1. Questions - Call 3 - 22/06/2021

Solutions to each question are included at the end of this document.

Call 3.

(1) **Q1-A**

Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

In this problem we find a power series solution of the differential equation

$$(1+x^2)y'' - 2y = 0$$

subject to the initial condition $y(0) = 3$, $y'(0) = 3$. Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ we obtain the formula

$$\sum_{n=0}^{\infty} \left[(n+\boxed{\text{a}})(n+1)a_{n+2} + (n-2)(n+\boxed{\text{b}})a_n \right] x^n = 0$$

where $\boxed{\text{a}}$: $\boxed{\begin{array}{ll} 2 & \checkmark \\ -2 & (50\%) \end{array}}$, $\boxed{\text{b}}$: $\boxed{\begin{array}{ll} 1 & \checkmark \\ -1 & (50\%) \end{array}}$ ($\frac{1}{2}$ point each). We can now produce a recursion formula for the coefficients. We calculate $a_0 = \boxed{\begin{array}{ll} 3 & \checkmark \\ -3 & (50\%) \end{array}}$, $a_2 = \boxed{\begin{array}{ll} 3 & \checkmark \\ -3 & (50\%) \end{array}}$, $a_4 = \boxed{0 \quad \checkmark}$, $a_6 = \boxed{0 \quad \checkmark}$ ($\frac{1}{2}$ point each). The general formula for the odd coefficients is (for all $n \geq 0$),

$$a_{2n+1} = \frac{-(-1)^n \, 3}{(\boxed{\text{c}}n + \boxed{\text{d}})(2n-1)}.$$

where $\boxed{\text{c}}$: $\boxed{\begin{array}{ll} 2 & \checkmark \\ -2 & (50\%) \end{array}}$, $\boxed{\text{d}}$: $\boxed{\begin{array}{ll} 1 & \checkmark \\ -1 & (50\%) \end{array}}$ (1 point each). The radius of convergence of this power series is $\boxed{1 \quad \checkmark}$ (1 point).

(2) **Q1-B**

Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

In this problem we find a power series solution of the differential equation

$$(1+x^2)y'' - 2y = 0$$

subject to the initial condition $y(0) = 4$, $y'(0) = 4$. Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ we obtain the formula

$$\sum_{n=0}^{\infty} \left[(n+\boxed{\text{a}})(n+1)a_{n+2} + (n-2)(n+\boxed{\text{b}})a_n \right] x^n = 0$$

where $\boxed{\text{a}}$: $\boxed{\begin{array}{ll} 2 & \checkmark \\ -2 & (50\%) \end{array}}$, $\boxed{\text{b}}$: $\boxed{\begin{array}{ll} 1 & \checkmark \\ -1 & (50\%) \end{array}}$ ($\frac{1}{2}$ point each). We can now produce a recursion formula for the coefficients. We calculate $a_0 = \boxed{\begin{array}{ll} 4 & \checkmark \\ -4 & (50\%) \end{array}}$, $a_2 = \boxed{\begin{array}{ll} 4 & \checkmark \\ -4 & (50\%) \end{array}}$, $a_4 = \boxed{0 \quad \checkmark}$, $a_6 = \boxed{0 \quad \checkmark}$ ($\frac{1}{2}$ point each). The general formula for the odd coefficients is (for all $n \geq 0$),

$$a_{2n+1} = \frac{-(-1)^n 4}{(\boxed{\text{c}}\, n + \boxed{\text{d}})(2n-1)}.$$

where $\boxed{\text{c}}$: $\boxed{\begin{matrix} 2 & \checkmark \\ -2 & (50\%) \end{matrix}}$, $\boxed{\text{d}}$: $\boxed{\begin{matrix} 1 & \checkmark \\ -1 & (50\%) \end{matrix}}$ (1 point each). The radius of convergence of this power series is $\boxed{1 \quad \checkmark}$ (1 point).

(3) **Q2-A**

(Fill in each of the following blanks with the correct **integer**, possibly zero or negative.) Let $g(x, y) := x^2 + xy + y^2 - 12$. We will find the points in the set $\{g(x, y) = 0\} \subset \mathbb{R}^2$ which are closest / furthest from the origin. Introduce a suitable function $f(x, y)$ and apply the Lagrange multiplier method with the constraint $g(x, y) = 0$ in order to find the extrema points. There are $\boxed{4 \quad \checkmark}$ extrema points (2 points). There is a single extrema point in the upper ~~lower~~ right quadrant and it is equal to ($\boxed{2 \quad \checkmark}$, $\boxed{\cancel{-2} \quad \checkmark}$) (1 point each). 2

The extrema points are:

- all equally the closest points to the origin
- some are the closest and some are the furthest ✓
- all equally the furthest points to the origin
- something else

(2 points).

(4) **Q2-B**

(Fill in each of the following blanks with the correct **integer**, possibly zero or negative.) Let $g(x, y) := x^2 + xy + y^2 - 27$. We will find the points in the set $\{g(x, y) = 0\} \subset \mathbb{R}^2$ which are closest / furthest from the origin. Introduce a suitable function $f(x, y)$ and apply the Lagrange multiplier method with the constraint $g(x, y) = 0$ in order to find the extrema points. There are $\boxed{4 \quad \checkmark}$ extrema points (2 points). There is a single extrema point in the upper ~~lower~~ right quadrant and it is equal to ($\boxed{3 \quad \checkmark}$, $\boxed{\cancel{-3} \quad \checkmark}$) (1 point each). 3

The extrema points are:

- all equally the closest points to the origin
- some are the closest and some are the furthest ✓
- all equally the furthest points to the origin
- something else

(2 points).

(5) **Q3-A**

Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

(a) If C is the path from $(0, 1)$ to $(1, e)$ along the curve $y = e^x$ and

$$\mathbf{f}(x, y) = \begin{pmatrix} 4x^2 \\ 2y \end{pmatrix}.$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{1 \quad \checkmark}\ /3 + e^2$.

(b) If C be the line segment from $(0, 2)$ to $(2, 4)$ and

$$\mathbf{f}(x, y) = \begin{pmatrix} x^3 y \\ xy \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 316 & \checkmark \\ -316 & (50\%) \end{array}}/15$.

(c) If Let C be the path from $(-1,-1)$ to $(1,1)$ along the curve $y = x^3$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} x^2 \\ 5y^2 \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 4 & \checkmark \\ -4 & (50\%) \end{array}}$.

(6) **Q3-B**

Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

(a) If C is the path from $(0,1)$ to $(1,e)$ along the curve $y = e^x$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} 5x^2 \\ 2y \end{pmatrix}.$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{2 \ \checkmark} /3 + e^2$.

(b) If C be the line segment from $(0,2)$ to $(2,4)$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} x^2y \\ xy \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 16 & \checkmark \\ -16 & (50\%) \end{array}}$.

(c) If Let C be the path from $(-1,-1)$ to $(1,1)$ along the curve $y = x^3$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} x^2 \\ 8y^2 \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 6 & \checkmark \\ -6 & (50\%) \end{array}}\ \pi$.

(7) **Q4-A**

Fill in the following blanks with the correct integers, possibly zero or negative. We wish to evaluate the integral

$$I = \iiint_V 5x + 4y + 3z\ dxdydz$$

where the integral is over the half ellipsoid

$$V = \left\{ (x,y,z) : \frac{x^2}{4} + y^2 + z^2 \leq 1, x \geq 0 \right\} \subset \mathbb{R}^3.$$

We choose a change of coordinates $x = r\cos\theta,\ y = \frac{1}{2} r \sin\theta,\ z = z$ under which V is sent to

$$W = \left\{ (r,\theta,z) : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq \boxed{\text{c}}, |z| \leq \sqrt{\boxed{\text{d}} - \frac{r^2}{\boxed{\text{e}}}} \right\}$$

and the Jacobian is $J(r,\theta,z) = \frac{r}{\boxed{\text{f}}}$. The missing values are (1 point each):

$\boxed{\text{c}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{d}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{e}}$: $\boxed{4 \ \checkmark}$ $\boxed{\text{f}}$: $\boxed{2 \ \checkmark}$ Evaluating the integral we obtain the final result $I = \boxed{5 \ \checkmark}\ \frac{\pi}{2}$ (2 points).

(8) **Q5**

Fill in each blank with the correct **integer**, possibly zero or negative.

Consider the surface $S = \{(x, y, z) : x^2 + y^2 = z, z \leq 16\}$. A possible choice for the parametric form of the surface S is to let $T = \{(r, \theta) : r \in [0, \boxed{4 \quad \checkmark}], \theta \in [0, 2\pi]\}$ ($\frac{1}{2}$ point) and

$$\mathbf{r} : (r, \theta) \mapsto \left(r \cos\theta, \boxed{\text{a}}, \boxed{\text{b}}\right).$$

For this parametric representation we calculate that

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \boxed{\text{c}} \\ \boxed{\text{d}} \\ \boxed{\text{e}} \end{pmatrix}.$$

The missing formulae are ($\frac{1}{2}$ point each):

$\boxed{\text{a}}$:

• $r \sin\theta$ ✓ • $r \cos\theta$ • r • r^2 • $r^2 \cos\theta$ • $r^2 \sin\theta$

$\boxed{\text{b}}$:

• $r \sin\theta$ • $r \cos\theta$ • r • r^2 ✓ • $r^2 \cos\theta$ • $r^2 \sin\theta$

$\boxed{\text{c}}$:

• $2r \sin\theta$ • r • $2r^2$ • $-r^2 \sin\theta$ • $-2r^2 \cos\theta$ ✓

$\boxed{\text{d}}$:

• $2r \sin\theta$ • r • r^2 • $-2r^2 \sin\theta$ ✓ • $-2r^2 \cos\theta$

$\boxed{\text{e}}$:

• $2r \sin\theta$ • r ✓ • r^2 • $-2r^2 \sin\theta$ • $-2r^2 \cos\theta$

Consider the vector field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} x \\ 0 \\ 2 \end{pmatrix}$$

and let $\mathbf{n}$ be the unit normal to S which has **negative** z-component. The surface integral $\iint_S \mathbf{f} \cdot \mathbf{n} \, dS = \boxed{\begin{array}{ll} 96 & \checkmark \\ -96 & (50\%) \end{array}}$ π (3 points).

Q1 Solution: In this problem we find a power series solution of the differential equation
$$(1+x^2)y'' - 2y = 0$$
subject to the initial condition $y(0) = a$, $y'(0) = a$. (Either $a = 3$ or $a = 6$.) To proceed we substitute $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ to obtain
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0.$$
Rearranging this is equivalent to
$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 2a_n\right] x^n = 0.$$
Consequently we know that, for all $n \geq 0$,
$$(n+2)(n+1)a_{n+2} = -\left(n(n-1) - 2\right)a_n = -(n-2)(n+1)a_n.$$
This leads to the recursion relation
$$a_{n+2} = -\frac{(n-2)}{(n+2)}a_n.$$
Using the initial conditions we have $a_0 = y(0) = a$, $a_1 = y'(0) = a$. Using the recursion relation we observe that $a_2 = a_0 = a$ but $a_4 = 0$, $a_6 = 0$, $a_8 = 0$, etc. For the odd terms we need to work slightly harder. Using the recursion formula we calculate that
$$\begin{aligned}
a_{2n+1} &= \left(-\frac{2n-3}{2n+1}\right)a_{2n-1}\\
&= \left(-\frac{2n-3}{2n+1}\right)\left(-\frac{2n-5}{2n-1}\right)a_{2n-3}\\
&= \left(-\frac{2n-3}{2n+1}\right)\left(-\frac{2n-5}{2n-1}\right)\left(-\frac{2n-7}{2n-3}\right)a_{2n-7}\\
&= \left(-\frac{2n-3}{2n+1}\right)\left(-\frac{2n-5}{2n-1}\right)\left(-\frac{2n-7}{2n-3}\right)\cdots\left(-\frac{5}{9}\right)\left(-\frac{3}{7}\right)\left(-\frac{1}{5}\right)\left(-\frac{-1}{3}\right)a_1.
\end{aligned}$$
The telescoping cancellations leads to the general formula
$$a_{2n+1} = \frac{-\left(-1\right)^n a}{(2n+1)(2n-1)}.$$
Using the ratio test with the recursion formula is convenient to deduce that the radius of convergence is equal to 1.

Q2 Solution: Let $g(x,y) := x^2 + xy + y^2 - b$. Either $b = 12$ or $b = 27$. One suitable choice of function for finding points closest / furthest from the origin is $f(x,y) = x^2 + y^2$. We calculate
$$\nabla g(x,y) = \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix}, \quad \nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$
According to the Lagrange multiplier method we introduce $\lambda \in \mathbb{R}$ and write
$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix}.$$

Multiplying the first line by y and the second line by x we obtain that $2xy = 2\lambda xy + \lambda y^2$ and $2xy = \lambda x^2 + 2\lambda xy$. Equating these implies that $2\lambda xy + \lambda y^2 = \lambda x^2 + 2\lambda xy$ and so $y^2 = x^2$. We treat the case $y = x$ and $y = -x$ independently.

Case $y = x$**:** Substituting into $x^2 + xy + y^2 - b = 0$ we obtain $(2+1)x^2 = b$. Consequently $x = \pm\sqrt{\frac{b}{3}}$. This gives two solutions: $(\sqrt{\frac{b}{3}}, \sqrt{\frac{b}{3}})$ and $(-\sqrt{\frac{b}{3}}, -\sqrt{\frac{b}{3}})$.

Case $y = -x$**:** Substituting into $x^2 + xy + y^2 - b = 0$ we obtain $(2-1)x^2 = b$. Consequently $x = \pm\sqrt{b}$. This gives two solutions: $(\sqrt{b}, -\sqrt{b})$ and $(-\sqrt{b}, \sqrt{b})$.

This set is an ellipse. Two extrema are the two points equally close as each other to the origin, the other two extrema are the two points equally far as each other from the origin.

Q3 Solution: **(a)** Let C be the path from $(0,1)$ to $(1,e)$ along the curve $y = e^x$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} ax^2 \\ 2y \end{pmatrix}.$$

Choose $\boldsymbol{\alpha}(t) := (t, e^t)$, $t \in [0,1]$. We calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 1 \\ e^t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} at^2 \\ 2e^t \end{pmatrix}.$$

Consequently $\boldsymbol{\alpha}(t) \cdot \mathbf{f}(\boldsymbol{\alpha}(t)) = at^2 + 2e^{2t}$. And so

$$\begin{aligned} \int \mathbf{f}\ d\boldsymbol{\alpha} &= \int_0^1 (at^2 + 2e^{2t})\ dt \\ &= \int_0^1 at^2 + 2e^{2t}\ dt = \left[\frac{a}{3}t^3 + e^{2t}\right]_0^1 = (\frac{a}{3} + e^2 - 1). \end{aligned}$$

(b) Let C be the line segment from $(0,2)$ to $(2,4)$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} x^a y \\ xy \end{pmatrix}.$$

Choose $\boldsymbol{\alpha}(t) := (2t, 2+2t)$, $t \in [0,1]$. We calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} 2^a t^a (2+2t) \\ (2t)(2+2t) \end{pmatrix} = \begin{pmatrix} 2 \cdot 2^a t^a (1+t) \\ 4t(1+t) \end{pmatrix}.$$

Consequently $\boldsymbol{\alpha}(t) \cdot \mathbf{f}(\boldsymbol{\alpha}(t)) = 4 \cdot 2^a t^a (1+t) + 8t(1+t) = 8t + 8t^2 + 4 \cdot 2^a t^a + 4 \cdot 2^a t^{a+1}$. And so

$$\begin{aligned} \int \mathbf{f}\ d\boldsymbol{\alpha} &= \int_0^1 (2^3 t + 2^3 t^2 + 2^{a+2} t^a + 2^{a+2} t^{a+1})\ dt \\ &= \left[2^2 t^2 + \frac{2^3}{3} t^3 + \frac{2^{a+2}}{a+1} t^{a+1} + \frac{2^{a+2}}{a+2} t^{a+2}\right]_0^1 \\ &= 2^2 + \frac{2^3}{3} + \frac{2^{a+2}}{a+1} + \frac{2^{a+2}}{a+2} = \frac{20}{3} + \frac{2^{a+2}}{a+1} + \frac{2^{a+2}}{a+2}. \end{aligned}$$

(c)

Let C be the path from $(-1,-1)$ to $(1,1)$ along the curve $y = x^3$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} x^2 \\ ay^2 \end{pmatrix}.$$

Choose $\boldsymbol{\alpha}(t) := (t, t^3)$, $t \in [-1, 1]$. We calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} t^2 \\ at^6 \end{pmatrix}.$$

Consequently $\boldsymbol{\alpha}(t) \cdot \mathbf{f}(\boldsymbol{\alpha}(t)) = t^2 + 3at^8$. And so

$$\begin{aligned} \int \mathbf{f}\ d\boldsymbol{\alpha} &= \int_{-1}^{1} (t^2 + 3at^8)\ dt \\ &= \left[\frac{1}{3}t^3 + \frac{a}{3}t^9\right]_{-1}^{1} = \frac{1}{3}(1 + a + 1 + a) = \frac{2}{3}(1 + a). \end{aligned}$$

Q4 Solution: Here we evaluate the integral

$$I = \iiint_V 5x + 4y + 3z\ dxdydz$$

where

$$V = \left\{(x, y, z) : \frac{x^2}{4} + y^2 + z^2 \leq 1, x \geq 0\right\} \subset \mathbb{R}^3.$$

We choose a change of coordinates $x = r\cos\theta$, $y = \frac{1}{2}r\sin\theta$, $z = z$.

We calculate the Jacobian determinant

$$J(r, \theta, z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \frac{1}{2}\sin\theta & \frac{1}{2}r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{r}{2}.$$

Observing the symmetry of the problem in y and z (try calculating the three terms separately and you find that these two terms are odd functions integrated over intervals symmetric on the origin),

$$I = \iiint_V 5x + 4y + 3z\ dxdydz = \iiint_V 5x\ dxdydz.$$

Using the change of variables we obtain

$$I = 5\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \int_{-\sqrt{1-\frac{r^2}{4}}}^{\sqrt{1-\frac{r^2}{4}}} \left(\frac{r}{2}\right)(r\cos\theta)\ dzdrd\theta$$

. Since $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\ d\theta = 2$ and $\int_{-\sqrt{1-\frac{r^2}{4}}}^{\sqrt{1-\frac{r^2}{4}}}\ dz = 2\sqrt{1 - \frac{r^2}{4}}$,

$$I = 5\int_0^2 r^2\sqrt{1 - \tfrac{r^2}{4}}\ dr.$$

It is convenient to change variables letting $r = 2\sin t$ and hence

$$I = 5\int_0^{\frac{\pi}{2}} 8\sin^2 t\cos^2 t\ dt.$$

Using the double angle formulae $\sin 2t = 2\sin t\cos t$ and $\cos 2t = 1 - 2\sin^2 t$ we know that $8\sin^2 t\cos^2 t = 1 - \cos 4t$ and so

$$I = 5\int_0^{\frac{\pi}{2}} 1 - \cos 4t\ dt = \frac{5}{2}\pi.$$

Q5 Solution: We choose the parametric form of the surface S by letting $T = \{(r,\theta) : r \in [0,4], \theta \in [0,2\pi]\}$ and

$$\mathbf{r} : (r,\theta) \mapsto (r\cos\theta, r\sin\theta, r^2).$$

We calculate

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}.$$

We observe that this corresponds to the opposite normal compared to the one that we want so we will need to add a minus sign.

$$\begin{aligned} \iint_S \mathbf{f} \cdot \mathbf{n} \; dS &= -\int_0^4 \int_0^{2\pi} \begin{pmatrix} r\cos\theta \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix} \; d\theta dr \\ &= \int_0^4 \int_0^{2\pi} 2r^3\cos^2\theta - 2r \; d\theta dr. \end{aligned}$$

We calculate that

$$\int_0^4 \int_0^{2\pi} (-2r) \; d\theta dr = -4\pi \left[\frac{1}{2}r^2\right]_0^4 = -2 \cdot 4^2 \pi.$$

On the other hand, using the indefinite integral $\int \cos^2\theta \; d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$, we calculate that

$$\int_0^{2\pi} \cos^2\theta \; d\theta = \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{2\pi} = \pi.$$

This means that

$$\int_0^4 \int_0^{2\pi} 2r^3\cos^2\theta \; d\theta dr = 2\pi \int_0^4 r^3 \; dr = \frac{\pi}{2}\left[r^4\right]_0^4 = \frac{4^4}{2}\pi.$$

Summing together the two parts of the integral we have

$$\iint_S \mathbf{f} \cdot \mathbf{n} \; dS = \left(\frac{4^4}{2} - 2 \cdot 4^2\right)\pi = 96\pi.$$