\boxed{\text{a}}}\left(\boxed{\text{b}}^{\frac{3}{2}} - 1\right)$$

where $\boxed{\text{a}}$ =

NUMERICAL | 3 points

24 ✓	

; $\boxed{\text{b}}$ =

NUMERICAL | 3 points

65 ✓	

. Fill in the above blanks with the correct **integers**, possibly zero or negative (3 points each). Hint: Observe that the surface has the form $z = f(x, y)$ for some function f. Furthermore the surface can be written as $\{(x, y, f(x, y)) : (x, y) \in D\}$ where D is the disk of radius 2.

Q1 Solution: Calculate the first few derivatives of the function and observe the general formula. Note, the first term doesn't fit the general pattern. For more details see: `https://tutorial.math.lamar.edu/Solutions/CalcII/TaylorSeries/Prob4.aspx`. The series diverges for $x = -r$, this is the *harmonic series.* Convergence at $x = r$ because this is the *alternating harmonic series.*

Q2 Solution: Let

$$\begin{aligned} g_1(x,y,z) &= x^2 + y^2 - xy - z^2 + 1 = 0, \\ g_2(x,y,z) &= x^2 + y^2 - 18 = 0, \\ f(x,y,z) &= x^2 + y^2 + z^2. \end{aligned}$$

We calculate the gradients:

$$\nabla g_1(x,y,z) = \begin{pmatrix} 2x - y \\ 2y - x \\ -2z \end{pmatrix}, \quad \nabla g_2(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}, \quad \nabla f(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}.$$

By the Lagrange multiplier method we wish to solve

$$\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \lambda_1 \begin{pmatrix} 2x - y \\ 2y - x \\ -2z \end{pmatrix} + \lambda_2 \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}.$$

From the last line, $z = -\lambda_1 z$ so either $z = 0$ or $\lambda_1 = -1$. However, if $z = 0$, then $g_1(x,y,z) = 0$ has no solution since $xy \leq x^2 + y^2$.[1] This means that $\lambda_1 = -1$. Considering the remaining pair of equations

$$\begin{aligned} 4x &= y + 2\lambda_2 x \\ 4y &= x + 2\lambda_2 y \end{aligned}$$

we deduce that $x = y$ or $x = -y$. In the first case $\lambda_2 = \frac{3}{2}$ and in the second case $\lambda_2 = \frac{5}{2}$. Using also that $g_2 \equiv 0$ we deduce that $x^2 = y^2 = 9$ and so $\boxed{\text{a}} = 3$. Finally, using that $g_1 \equiv 0$, we obtain the z values of the extremal points: We know that $z^2 = x^2 + y^2 - xy + 1$ and so, in the case $x = y$, $z^2 = 3^2 + 3^2 - 3^2 + 1 = 10$, on the other hand, in the case $x = -y$, $z^2 = 3^2 + 3^2 + 3^2 + 1 = 28$.

Q3 Solution: Consider the curve $C = \left\{(x,y) : \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1, x \geq 0\right\}$. A parametrization of C, starting at $(0,3)$ and finishing at $(0,-3)$ is

$$\boldsymbol{\alpha}(t) := (2\cos t, -3\sin t), \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

We can check that this is orientated correct by calculating $\boldsymbol{\alpha}(-\frac{\pi}{2})$, $\boldsymbol{\alpha}(0)$, $\boldsymbol{\alpha}(\frac{\pi}{2})$. Consider the vector field $\mathbf{f}(x,y) = \begin{pmatrix} x^2 + y \\ y^2 \end{pmatrix}$. We calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} -2\sin t \\ -3\cos t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} 4\cos^2 t - 3\sin t \\ 9\sin^2 t \end{pmatrix}.$$

[1] Either $x \leq y$ or $y \leq x$. In the first case $xy \leq y^2$ and in the second case $xy \leq x^2$. Consequently $xy \leq x^2 + y^2$.

Consequently $\boldsymbol{\alpha}(t) \cdot \mathbf{f}(\boldsymbol{\alpha}(t)) = -8\sin t\cos^2 t + 6\sin^2 t - 27\cos t\sin^2 t$. And so

$$\begin{aligned}
\int \mathbf{f}\ d\boldsymbol{\alpha} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-8\sin t\cos^2 t + 6\sin^2 t - 27\cos t\sin^2 t)\ dt \\
&= -8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t\cos^2 t)\ dt + 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t\ dt - 27\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t\sin^2 t)\ dt \\
&= 6\left[\frac{t}{2} - \frac{1}{4}\sin(2t)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 27\left[\frac{1}{3}\sin^3 t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 3\pi - 18.
\end{aligned}$$

Let $\boldsymbol{\beta}(t) := (0, -t),\ t \in (-b, b)$ and compute $\int \mathbf{f}\ d\boldsymbol{\beta}$. The vector field $\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$ is conservative and so this part of the line integral is equal to the other. The vector field $\begin{pmatrix} y \\ 0 \end{pmatrix}$ doesn't contribute in this integral which leaves -18. (Or simply compute the integral directly without these observations.)

Q4 Solution: `https://tutorial.math.lamar.edu/Solutions/CalcII/CenterOfMass/Prob1.aspx`

Q5 Solution: `https://tutorial.math.lamar.edu/Solutions/CalcIII/SurfaceArea/Prob4.aspx`