1. Questions - Call 2 - 15/02/2021

Solutions to each question are included at the end of this document.

Call 2.

(1) **Q1-A**
Fill in the blanks with **integers**, possibly 0 or negative ($\frac{1}{4}$ point each). In this question we calculate the Taylor expansion of $x\cos^2(x)$ about the point $x=0$. Consider first $\cos(2x)$. Its Taylor expansion is

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{\boxed{\text{a}}^n}{(2n)!} x^{\boxed{\text{b}}n}$$

where $\boxed{\text{a}}$: $\boxed{-4 \quad \checkmark}$ $\boxed{\text{b}}$: $\boxed{2 \quad \checkmark}$. Additionally we know that

$$\cos^2 x = \frac{1}{\boxed{\text{c}}} + \frac{\boxed{\text{d}}}{2}\cos(2x)$$

where $\boxed{\text{c}}$: $\boxed{2 \quad \checkmark}$ $\boxed{\text{d}}$: $\boxed{1 \quad \checkmark}$. From this we obtain the Taylor expansion of $x\cos^2 x = \sum_{n=0}^{\infty} a_n x^n$. The first few terms are

$$x\cos^2 x = \boxed{\text{e}} + x + \boxed{\text{f}}x^2 - x^3 + \boxed{\text{g}}x^4 + \frac{\boxed{\text{h}}}{3}x^5 + \boxed{\text{i}}x^6 + \frac{\boxed{\text{j}}}{45}x^7 + o(x^7).$$

$\boxed{\text{e}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{f}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{g}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{h}}$: $\boxed{1 \quad \checkmark}$ $\boxed{\text{i}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{j}}$: $\boxed{-2 \quad \checkmark}$.
We compute the integral:

$$\int_0^x t\cos^2 t\, dt = \boxed{\text{k}} + \boxed{\text{l}}x + \frac{1}{2}x^2 + \boxed{\text{m}}x^3 + \frac{\boxed{\text{n}}}{4}x^4 + \boxed{\text{o}}x^5 + \frac{1}{\boxed{\text{p}}}x^6 + o(x^6)$$

where $\boxed{\text{k}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{l}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{m}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{n}}$: $\boxed{-1 \quad \checkmark}$ $\boxed{\text{o}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{p}}$: $\boxed{18 \quad \checkmark}$. The radius of convergence of the series $x\cos^2 x = \sum_{n=0}^{\infty} a_n x^n$ is (1 point).

0
1
4
infinite ✓

(2) **Q1-B**
Fill in the blanks with integers, possibly 0 or negative ($\frac{1}{4}$ point each). In this question we calculate the Taylor expansion of $x\sin^2(x)$ about the point $x=0$. Consider first $\cos(2x)$. Its Taylor expansion is

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{\boxed{\text{a}}^n}{(2n)!} x^{\boxed{\text{b}}n}$$

where $\boxed{\text{a}}$: $\boxed{-4 \quad \checkmark}$ $\boxed{\text{b}}$: $\boxed{2 \quad \checkmark}$. Additionally we know that

$$\sin^2 x = \frac{1}{\boxed{\text{c}}} + \frac{\boxed{\text{d}}}{2}\cos(2x)$$

where $\boxed{\text{c}}$: $\boxed{2 \quad \checkmark}$ $\boxed{\text{d}}$: $\boxed{-1 \quad \checkmark}$. From this we obtain the Taylor expansion of $x \sin^2 x = \sum_{n=0}^{\infty} a_n x^n$. The first few terms are

$$x \sin^2 x = \boxed{\text{e}}x + \boxed{\text{f}}x^2 + x^3 + \boxed{\text{g}}x^4 + \frac{\boxed{\text{h}}}{3}x^5 + \boxed{\text{i}}x^6 + \frac{\boxed{\text{j}}}{45}x^7 + o(x^7).$$

$\boxed{\text{e}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{f}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{g}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{h}}$: $\boxed{-1 \quad \checkmark}$ $\boxed{\text{i}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{j}}$: $\boxed{2 \quad \checkmark}$. We compute the integral:

$$\int_0^x t \cos^2 t \, dt = \boxed{\text{k}}x + \boxed{\text{l}}x^2 + \boxed{\text{m}}x^3 + \frac{\boxed{\text{n}}}{4}x^4 + \boxed{\text{o}}x^5 - \frac{1}{\boxed{\text{p}}}x^6 + o(x^6)$$

where $\boxed{\text{k}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{l}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{m}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{n}}$: $\boxed{1 \quad \checkmark}$ $\boxed{\text{o}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{p}}$: $\boxed{18 \quad \checkmark}$. The radius of convergence of the series $x \sin^2 x = \sum_{n=0}^{\infty} a_n x^n$ is

0	(1 point).
1	
4	
infinite ✓	

(3) **Q2-A**

In this question we will find and classify the extrema points of $f(x,y) = x^4 + 3xy + 2y^2$. The gradient of this function is

$$\nabla f(x,y) = \begin{pmatrix} \boxed{\text{a}}x^3 + \boxed{\text{b}}y \\ \boxed{\text{c}}x + \boxed{\text{d}}y \end{pmatrix},$$

where $\boxed{\text{a}}$: $\boxed{4\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{3\ \checkmark}$ $\boxed{\text{c}}$: $\boxed{3\ \checkmark}$ $\boxed{\text{d}}$: $\boxed{4\ \checkmark}$.

There are three stationary points: $(-\frac{3}{4}, \frac{\boxed{\text{e}}}{16})$, $(0, \boxed{\text{f}})$ and $(\frac{\boxed{\text{g}}}{4}, \frac{\boxed{\text{h}}}{16})$ where $\boxed{\text{e}}$: $\boxed{9\ \checkmark}$ $\boxed{\text{f}}$: $\boxed{0\ \checkmark}$ $\boxed{\text{g}}$: $\boxed{3\ \checkmark}$ $\boxed{\text{h}}$: $\boxed{-9\ \checkmark}$.

Computing the Hessian at each stationary points we deduce that there are $\boxed{2\ \checkmark}$ relative minima, $\boxed{1\ \checkmark}$ saddle point and $\boxed{0\ \checkmark}$ relative maxima. Moreover $f(x,y)$ is

bounded
unbounded ✓

. Fill in the blanks with **integers**, possibly 0 or negative. Each part is worth $\frac{1}{2}$ point.

(4) **Q2-B**

In this question we will find and classify the extrema points of $f(x,y) = x^4 + 4xy + \frac{9}{2}y^2$. The gradient of this function is

$$\nabla f(x,y) = \begin{pmatrix} \boxed{\text{a}}x^3 + \boxed{\text{b}}y \\ \boxed{\text{c}}x + \boxed{\text{d}}y \end{pmatrix},$$

where $\boxed{\text{a}}$: $\boxed{4\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{4\ \checkmark}$ $\boxed{\text{c}}$: $\boxed{4\ \checkmark}$ $\boxed{\text{d}}$: $\boxed{9\ \checkmark}$.

There are three stationary points: $(-\frac{2}{3}, \frac{\boxed{\text{e}}}{27})$, $(0, \boxed{\text{f}})$ and $(\frac{\boxed{\text{g}}}{3}, \frac{\boxed{\text{h}}}{27})$ where $\boxed{\text{e}}$: $\boxed{8\ \checkmark}$ $\boxed{\text{f}}$: $\boxed{0\ \checkmark}$ $\boxed{\text{g}}$: $\boxed{2\ \checkmark}$ $\boxed{\text{h}}$: $\boxed{-8\ \checkmark}$.

Computing the Hessian at each stationary points we deduce that there are $\boxed{2\ \checkmark}$ relative minima, $\boxed{1\ \checkmark}$ saddle point and $\boxed{0\ \checkmark}$ relative maxima. Moreover $f(x,y)$ is

bounded
unbounded ✓

. Fill in the blanks with **integers**, possibly 0 or negative. Each part is worth $\frac{1}{2}$ point.

(5) **Q3-A**

Choose the correct option in each of the following four places (1 point each). The vector-field

$$\mathbf{f}(x,y) = \begin{pmatrix} 2y(1+x)e^x \\ 2xe^x \end{pmatrix}$$

is ✓
is not

conservative on $\mathbb{R}^2$. The vector-field

$$\mathbf{g}(x,y) = \begin{pmatrix} 2y^2 \\ x+2 \end{pmatrix}$$

is
is not ✓

conservative on $\mathbb{R}^2$. The vector-field

$$\mathbf{h}(x,y) = \begin{pmatrix} 3y(x^2+y^2)^{-1} \\ 3x(x^2+y^2)^{-1} \end{pmatrix}$$

is
is not ✓

conservative on the domain $\{(x,y) : |y| > 0\}$ and

is
is not ✓

conservative on the annular domain $\{(x,y) : 1 \leq x^2+y^2 \leq 4\}$.

Let $\boldsymbol{\alpha}$ denote the anticlockwise triangular path with three straight segments and vertices $(0,0)$, $(1,0)$, $(0,1)$. With $\mathbf{g}$ the vector-field defined above, calculate the line integral $\int \mathbf{g}\ d\boldsymbol{\alpha} =$

-1 ✓
1 (50%)

/6. Fill in the blank with the correct integer, possibly zero or negative (2 points).

(6) **Q3-B**

Choose the correct option in each of the following four places (1 point each). The vector-field

$$\mathbf{f}(x,y) = \begin{pmatrix} 3y(1+x)e^x \\ 3xe^x \end{pmatrix}$$

is ✓
is not

conservative on $\mathbb{R}^2$. The vector-field

$$\mathbf{g}(x,y) = \begin{pmatrix} 4y^2 \\ x+4 \end{pmatrix}$$

is
is not ✓

conservative on $\mathbb{R}^2$. The vector-field

$$\mathbf{h}(x,y) = \begin{pmatrix} 2y(x^2+y^2)^{-1} \\ 2x(x^2+y^2)^{-1} \end{pmatrix}$$

is
is not ✓

conservative on the domain $\{(x,y) : |y| > 0\}$ and

is
is not ✓

conservative on the annular domain $\{(x,y) : 1 \leq x^2+y^2 \leq 4\}$.

Let $\boldsymbol{\alpha}$ denote the anticlockwise triangular path with three straight segments and vertices $(0,0)$, $(1,0)$, $(0,1)$. With $\mathbf{g}$ the vector-field defined above, calculate the line integral $\int \mathbf{g}\ d\boldsymbol{\alpha} =$

-5 ✓
5 (50%)

/6. Fill in the blank with the correct integer, possibly zero or negative (2 points).

(7) **Q4-A**

The set $V = \{(x,y,z) : x^2+y^2 \le 3^2, 0 \le z \le 3 - \sqrt{x^2+y^2}\}$ is a cone of height 3 with base in the xy-plane. The set $W = \{(x,y,z) : (x-\frac{3}{2})^2 + y^2 \le (\frac{3}{2})^2\}$ is a cylinder. Let $D \subset \mathbb{R}^3$ be the subset of the cone V which is contained within the cylinder W. We will calculate the volume of D.

Let $S = \left\{(x,y) : (x-\frac{3}{2})^2 + y^2 \le (\frac{3}{2})^2\right\}$. We can then write

$$D = \left\{(x,y,z) : (x,y) \in S, 0 \le z \le \boxed{\text{a}} - \sqrt{x^2+y^2}\right\}$$

where $\boxed{\text{a}}$: $\boxed{3 \ \checkmark}$ (1 point). This is convenient since the volume of D is equal to $\iint_S \boxed{\text{a}} - \sqrt{x^2+y^2}\ dxdy$.

To proceed we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ which means that the Jacobian is $J(r,\theta) = r$ and the corresponding region is

$$\widetilde{S} = \left\{(r,\theta) : \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], 0 \le r \le \boxed{\text{b}}\cos\theta\right\}$$

where $\boxed{\text{b}}$: $\boxed{3 \ \checkmark}$ (1 point).

This all means that the volume of D is equal to

$$\iint_{\widetilde{S}} r(\boxed{\text{a}} - r)\ drd\theta.$$

Evaluating the integral we show that the volume of D is $\mathrm{Vol}(D) = \boxed{\text{c}}\frac{\pi}{4} + \boxed{\text{d}}$ where $\boxed{\text{c}}$: $\boxed{27 \ \checkmark}$, $\boxed{\text{d}}$: $\boxed{-12 \ \checkmark}$ (2 points each).

(8) **Q4-B**

The set $V = \{(x,y,z) : x^2+y^2 \le 4^2, 0 \le z \le 4 - \sqrt{x^2+y^2}\}$ is a cone of height 4 with base in the xy-plane. The set $W = \{(x,y,z) : (x-2)^2 + y^2 \le 2^2\}$ is a cylinder. Let $D \subset \mathbb{R}^3$ be the subset of the cone V which is contained within the cylinder W. We will calculate the volume of D.

Let $S = \left\{(x,y) : (x-2)^2 + y^2 \le 2^2\right\}$. We can then write

$$D = \left\{(x,y,z) : (x,y) \in S, 0 \le z \le \boxed{\text{a}} - \sqrt{x^2+y^2}\right\}$$

where $\boxed{\text{a}}$: $\boxed{4 \ \checkmark}$ (1 point). This is convenient since the volume of D is equal to $\iint_S \boxed{\text{a}} - \sqrt{x^2+y^2}\ dxdy$.

To proceed we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ which means that the Jacobian is $J(r,\theta) = r$ and the corresponding region is

$$\widetilde{S} = \left\{(r,\theta) : \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], 0 \le r \le \boxed{\text{b}}\cos\theta\right\}$$

where $\boxed{\text{b}}$: $\boxed{4 \ \checkmark}$ (1 point).

This all means that the volume of D is equal to

$$\iint_{\widetilde{S}} r(\boxed{\text{a}} - r)\ drd\theta.$$

Evaluating the integral we show that the volume of D is $\mathrm{Vol}(D) = \boxed{\text{c}}\pi + \frac{\boxed{\text{d}}}{9}$ where $\boxed{\text{c}}$: $\boxed{16 \ \checkmark}$, $\boxed{\text{d}}$: $\boxed{-256 \ \checkmark}$ (2 points each).

(9) **Q5-A**
Consider the parametric surface $\mathbf{r}(S)$ where

$$\mathbf{r}(u, v) = (2u\cos v, 2u\sin v, u^2),$$

and $S = \{(u, v) : 0 \le u \le 1, 0 \le v \le 2\pi\}$. This surface has the Cartesian equation $x^2 + y^2 = \boxed{4 \quad \checkmark}\ z$ (1 point). The fundamental vector product is

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -4u^2\cos v \\ \boxed{\text{a}}\,u^2 \sin v \\ \boxed{\text{b}}\,u \end{pmatrix}$$

where $\boxed{\text{a}}$: $\boxed{-4 \quad \checkmark}$, $\boxed{\text{b}}$: $\boxed{4 \quad \checkmark}$ (1 point each). Now calculate $\left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\|$ and then integrate to calculate the area of the surface Area $(\mathbf{r}(S)) = \boxed{16 \quad \checkmark}$ $\frac{\sqrt{2}}{3}\pi - \frac{8}{3}\pi$ (3 points). (Fill in the blanks with the correct **integers**, possibly zero or negative.)

(10) **Q5-B**
Consider the parametric surface $\mathbf{r}(S)$ where

$$\mathbf{r}(u, v) = (3u\cos v, 3u\sin v, u^2),$$

and $S = \{(u, v) : 0 \le u \le 1, 0 \le v \le 2\pi\}$. This surface has the Cartesian equation $x^2 + y^2 = \boxed{9 \quad \checkmark}\ z$ (1 point). The fundamental vector product is

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -6u^2\cos v \\ \boxed{\text{a}}\,u^2 \sin v \\ \boxed{\text{b}}\,u \end{pmatrix}$$

where $\boxed{\text{a}}$: $\boxed{-6 \quad \checkmark}$, $\boxed{\text{b}}$: $\boxed{9 \quad \checkmark}$ (1 point each). Now calculate $\left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\|$ and then integrate to calculate the area of the surface Area $(\mathbf{r}(S)) = \boxed{13 \quad \checkmark}$ $\frac{\sqrt{13}}{2}\pi - \frac{27}{2}\pi$ (3 points). (Fill in the blanks with the correct **integers**, possibly zero or negative.)

Q1 Solution: In this question we calculate the Taylor expansion of $x\cos^2 x$ or $x\sin^2 x$ about the point $x = 0$.

(1) We recall or calculate that $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ and so

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} x^{2n}$$

.

(2) Additionally we know that $\cos(2x) = \cos^2 x - \sin^2 x$ and hence

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x) \quad \text{and} \quad \sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x).$$

(3) Combining the above we obtain

$$\begin{aligned} x\cos^2 x &= \frac{x}{2} + \frac{x}{2}\cos(2x) = \frac{x}{2} + \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} x^{2n} \\ &= \frac{x}{2} + \frac{x}{2}\left(1 + \frac{-4}{2!}x^2 + \frac{(-4)^2}{4!}x^4 + \frac{(-4)^3}{(6)!}x^6 + \cdots\right) \\ &= x - x^3 + \frac{1}{3}x^5 - \frac{2}{45}x^7 + \cdots \end{aligned}$$

On the other hand

$$\begin{aligned} x\sin^2 x &= \frac{x}{2} - \frac{x}{2}\cos(2x) = \frac{x}{2} - \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} x^{2n} \\ &= \frac{x}{2} - \frac{x}{2}\left(1 + \frac{-4}{2!}x^2 + \frac{(-4)^2}{4!}x^4 + \frac{(-4)^3}{(6)!}x^6 + \cdots\right) \\ &= x^3 - \frac{1}{3}x^5 + \frac{2}{45}x^7 + \cdots \end{aligned}$$

(4) To integrate we use the fact that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ implies $\int_0^x f(t)\,dt = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n!} x^n$ (or simply integrate the above term-by-term) and so

$$\int_0^x t\cos^2 t\;dt = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{18}x^6 - \frac{1}{180}x^8 + \cdots$$

and

$$\int_0^x t\sin^2 t\;dt = \frac{1}{4}x^4 - \frac{1}{18}x^6 + \frac{1}{180}x^8 + \cdots$$

(5) The Taylor expansion for $\cos x$ converges for all x and consequently the Taylor expansions for $\frac{1}{2}(1 + \cos(2x))$ and for $\frac{1}{2}(1 - \cos(2x))$ also converge for all x.

Q2-A Solution: Let $f(x, y) = x^4 + 3xy + 2y^2$.

(1) We calculate the gradient of this function

$$\nabla f(x, y) = \begin{pmatrix} 4x^3 + 3y \\ 3x + 4y \end{pmatrix}.$$

(2) To find the stationary points we suppose $\nabla f(x, y) = 0$ and solve for (x, y). The second equation $(3x + 4y = 0)$ implies that $y = -\frac{3x}{4}$. Substituting this into the first equation $(4x^3 + 3y = 0)$ we obtain $4x^3 - \frac{9}{4}x = 0$. Consequently, either $x = 0$ or $x^2 - \frac{9}{16} = 0$. In the first case we obtain the solution $(0, 0)$. In the second case we have $x = \pm\frac{3}{4}$. Using again that $y = -\frac{3x}{4}$ we obtain the solutions $(-\frac{3}{4}, \frac{9}{16})$ and $(\frac{3}{4}, -\frac{9}{16})$.

(3) We calculate the Hessian matrix

$$\mathbf{H}f(x, y) = \begin{pmatrix} 12x^2 & 3 \\ 3 & 4 \end{pmatrix}.$$

This means that

$$\mathbf{H}f(0, 0) = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix},$$

and

$$\mathbf{H}f(-\tfrac{3}{4}, \tfrac{9}{16}) = \mathbf{H}f(\tfrac{3}{4}, -\tfrac{9}{16}) = \begin{pmatrix} \frac{27}{4} & 3 \\ 3 & 4 \end{pmatrix}.$$

(4) The eigenvalues of $\mathbf{H}f(0, 0)$ are $\lambda = 2 \pm \sqrt{13}$. In particular one is positive and the other is negative so this is a saddle. The eigenvalues of $\mathbf{H}f(-\frac{3}{4}, \frac{9}{16}) = \mathbf{H}f(\frac{3}{4}, -\frac{9}{16})$ are both positive and so these two points are relative minima.

Q2-B Solution: Let $f(x,y) = x^4 + 4xy + \frac{9}{2}y^2$.

(1) We calculate the gradient of this function
$$\nabla f(x,y) = \begin{pmatrix} 4x^3 + 4y \\ 4x + 9y \end{pmatrix}.$$

(2) To find the stationary points we suppose $\nabla f(x,y) = 0$ and solve for (x,y). The second equation $(4x + 9y = 0)$ implies that $y = -\frac{4x}{9}$. Substituting this into the first equation $(4x^3 + 4y = 0)$ we obtain $x^3 - \frac{4}{9}x = 0$. Consequently, either $x = 0$ or $x^2 - \frac{4}{9} = 0$. In the first case we obtain the solution $(0,0)$. In the second case we have $x = \pm\frac{2}{3}$. Using again that $y = -\frac{4x}{9}$ we obtain the solutions $(-\frac{2}{3}, \frac{8}{27})$ and $(\frac{2}{3}, -\frac{8}{27})$.

(3) We calculate the Hessian matrix
$$\mathbf{H}f(x,y) = \begin{pmatrix} 12x^2 & 4 \\ 4 & 9 \end{pmatrix}.$$
This means that
$$\mathbf{H}f(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 9 \end{pmatrix},$$
and
$$\mathbf{H}f(-\tfrac{2}{3}, \tfrac{8}{27}) = \mathbf{H}f(\tfrac{2}{3}, -\tfrac{8}{27}) = \begin{pmatrix} \frac{16}{3} & 4 \\ 4 & 9 \end{pmatrix}.$$

(4) The eigenvalues of $\mathbf{H}f(0,0)$ are $\lambda = \frac{9}{2} \pm \sqrt{(\frac{9}{2})^2 + 16}$. In particular one is positive and the other is negative so this is a saddle. The eigenvalues of $\mathbf{H}f(-\frac{2}{3}, \frac{8}{27}) = \mathbf{H}f(\frac{2}{3}, -\frac{8}{27})$ are both positive and so these two points are relative minima.

Q3-A Solution: We see that $\mathbf{f}(x,y)$ is conservative because, if $\varphi(x,y) = 2xye^x$ then

$$\nabla\varphi(x,y) = \begin{pmatrix} 2y(1+x)e^x \\ 2xe^x \end{pmatrix}.$$

Comparing the y derivative of the first component and the x derivative of the second component we see that the other two vector fields are not conservative on any domain.

Let's calculate the line integral.

(1) It is convenient to divide the path $\boldsymbol{\alpha}$ into three pieces:
- $\boldsymbol{\alpha}_1(t) = (t,0)$, $t \in [0,1]$,
- $\boldsymbol{\alpha}_2(t) = (1-t,t)$, $t \in [0,1]$,
- $\boldsymbol{\alpha}_3(t) = (0,1-t)$, $t \in [0,1]$.

(2) This in turn implies that

$$\boldsymbol{\alpha}_1'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\alpha}_2'(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\alpha}_3'(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

(3) Since

$$\mathbf{g}(x,y) = \begin{pmatrix} 2y^2 \\ x+2 \end{pmatrix},$$

$$\mathbf{g}(\boldsymbol{\alpha}_1(t)) = \begin{pmatrix} 0 \\ t+2 \end{pmatrix}, \quad \mathbf{g}(\boldsymbol{\alpha}_2(t)) = \begin{pmatrix} 2t^2 \\ 1-t+2 \end{pmatrix}, \quad \mathbf{g}(\boldsymbol{\alpha}_3(t)) = \begin{pmatrix} 2(1-t)^2 \\ 2 \end{pmatrix}.$$

And so

$$\begin{aligned} \boldsymbol{\alpha}_1'(t) \cdot \mathbf{g}(\boldsymbol{\alpha}_1(t)) &= 0, \\ \boldsymbol{\alpha}_2'(t) \cdot \mathbf{g}(\boldsymbol{\alpha}_2(t)) &= 1+2-t-2t^2, \\ \boldsymbol{\alpha}_3'(t) \cdot \mathbf{g}(\boldsymbol{\alpha}_3(t)) &= -2. \end{aligned}$$

(4) Finally

$$\begin{aligned} \int \mathbf{g}\ d\boldsymbol{\alpha} &= \int_0^1 1-t-2t^2\ dt \\ &= \left[t - \frac{t^2}{2} - \frac{2t^3}{3}\right]_0^1 = 1 - \frac{1}{2} - \frac{2}{3} = \frac{3-4}{6} = -\frac{1}{6}. \end{aligned}$$

Q3-B Solution: We see that $\mathbf{f}(x,y)$ is conservative because, if $\varphi(x,y) = 3xye^x$ then

$$\nabla\varphi(x,y) = \begin{pmatrix} 3y(1+x)e^x \\ 3xe^x \end{pmatrix}.$$

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Let's calculate the line integral.

(1) It is convenient to divide the path $\boldsymbol{\alpha}$ into three pieces:
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- $\boldsymbol{\alpha}_3(t) = (0,1-t)$, $t \in [0,1]$.

(2) This in turn implies that

$$\boldsymbol{\alpha}_1'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\alpha}_2'(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\alpha}_3'(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

(3) Since

$$\mathbf{g}(x,y) = \begin{pmatrix} 4y^2 \\ x+4 \end{pmatrix},$$

$$\mathbf{g}(\boldsymbol{\alpha}_1(t)) = \begin{pmatrix} 0 \\ t+4 \end{pmatrix}, \quad \mathbf{g}(\boldsymbol{\alpha}_2(t)) = \begin{pmatrix} 4t^2 \\ 1-t+4 \end{pmatrix}, \quad \mathbf{g}(\boldsymbol{\alpha}_3(t)) = \begin{pmatrix} 4(1-t)^2 \\ 4 \end{pmatrix}.$$

And so

$$\begin{aligned} \boldsymbol{\alpha}_1'(t) \cdot \mathbf{g}(\boldsymbol{\alpha}_1(t)) &= 0, \\ \boldsymbol{\alpha}_2'(t) \cdot \mathbf{g}(\boldsymbol{\alpha}_2(t)) &= 1+4-t-4t^2, \\ \boldsymbol{\alpha}_3'(t) \cdot \mathbf{g}(\boldsymbol{\alpha}_3(t)) &= -4. \end{aligned}$$

(4) Finally

$$\begin{aligned} \int \mathbf{g}\, d\boldsymbol{\alpha} &= \int_0^1 1 - t - 4t^2 \, dt \\ &= \left[t - \frac{t^2}{2} - \frac{4t^3}{3} \right]_0^1 = 1 - \frac{1}{2} - \frac{4}{3} = \frac{3-8}{6} = -\frac{5}{6}. \end{aligned}$$

Q4-A Solution: The set $V = \{(x,y,z) : x^2+y^2 \leq 3^2, 0 \leq z \leq 3 - \sqrt{x^2+y^2}\}$ is a cone of height 3 with base in the xy-plane. The set $W = \{(x,y,z) : (x-\frac{3}{2})^2 + y^2 \leq (\frac{3}{2})^2\}$ is a cylinder. Let $D \subset \mathbb{R}^3$ be the subset of the cone V which is contained within the cylinder W. We will calculate the volume of D.

(1) We define $S = \left\{(x,y) : (x-\frac{3}{2})^2 + y^2 \leq (\frac{3}{2})^2\right\}$ (the projection of the cylinder on to the xy-plane). Consequently
$$D = \left\{(x,y,z) : (x,y) \in S, 0 \leq z \leq 3 - \sqrt{x^2+y^2}\right\}.$$
In particular the volume of D is equal to $\iint_S 3 - \sqrt{x^2+y^2}\ dxdy$.

(2) To proceed we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ which means that the Jacobian is $J(r,\theta) = r$ and the corresponding region is (it helps to sketch a picture here)
$$\widetilde{S} = \left\{(r,\theta) : \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right], 0 \leq r \leq 3\cos\theta\right\}.$$
The condition on r is because $(x-\frac{3}{2})^2 + y^2 \leq (\frac{3}{2})^2$ implies $(r\cos\theta - \frac{3}{2})^2 + r^2\sin^2\theta \leq (\frac{3}{2})^2$ which in turn implies that $r - 3\cos\theta \leq 0$.

(3) This all means that the volume of D is equal to
$$\iint_{\widetilde{S}} r(3-r)\ drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{3\cos\theta} 3r - r^2\ dr\right] d\theta$$

(4) For the inner integral we calculate
$$\int_0^{3\cos\theta} 3r - r^2\ dr = \left[\frac{3}{2}r^2 - \frac{1}{3}r^3\right]_0^{3\cos\theta} = \frac{27}{2}\cos^2\theta - \frac{27}{3}\cos^3\theta.$$

(5) Consequently the volume of D is equal to
$$27\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\ d\theta - \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\ d\theta\right).$$
Either from memory or from calculation $\int \cos^2\ d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$ and $\int \cos^3\theta\ d\theta = \sin\theta - \frac{1}{3}\sin^3\theta$. It is also convenient to note that both $\cos^2\theta$ and $\cos^3\theta$ are even.

(6) Putting everything together we have calculated that the volume of D is equal to
$$27\left(\left[\tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\cos\theta\right]_0^{\frac{\pi}{2}} - \tfrac{2}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{2}}\right).$$
And so $\mathrm{Vol}(D) = 27(\frac{\pi}{4} - \frac{4}{9}) = 27\frac{\pi}{4} - 12$.

Q4-B Solution: The set $V = \{(x,y,z) : x^2+y^2 \le 4^2, 0 \le z \le 4 - \sqrt{x^2+y^2}\}$ is a cone of height 4 with base in the xy-plane. The set $W = \{(x,y,z) : (x-2)^2+y^2 \le 2^2\}$ is a cylinder. Let $D \subset \mathbb{R}^3$ be the subset of the cone V which is contained within the cylinder W. We will calculate the volume of D.

(1) We define $S = \left\{(x,y) : (x-2)^2+y^2 \le 2^2\right\}$ (the projection of the cylinder on to the xy-plane). Consequently
$$D = \left\{(x,y,z) : (x,y) \in S, 0 \le z \le 4 - \sqrt{x^2+y^2}\right\}.$$
In particular the volume of D is equal to $\iint_S 4 - \sqrt{x^2+y^2}\ dxdy$.

(2) To proceed we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ which means that the Jacobian is $J(r,\theta) = r$ and the corresponding region is (it helps to sketch a picture here)
$$\widetilde{S} = \left\{(r,\theta) : \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right], 0 \le r \le 4\cos\theta\right\}.$$
The condition on r is because $(x-2)^2+y^2 \le 2^2$ implies $(r\cos\theta-2)^2 + r^2\sin^2\theta \le 2^2$ which in turn implies that $r - 4\cos\theta \le 0$.

(3) This all means that the volume of D is equal to
$$\iint_{\widetilde{S}} r(4-r)\ drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{a\cos\theta} 4r - r^2\ dr\right] d\theta$$

(4) For the inner integral we calculate
$$\int_0^{4\cos\theta} 4r - r^2\ dr = \left[2r^2 - \frac{1}{3}r^3\right]_0^{4\cos\theta} = \frac{64}{2}\cos^2\theta - \frac{64}{3}\cos^3\theta.$$

(5) Consequently the volume of D is equal to
$$64\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\ d\theta - \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\ d\theta\right).$$
Either from memory or from calculation $\int \cos^2\ d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$ and $\int \cos^3\theta\ d\theta = \sin\theta - \frac{1}{3}\sin^3\theta$. It is also convenient to note that both $\cos^2\theta$ and $\cos^3\theta$ are even.

(6) Putting everything together we have calculated that the volume of D is equal to
$$64\left(\left[\tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\cos\theta\right]_0^{\frac{\pi}{2}} - \tfrac{2}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{2}}\right).$$
And so $\mathrm{Vol}(D) = 64(\frac{\pi}{4} - \frac{4}{9}) = 16\pi - \frac{256}{9}$.

Q5-A Solution: Consider the parametric surface $\mathbf{r}(S)$ where

$$\mathbf{r}(u,v) = (2u\cos v, 2u\sin v, u^2),$$

and $S = \{(u,v) : 0 \le u \le 1, 0 \le v \le 2\pi\}$.

(1) We observe that $(2u\cos v)^2 + (2u\sin v)^2 = 4u^2$ and so this surface has the Cartesian equation $x^2 + y^2 = 4z$.

(2) We calculate that

$$\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 2\cos v \\ 2\sin v \\ 2u \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -2u\sin v \\ 2u\cos v \\ 0 \end{pmatrix}$$

and so the fundamental vector product is

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -4u^2\cos v \\ -4u^2\sin v \\ 4u \end{pmatrix}.$$

(3) Hence

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \left(16u^4 + 16u^2\right)^{\frac{1}{2}} = 4u\left(u^2+1\right)^{\frac{1}{2}}.$$

(4) Using this we calculate the surface area

$$\begin{aligned} \operatorname{Area}(\mathbf{r}(S)) &= \iint_S \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, dudv \\ &= \int_0^{2\pi} \left[\int_0^1 \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, du \right] \, dv \\ &= 2\pi \int_0^1 \left(4u\left(u^2+1\right)^{\frac{1}{2}} \right) \, du \\ &= 2\pi \left[\frac{4}{3}\left(u^2+1\right)^{\frac{3}{2}} \right]_0^1 = \frac{8\pi}{3}\left(2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{\pi}{3}\left(16\sqrt{2} - 8\right). \end{aligned}$$

Q5 Solution: Consider the parametric surface $\mathbf{r}(S)$ where

$$\mathbf{r}(u,v) = (3u\cos v, 3u\sin v, u^2),$$

and $S = \{(u,v) : 0 \le u \le 1, 0 \le v \le 2\pi\}$.

(1) We observe that $(3u\cos v)^2 + (3u\sin v)^2 = 9u^2$ and so this surface has the Cartesian equation $x^2 + y^2 = 9z$.

(2) We calculate that

$$\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 3\cos v \\ 3\sin v \\ 2u \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -3u\sin v \\ 3u\cos v \\ 0 \end{pmatrix}$$

and so the fundamental vector product is

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} -6u^2\cos v \\ -6u^2\sin v \\ 9u \end{pmatrix}.$$

(3) Hence

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \left(36u^4 + 81u^2\right)^{\frac{1}{2}} = 3u\left(4u^2 + 9\right)^{\frac{1}{2}}.$$

(4) Using this we calculate the surface area

$$\begin{aligned}
\mathrm{Area}\,(\mathbf{r}(S)) &= \iint_S \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, du dv \\
&= \int_0^{2\pi} \left[\int_0^1 \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, du \right] \, dv \\
&= 2\pi \int_0^1 \left(3u\left(4u^2 + 9\right)^{\frac{1}{2}} \right) \, du \\
&= 2\pi \left[\frac{3}{12}\left(4u^2 + 9\right)^{\frac{3}{2}} \right]_0^1 = \frac{\pi}{2}\left(13^{\frac{3}{2}} - 9^{\frac{3}{2}}\right) = \frac{\pi}{2}\left(13\sqrt{13} - 27\right).
\end{aligned}$$