1. Questions - Call 4 - 07/07/2021

Solutions to each question are included at the end of this document.

Call 4.

(1) **Q1**

CLOZE | 1 point | 0.10 penalty

(Fill in each of the following blanks with the correct **integer**, possibly zero or negative.) We will find a power series solution to the differential equation

$$2x^2y'' + xy' + x^2y = 0$$

under the assumption that $f(0) = 336$. By substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ one obtains the equation

$$\sum_{n=2}^{\infty} \left[(2n^2 + \boxed{\text{a}}\, n)a_n + a_{n-2}\right] x^n + a_1 x + \boxed{\text{b}}\, a_0 = 0$$

where $\boxed{\text{a}}$:

NUMERICAL | 2 points

-1 ✓	

, $\boxed{\text{b}}$:

NUMERICAL | 2 points

0 ✓	

(1 point each). Using the above equation and also the given initial value we know that $a_0 =$

NUMERICAL | 1 point

336 ✓	

and $a_1 =$

NUMERICAL | 1 point

0 ✓	

($\frac{1}{2}$ point each). Furthermore, the above equation implies a recurrence relation between a_{n-2} and a_n which holds for all $n \geq 2$. Derive this recurrence relation and use it to calculate the following coefficients of the power series solution: $a_2 =$

NUMERICAL | 1 point

-56 ✓	

, $a_3 =$

NUMERICAL | 1 point

0 ✓	

, $a_4 =$

NUMERICAL | 1 point

2 ✓	

, $a_5 =$

NUMERICAL | 1 point

0 ✓	

($\frac{1}{2}$ point each). The radius of convergence of the power series solution is

MULTI | 2 points | Single | Shuffle

0	
1	
336	
infinite ✓	

(1 point).

(2) **Q2**

CLOZE | 1 point | 0.10 penalty

Fill in the blanks with integers, possibly 0 or negative. Find all stationary points of the function below, following the suggested steps.

$$f(x,y,z) = 2e^{(x-1)^2}(y^4 - 4yz + 2z^2).$$

First, we compute the gradient ∇f:

$$\nabla f(x,y,z) = \begin{pmatrix} \boxed{a}(x-1)e^{(x-1)^2}(y^4 - 4yz + 2z^2) \\ e^{(x-1)^2}(\boxed{b}y^{\boxed{c}} + \boxed{d}z) \\ e^{(x-1)^2}(\boxed{e}y + \boxed{f}z) \end{pmatrix}.$$

a:

NUMERICAL | 1 point

4 ✓	

b:

NUMERICAL | 1 point

8 ✓	

c:

NUMERICAL | 1 point

3 ✓	

d:

NUMERICAL | 1 point

-8 ✓	

e:

NUMERICAL | 1 point

-8 ✓	

f:

NUMERICAL | 1 point

8 ✓	

The equation $\nabla f(x,y,z) = \mathbf{0}$ has three solutions. They are $(x,y,z) = (\boxed{g}, \sqrt{\boxed{h}}, \sqrt{\boxed{i}}), (\boxed{j}, -\sqrt{\boxed{k}}, -\sqrt{\boxed{l}})$ and $(1,0,0)$ where g:

NUMERICAL | 1 point

1 ✓	

, h:

NUMERICAL | 1 point

2 ✓	

, i:

NUMERICAL | 1 point

2 ✓	

, j :

NUMERICAL 1 point

1 ✓	

, k :

NUMERICAL 1 point

2 ✓	

, l :

NUMERICAL 1 point

2 ✓	

.

Consider the first of them (g , $\sqrt{\text{h}}$, $\sqrt{\text{i}}$). The determinant of the Hessian at this point is

MULTI 1 point Single Shuffle

- positive ✓
- 0
- negative

At this point, the function $f(x, y, z)$ takes a

MULTI 2 points Single Shuffle

- local minimum
- saddle point
- local maximum ✓

Consider the solution $(1, 0, 0)$. The determinant of the Hessian at this point is

MULTI 1 point Single Shuffle

- positive ✓
- 0
- negative

At this point, the function $f(x, y)$ has a

MULTI 2 points Single Shuffle

- local minimum
- saddle point ✓
- local maximum

(3) **Q3**

CLOZE 1 point 0.10 penalty

Determine which of the following is a parametrization of the path

$$C = \{(x, y) : 4x^2 + y^2 = 4, y \leq 0\} \subset \mathbb{R}^2,$$

starting at $(-1, 0)$ and finishing at $(1, 0)$ ($\frac{1}{2}$ point each):

- $(\cos t, 2\sin t)$, $t \in [-\pi, 0]$

  MULTI 1 point Single Shuffle

is ✓	
is not	

- $(\sin t, 2\cos t)$, $t \in [-\pi, 0]$

  MULTI 1 point Single Shuffle

is	
is not ✓	

- $(t, -2\sqrt{1-t^2})$, $t \in [-1, 1]$

MULTI 1 point Single Shuffle

is ✓	
is not	

- $(\frac{2t}{t^2+1}, \frac{2t^2-2}{t^2+1})$, $t \in [-1, 1]$

MULTI 1 point Single Shuffle

is	
is not ✓	

- $(\frac{2t}{t^2+1}, \frac{2-2t^2}{t^2+1})$, $t \in [-1, 1]$

MULTI 1 point Single Shuffle

is ✓	
is not	

- $(2t, 2-2\sqrt{1-t^2})$, $t \in [-1, 1]$

MULTI 1 point Single Shuffle

is	
is not ✓	

If C is the path above and

$$\mathbf{f}(x,y) = \begin{pmatrix} y^2 \\ 3x^2 \end{pmatrix}$$

is a vector field on $\mathbb{R}^2$ calculate $\int_C \mathbf{f}\, d\boldsymbol{\alpha} =$

NUMERICAL 6 points

16 ✓	
-16 (50%)	

/3. Fill in the blank with the correct **integer**, possibly zero or negative (3 points).

(4) **Q4**

CLOZE 1 point 0.10 penalty

Let S be the the triangular region of the xy-plane which is bounded by the three lines $x = 0$, $y = x$ and $y = 2 - x$. Let V be the solid under the paraboloid $z = x^2 + y^2$ and above the region S. The volume of V is

NUMERICAL 6 points

4 ✓	

/3. Fill in the blank with the correct **integer**, possibly zero or negative (6 points).

(5) **Q5**

CLOZE 1 point 0.10 penalty

Consider the the sphere $S = \{(x,y,z) : x^2 + y^2 + z^2 = 1/4\}$ and the vector-field

$$\mathbf{f}(x,y,z) = \begin{pmatrix} 2x^3 \\ 2y^3 \\ 2z^3 \end{pmatrix}.$$

We wish to compute $\iint_S \mathbf{f}\cdot\mathbf{n}\,dS$ where $\mathbf{n}$ is the outgoing unit normal on S. If we let $V = \{(x,y,z) : x^2+y^2+z^2 \le 1/4\}$ then, by Gauss, we know that

$$\iint_S \mathbf{f}\cdot\mathbf{n}\,dS = \iiint_V \nabla\cdot\mathbf{f}\,dV$$

($\nabla\cdot\mathbf{f}$ denotes the divergence of $\mathbf{f}$). In this case $\nabla\cdot\mathbf{f}$ is equal to

MULTI | 1 point | Single | Shuffle

- $3(x^2+y^2+z^2)$,
- $6(x^2+y^2+z^2)$, ✓
- $x^3+y^3+z^3$,
- $(x^4+y^4+z^4)/4$.

To evaluate the three-dimensional integral use spherical coordinates $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$. The Jacobian $J(r,\theta,\varphi)$ is equal to

MULTI | 1 point | Single | Shuffle

- $r\cos\theta$,
- $r^2\sin\theta$, ✓
- $2r^2\sin\theta$,
- $r^3\tan\varphi$.

Consequently

$$\iiint_V \nabla\cdot\mathbf{f}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1/2} \boxed{?}\,dr d\theta d\varphi$$

where the blank $\boxed{?}$ should be

MULTI | 2 points | Single | Shuffle

- $3r^4\sin\theta$,
- $6r^4\sin\theta$, ✓
- $2r^4\sin\theta$,
- $r^3\sin\theta$.

Evaluate the integral and hence calculate $\iint_S \mathbf{f}\cdot\mathbf{n}\,dS =$

NUMERICAL | 1 point

6 ✓	

$\frac{\pi}{40}+$

NUMERICAL | 1 point

0 ✓	

(fill in the blanks with the correct integers, possibly zero or negative).

Q1 Solution: We consider, the equation

$$2x^2y'' + xy' + x^2y = 0.$$

Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ one obtains the equation

$$2x^2 \left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \right) + x \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0$$

and so

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Shifting the index in the third sum

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

Consequently, separating the terms of x^1 and x^0,

$$\sum_{n=2}^{\infty} \left[2n(n-1)a_n + n a_n + a_{n-2}\right] x^n + a_1 x = 0.$$

Equivalently

$$\sum_{n=2}^{\infty} \left[(2n^2 - n)a_n + a_{n-2}\right] x^n + a_1 x = 0.$$

This means that $a_1 = 0$ and, as an immediate consequence of the given initial value, $a_0 = 336$. Considering the first sum in the above equation, we see that, for all $n \geq 2$, $(2n^2 - n)a_n + a_{n-2} = 0$ and so

$$a_n = -\frac{a_{n-2}}{2n^2 - n}.$$

The recurrence relation would allow us to determine the coefficients of the power series solution to this differential equation. For the first few terms we calculate that

$$\begin{aligned}
a_2 &= -\frac{1}{2 \cdot 2^2 - 2} a_0 = -\frac{1}{6} a_0, \\
a_3 &= -\frac{1}{2 \cdot 3^2 - 3} a_1 = 0, \\
a_4 &= -\frac{1}{2 \cdot 4^2 - 4} a_2 = -\frac{1}{28} a_2, \\
a_5 &= -\frac{1}{2 \cdot 5^2 - 5} a_3 = 0.
\end{aligned}$$

Using the ratio test on the recurrence relation shows that the radius of converge of the power series solution is infinite.

Q3 Solution: Consider the vector field

$$\mathbf{f}(x,y) = \begin{pmatrix} ay^2 \\ (4-a)x^2 \end{pmatrix}$$

Picking the parametrization $\boldsymbol{\alpha}(t) = (\cos t, 2\sin t), \quad t \in [-\pi, 0]$ we calculate that

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} -\sin t \\ 2\cos t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} 4a\sin^2 t \\ (4-a)\cos^2 t \end{pmatrix}$$

and so $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = 2(4-a)\cos^3 t - 4a\sin^3 t$. Consequently

$$\int_C \mathbf{f}\ d\boldsymbol{\alpha} = 2(4-a)\int_{-\pi}^{0} \cos^3 t\ dt - 4a\int_{-\pi}^{0} \sin^3 t\ dt.$$

To proceed we note the indefinite integrals $\int \cos^3 t\ dt = -\frac{1}{3}\sin^3 t + \sin t + C$ and $\int \sin^3 t\ dt = \frac{1}{3}\cos^3 t - \cos t + C$. Consequently

$$\begin{aligned}\int_C \mathbf{f}\ d\boldsymbol{\alpha} &= \frac{2(4-a)}{3}\left[-\sin^3 t + 3\sin t\right]_{-\pi}^{0} - \frac{4a}{3}\left[\cos^3 t - 3\cos t\right]_{-\pi}^{0} \\ &= 0 - \frac{4a}{3}(1 - 3 + 1 - 3) = \frac{16a}{3}.\end{aligned}$$

Q5 Solution. Consider the the sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1/4\}$ and the vector-field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}.$$

We wish to compute $\iint_S \mathbf{f} \cdot \mathbf{n} \; dS$ where $\mathbf{n}$ is the outgoing unit normal on S. If we let $V = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1/4\}$ then, by Gauss, we know that

$$\iint_S \mathbf{f} \cdot \mathbf{n} \; dS = \iiint_V \nabla \cdot \mathbf{f} \; dV$$

($\nabla \cdot \mathbf{f}$ denotes the divergence of $\mathbf{f}$). In this case $\nabla \cdot \mathbf{f}$ is equal to $3(x^2 + y^2 + z^2)$, To evaluate the three-dimensional integral use spherical coordinates $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$. The Jacobian $J(r, \theta, \varphi)$ is equal to $r^2 \sin\theta$, Consequently

$$\iiint_V \nabla \cdot \mathbf{f} \; dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{1/2} \boxed{?} \; dr d\theta d\varphi$$

where the blank $\boxed{?}$ should be $3r^4 \sin\theta$, Evaluate the integral and hence calculate $\iint_S \mathbf{f} \cdot \mathbf{n} \; dS = 3 \; \frac{\pi}{40}$ (fill in the blanks with the correct integers, possibly zero or negative).

$$\begin{aligned} \iiint_V \nabla \cdot \mathbf{f} \; dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{1/2} 3r^4 \sin\theta \; dr d\theta d\varphi \\ &= 2\pi \left(\int_0^{\pi} \sin\theta \; d\theta \right) \left(\int_0^{1/2} 3r^4 \; dr \right) \\ &= 2\pi \cdot 2 \cdot \frac{3}{5 \cdot 2^5} = \frac{3}{40}\pi. \end{aligned}$$