1. Questions - Call 1 - 18/01/2021

Solutions to each question are included at the end of this document.

Call 1.

(1) **Q1-A**

(Fill in each of the following blanks with the correct **integer**, possibly zero or negative.) We will find a power series solution to the differential equation

$$x^2y'' + xy' + x^2y = 0$$

under the assumption that $f(0) = 192$. By substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ one obtains the equation

$$\sum_{n=2}^{\infty} \left[n^2 a_n + a_{n-2}\right] x^n + \boxed{\text{a}} a_1 x + \boxed{\text{b}} a_0 = 0$$

where $\boxed{\text{a}}$: $\boxed{1 \ \checkmark}$, $\boxed{\text{b}}$: $\boxed{0 \ \checkmark}$ (1 point each). Using the above equation and also the given initial value we know that $a_0 = \boxed{192 \ \checkmark}$ and $a_1 = \boxed{0 \ \checkmark}$ ($\frac{1}{2}$ point each). Furthermore, the above equation implies a recurrence relation between a_{n-2} and a_n which holds for all $n \geq 2$. Derive this recurrence relation and use it to calculate the following coefficients of the power series solution: $a_2 = \boxed{-48 \ \checkmark}$, $a_3 = \boxed{0 \ \checkmark}$, $a_4 = \boxed{3 \ \checkmark}$, $a_5 = \boxed{0 \ \checkmark}$ ($\frac{1}{2}$ point each). The radius of convergence of the power series solution is (1 point).

0
1
192
infinite ✓

(2) **Q1-B**

(Fill in each of the following blanks with the correct **integer**, possibly zero or negative.) We will find a power series solution to the differential equation

$$x^2y'' + xy' + (x^2 - 1)y = 0$$

under the assumption that $f'(0) = 192$. By substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ one obtains the equation

$$\sum_{n=2}^{\infty} \left[(n^2 - 1)a_n + a_{n-2}\right] x^n + \boxed{\text{a}} a_1 x + \boxed{\text{b}} a_0 = 0$$

where $\boxed{\text{a}}$: $\boxed{0 \ \checkmark}$, $\boxed{\text{b}}$: $\boxed{-1 \ \checkmark}$ (1 point each). Using the above equation and also the given initial value we know that $a_0 = \boxed{0 \ \checkmark}$ and $a_1 = \boxed{192 \ \checkmark}$ ($\frac{1}{2}$ point each). Furthermore, the above equation implies a recurrence relation between a_{n-2} and a_n which holds for all $n \geq 2$. Derive this recurrence relation and use it to calculate the following coefficients of the power series solution: $a_2 = \boxed{0 \ \checkmark}$, $a_3 = \boxed{-24 \ \checkmark}$, $a_4 = \boxed{0 \ \checkmark}$, $a_5 = \boxed{1 \ \checkmark}$ ($\frac{1}{2}$ point each). The radius of convergence of the power series solution is (1 point).

0
1
192
infinite ✓

(3) **Q2-A**
(Fill in each of the following blanks with the correct **integer**, possibly zero or negative.) Let $g(x, y) := x^2 + 3xy + y^2 - 45$. We will find the points in the set $\{g(x, y) = 0\} \subset \mathbb{R}^2$ which are closest / furthest from the origin. Introduce a suitable function $f(x, y)$ and apply the Lagrange multiplier method with the constraint $g(x, y) = 0$ in order to find the extrema points. There are $\boxed{2 \quad \checkmark}$ extrema points (2 points). There is a single extrema point in the upper right quadrant and it is equal to ($\boxed{3 \quad \checkmark}$,$\boxed{3 \quad \checkmark}$) (1 point each). The extrema points are:

- all equally the closest points to the origin ✓
- some are the closest and some are the furthest
- all equally the furthest points to the origin
- something else

(2 point). Hint: draw a sketch of the set.

(4) **Q2-B**
(Fill in each of the following blanks with the correct **integer**, possibly zero or negative.) Let $g(x, y) := x^2 + 4xy + y^2 - 24$. We will find the points in the set $\{g(x, y) = 0\} \subset \mathbb{R}^2$ which are closest / furthest from the origin. Introduce a suitable function $f(x, y)$ and apply the Lagrange multiplier method with the constraint $g(x, y) = 0$ in order to find the extrema points. There are $\boxed{2 \quad \checkmark}$ extrema points (2 points). There is a single extrema point in the upper right quadrant and it is equal to ($\boxed{2 \quad \checkmark}$,$\boxed{2 \quad \checkmark}$) (1 point each). The extrema points are:

- all equally the closest points to the origin ✓
- some are the closest and some are the furthest
- all equally the furthest points to the origin
- something else

(2 point). Hint: draw a sketch of the set.

(5) **Q3-A**
Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

(a) If C is the path from $(0, 0)$ to $(\pi, 0)$ along the curve $y = \sin x$ and

$$\mathbf{f}(x, y) = \begin{pmatrix} 2x^2 \\ 4 \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\, d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 2 & \checkmark \\ -2 & (50\%) \end{array}}\ \frac{\pi^3}{3}$.

(b) If C is the line segment from $(1, 0, 1)$ to $(3, 2, 3)$ and

$$\mathbf{f}(x, y) = \begin{pmatrix} xy \\ z \\ y \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\, d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 32 & \checkmark \\ -32 & (50\%) \end{array}}/3$.

(c) If C is the circle $x^2+y^2=1$ traversed once in a counter clockwise direction and

$$\mathbf{f}(x,y)=\begin{pmatrix}(x+y)/(x^2+y^2)\\-(x-y)/(x^2+y^2)\end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll}-2 & \checkmark\\ 2 & (50\%)\end{array}}\ \pi.$

(6) **Q3-B**

Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

(a) If C is the path from $(0,0)$ to $(\pi,0)$ along the curve $y=\sin x$ and

$$\mathbf{f}(x,y)=\begin{pmatrix}3x^2\\6\end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll}3 & \checkmark\\ -3 & (50\%)\end{array}}\ \frac{\pi^3}{3}.$

(b) If C is the line segment from $(1,0,1)$ to $(4,2,4)$ and

$$\mathbf{f}(x,y)=\begin{pmatrix}xy\\z\\y\end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll}17 & \checkmark\\ -17 & (50\%)\end{array}}.$

(c) If C is the circle $x^2+y^2=1$ traversed once in a counter clockwise direction and

$$\mathbf{f}(x,y)=\begin{pmatrix}(x+y)/(x^2+y^2)\\-(x-y)/(x^2+y^2)\end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll}-2 & \checkmark\\ 2 & (50\%)\end{array}}\ \pi.$

(7) **Q4-A**

Fill in the blanks with the correct **integer**, possibly zero or negative. Let V be the solid bounded above by the sphere $x^2+y^2+z^2=4$ and below by the cone $z=\sqrt{x^2+y^2}$. We can write

$$V=\left\{(x,y,z):(x,y)\in D,\ \sqrt{x^2+y^2}\le z\le\sqrt{\boxed{?}-(x^2+y^2)}\right\}\subset\mathbb{R}^3$$

where $\boxed{?}$: $\boxed{4\ \checkmark}$ and $D=\{(x,y): x^2+y^2\le \boxed{2\ \checkmark}\ \}\subset\mathbb{R}^2$ ($\frac{1}{2}$ point each). In order to find the volume of V by evaluating the triple integral $\iiint_V dV$ we change to cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$, $z=z$. The Jacobian $|J(r,\theta,z)|$ is equal to (1 point)

- $r\cos\theta$,
- $r^2\sin\theta$,
- $r\sin\theta$,
- r. ✓

Complete the triple integral and show that the volume of V is equal to $(\boxed{-8\ \checkmark}\sqrt{2}+\boxed{16\ \checkmark})\frac{\pi}{3}$ (2 points each part).

(8) **Q4-B**
Fill in the blanks with the correct **integer**, possibly zero or negative. Let V be the solid bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the cone $z = \sqrt{x^2 + y^2}$. We can write

$$V = \left\{ (x, y, z) : (x, y) \in D, \sqrt{x^2 + y^2} \le z \le \sqrt{\boxed{?} - (x^2 + y^2)} \right\} \subset \mathbb{R}^3$$

where $\boxed{?}$: $\boxed{2 \quad \checkmark}$ and $D = \{(x, y) : x^2 + y^2 \le \boxed{1 \quad \checkmark}\ \} \subset \mathbb{R}^2$ ($\frac{1}{2}$ point each). In order to find the volume of V by evaluating the triple integral $\iiint_V dV$ we change to cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The Jacobian $|J(r, \theta, z)|$ is equal to (1 point)

- $r\cos\theta$,
- $r^2 \sin\theta$,
- $r \sin\theta$,
- r. ✓

Complete the triple integral and show that the volume of V is equal to $(\boxed{4 \quad \checkmark}\sqrt{2} + \boxed{-4 \quad \checkmark})\frac{\pi}{3}$ (2 points each part).

(9) **Q5-A**
Fill in each blank with the correct **integer**, possibly zero or negative.
Consider the surface $S = \{(x, y, z) : x^2 + y^2 = z, z \le 9\}$. A possible choice for the parametric form of the surface S is to let $T = \{(r, \theta) : r \in [0, \boxed{3 \quad \checkmark}\ ], \theta \in [0, 2\pi]\}$ ($\frac{1}{2}$ point) and

$$\mathbf{r} : (r, \theta) \mapsto \left(r\cos\theta, \boxed{a}, \boxed{b} \right).$$

For this parametric representation we calculate that

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \boxed{c} \\ \boxed{d} \\ \boxed{e} \end{pmatrix}.$$

The missing formulae are ($\frac{1}{2}$ point each):

$\boxed{a}$:
• $r\sin\theta$ ✓ • $r\cos\theta$ • r • r^2 • $r^2\cos\theta$ • $r^2\sin\theta$

$\boxed{b}$:
• $r\sin\theta$ • $r\cos\theta$ • r • r^2 ✓ • $r^2\cos\theta$ • $r^2\sin\theta$

$\boxed{c}$:
• $2r\sin\theta$ • r • $2r^2$ • $-r^2\sin\theta$ • $-2r^2\cos\theta$ ✓

$\boxed{d}$:
• $2r\sin\theta$ • r • r^2 • $-2r^2\sin\theta$ ✓ • $-2r^2\cos\theta$

$\boxed{e}$:
• $2r\sin\theta$ • r ✓ • r^2 • $-2r^2\sin\theta$ • $-2r^2\cos\theta$

Consider the vector field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} y^2 \\ 0 \\ z \end{pmatrix}$$

and let $\mathbf{n}$ be the unit normal to S which has **positive** z-component. The surface integral $\iint_S \mathbf{f} \cdot \mathbf{n}\, dS = \boxed{\begin{array}{ll} 81 & \checkmark \\ -81 & (50\%) \end{array}}\ \frac{\pi}{2}$ (3 points).

(10) **Q5-B**

Fill in each blank with the correct **integer**, possibly zero or negative.

Consider the surface $S = \{(x, y, z) : x^2 + y^2 = z, z \leq 16\}$. A possible choice for the parametric form of the surface S is to let $T = \{(r, \theta) : r \in [0,$ $\boxed{4 \quad \checkmark}$ $], \theta \in [0, 2\pi]\}$ ($\frac{1}{2}$ point) and

$$\mathbf{r} : (r, \theta) \mapsto \left(r\cos\theta, \boxed{\text{a}}, \boxed{\text{b}}\right).$$

For this parametric representation we calculate that

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \boxed{\text{c}} \\ \boxed{\text{d}} \\ \boxed{\text{e}} \end{pmatrix}.$$

The missing formulae are ($\frac{1}{2}$ point each):

$\boxed{\text{a}}$:

• $r\sin\theta$ ✓ • $r\cos\theta$ • r • r^2 • $r^2\cos\theta$ • $r^2\sin\theta$

$\boxed{\text{b}}$:

• $r\sin\theta$ • $r\cos\theta$ • r • r^2 ✓ • $r^2\cos\theta$ • $r^2\sin\theta$

$\boxed{\text{c}}$:

• $2r\sin\theta$ • r • $2r^2$ • $-r^2\sin\theta$ • $-2r^2\cos\theta$ ✓

$\boxed{\text{d}}$:

• $2r\sin\theta$ • r • r^2 • $-2r^2\sin\theta$ ✓ • $-2r^2\cos\theta$

$\boxed{\text{e}}$:

• $2r\sin\theta$ • r ✓ • r^2 • $-2r^2\sin\theta$ • $-2r^2\cos\theta$

Consider the vector field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} y^2 \\ 0 \\ z \end{pmatrix}$$

and let $\mathbf{n}$ be the unit normal to S which has **positive** z-component. The surface integral $\iint_S \mathbf{f} \cdot \mathbf{n}\, dS =$ $\boxed{\begin{array}{ll} 128 & \checkmark \\ -128 & (50\%) \end{array}}$ π (3 points).

Q1 Solution:

We consider, for $m \in \{0,1\}$, the equation[1]

$$x^2y'' + xy' + (x^2 - m^2)y = 0.$$

Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ one obtains the equation

$$x^2\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) + x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) - m^2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

and so

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=0}^{\infty} m^2 a_n x^n = 0.$$

Shifting the index in the third sum

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n - \sum_{n=0}^{\infty} m^2 a_n x^n = 0.$$

Consequently, separating the terms of x^1 and x^0,

$$\sum_{n=2}^{\infty} \left[n(n-1)a_n + n a_n + a_{n-2} - m^2 a_n\right] x^n + a_1 x - m^2 a_1 x - m^2 a_0 = 0.$$

Equivalently

$$\sum_{n=2}^{\infty} \left[(n^2 - m^2)a_n + a_{n-2}\right] x^n + (1 - m^2)a_1 x - m^2 a_0 = 0.$$

In the case that $m = 0$ this means that $a_1 = 0$ and, as an immediate consequence of the given initial value, $a_0 = 192$. On the other hand, in the case that $m = 1$ this means that $a_0 = 0$ and, as an immediate consequence of the given initial value, $a_1 = 192$. Considering the first sum in the above equation, we see that, for all $n \geq 2$, $(n^2 - m^2)a_n + a_{n-2} = 0$ and so

$$a_n = -\frac{a_{n-2}}{n^2 - m^2}.$$

The recurrence relation would allow us to determine the coefficients of the power series solution to this differential equation. For the first few terms we calculate that

$$\begin{aligned}
a_2 &= -\frac{1}{2^2 - m^2} a_0, \\
a_3 &= -\frac{1}{3^2 - m^2} a_1, \\
a_4 &= -\frac{1}{4^2 - m^2} a_2 = \frac{1}{(4^2 - m^2)(2^2 - m^2)} a_0, \\
a_5 &= -\frac{1}{5^2 - m^2} a_3 = \frac{1}{(5^2 - m^2)(3^2 - m^2)} a_1.
\end{aligned}$$

Using the ratio test on the recurrence relation shows that the radius of converge of the power series solution is infinite.

---

[1] This equation, for some constant m, is called Bessel's differential equation and has various applications including: electromagnetic waves in a cylindrical waveguide; heat conduction in a cylindrical object and the modes of vibration of a circular drum.

Q2 SOLUTION:

Let $g(x,y) := x^2 + axy + y^2 - b$. Either $(a = 3, b = 45)$ or $(a = 4, b = 24)$. One suitable choice of function for finding points closest / furthest from the origin is $f(x,y) = x^2 + y^2$. We calculate

$$\nabla g(x,y) = \begin{pmatrix} 2x + ay \\ ax + 2y \end{pmatrix}, \quad \nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$

According to the Lagrange multiplier method we introduce $\lambda \in \mathbb{R}$ and write

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 2x + ay \\ ax + 2y \end{pmatrix}.$$

Multiplying the first line by y and the second line by x we obtain that $2xy = 2\lambda xy + a\lambda y^2$ and $2xy = \lambda a x^2 + 2\lambda xy$. Equating these implies that $2\lambda xy + a\lambda y^2 = \lambda a x^2 + 2\lambda xy$ and so $y^2 = x^2$. We treat the case $y = x$ and $y = -x$ independently.

Case $y = x$: Substituting into $x^2 + axy + y^2 - b = 0$ we obtain $(2+a)x^2 = b$. Consequently $x = \pm\frac{b}{2+a}$. This gives two solutions: $(\sqrt{\frac{b}{2+a}}, \sqrt{\frac{b}{2+a}})$ and $(-\sqrt{\frac{b}{2+a}}, -\sqrt{\frac{b}{2+a}})$.

Case $y = -x$: Substituting into $x^2 + axy + y^2 - b = 0$ we obtain $(2-a)x^2 = b$. However $(2-a)$ is negative since $a > 2$ and so there are no solutions in this case.

This set consists of two curves and has mirror symmetry along the line $y = x$ and the line $y = -x$. The set is unbounded, it contains points infinitely far from the origin. The two extrema are the two points equally close to the origin.

Q3 Solution:

(a) Let C be the path from $(0,0)$ to $(\pi,0)$ along the curve $y=\sin x$ and

$$\mathbf{f}(x,y)=\begin{pmatrix} ax^2 \\ 2a \end{pmatrix}.$$

Choose $\boldsymbol{\alpha}(t):=(t,\sin t)$, $t\in[0,\pi]$. We calculate

$$\boldsymbol{\alpha}'(t)=\begin{pmatrix} 1 \\ \cos t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t))=\begin{pmatrix} at^2 \\ 2a \end{pmatrix}.$$

Consequently $\boldsymbol{\alpha}'(t)\cdot\mathbf{f}(\boldsymbol{\alpha}(t))=at^2+2a\cos t$. And so

$$\begin{aligned}\int \mathbf{f}\ d\boldsymbol{\alpha} &= \int_0^\pi (at^2+2a\cos t)\ dt \\ &= a\int_0^\pi t^2\ dt+2a\int_0^\pi \cos t\ dt = a\left[\frac{1}{3}t^3\right]_0^\pi+0=a\frac{\pi^3}{3}.\end{aligned}$$

(b) Let C be the line segment from $(1,0,1)$ to $(a+1,2,a+1)$ and

$$\mathbf{f}(x,y)=\begin{pmatrix} xy \\ z \\ y \end{pmatrix}.$$

Choose $\boldsymbol{\alpha}(t):=(1+ta,2t,1+ta)$, $t\in[0,1]$. We calculate

$$\boldsymbol{\alpha}'(t)=\begin{pmatrix} a \\ 2 \\ a \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t))=\begin{pmatrix} 2t(1+ta) \\ 1+ta \\ 2t \end{pmatrix}.$$

Consequently $\boldsymbol{\alpha}'(t)\cdot\mathbf{f}(\boldsymbol{\alpha}(t))=2at(1+ta)+2(1+ta)+2at=2a^2t^2+6at+2$. And so

$$\begin{aligned}\int \mathbf{f}\ d\boldsymbol{\alpha} &= \int_0^1 (2a^2t^2+6at+2)\ dt \\ &= \left[\frac{2a^2}{3}t^3+3at^2+2t\right]_0^1=\frac{2a^2}{3}+3a+2.\end{aligned}$$

(c) Let $\boldsymbol{\alpha}(t)=(\cos t,\sin t)$, $t\in[0,2\pi]$. Calculate

$$\boldsymbol{\alpha}'(t)=\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t))=\begin{pmatrix} \cos t+\sin t \\ -\cos t+\sin t \end{pmatrix}.$$

Moreover $\mathbf{f}(\boldsymbol{\alpha}(t))\cdot\boldsymbol{\alpha}'(t)=-\sin t\cos t-\sin^2 t-\cos^2 t+\sin t\cos t=-1$. Consequently

$$\int_C \mathbf{f}\ d\boldsymbol{\alpha}=\int_0^{2\pi}(-1)\ dt=-2\pi.$$

Q4 SOLUTION:

Let V be the solid bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by the cone $z=\sqrt{x^2+y^2}$. (Either $a=2$ or $a=\sqrt{2}$.) We can write

$$V=\left\{(x,y,z):(x,y)\in D, \sqrt{x^2+y^2}\le z\le \sqrt{a^2-(x^2+y^2)}\right\}\subset \mathbb{R}^3$$

where $D=\{(x,y): x^2+y^2\le \frac{a^2}{2}\}$. In cylindrical coordinates the solid V corresponds to the set $\widetilde{V}$ where,

$$\widetilde{V}=\left\{(r,\theta,z): r\in\left[0,\frac{a}{\sqrt{2}}\right], \theta\in[0,2\pi], r\le z\le\sqrt{a^2-r^2}\right\},$$

The volume integral is

$$\begin{aligned}\iiint_V dV &= 2\pi\int_0^{\frac{a}{\sqrt{2}}} r\left(\sqrt{a^2-r^2}-r\right)\,dr\\ &= 2\pi\int_0^{\frac{a}{\sqrt{2}}} r(a^2-r^2)^{\frac{1}{2}}\,dr-2\pi\int_0^{\frac{a}{\sqrt{2}}} r^2\,dr.\end{aligned}$$

Observe the indefinite integral $\int r(a^2-r^2)^{\frac{1}{2}}\,dr=-\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}+C$. This means that

$$\begin{aligned}\iiint_V dV &= -\frac{2}{3}\pi\left[(a^2-r^2)^{\frac{3}{2}}\right]_0^{\frac{a}{\sqrt{2}}}-2\pi\left[\frac{r^3}{3}\right]_0^{\frac{a}{\sqrt{2}}}\\ &= -\frac{2}{3}\pi\left(\left(a^2-\frac{a^2}{2}\right)^{\frac{3}{2}}-\left(a^2\right)^{\frac{3}{2}}\right)-\frac{2}{3}\pi\left(\frac{a^3}{2\sqrt{2}}\right)\\ &= -\frac{2}{3}\pi\left(\frac{a^3}{2\sqrt{2}}-a^3+\frac{a^3}{2\sqrt{2}}\right)=\frac{\pi}{3}\left(2-\sqrt{2}\right)a^3.\end{aligned}$$

Note that $\left(2-\sqrt{2}\right)(\sqrt{2})^3=4\sqrt{2}-4$ and $\left(2-\sqrt{2}\right)(2)^3=16-8\sqrt{2}$.

Q5 Solution:

Consider the surface $S = \left\{(x, y, z) : x^2 + y^2 = z, z \leq a^2\right\}$. Either $a = 3$ or $a = 4$. We choose the parametric form of the surface S by letting $T = \{(r, \theta) : r \in [0, a], \theta \in [0, 2\pi]\}$ and

$$\mathbf{r} : (r, \theta) \mapsto (r\cos\theta, r\sin\theta, r^2).$$

We calculate

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}.$$

We observe that this corresponds to the required normal. Since $\mathbf{f}(x, y, z) = \left(\begin{smallmatrix} y^2 \\ 0 \\ z \end{smallmatrix}\right)$,

$$\mathbf{f}(\mathbf{r}(r, \theta)) = \left(\begin{smallmatrix} r^2\sin^2\theta \\ 0 \\ r^2 \end{smallmatrix}\right).$$

Consequently

$$\begin{aligned} \iint_S \mathbf{f} \cdot \mathbf{n}\ dS &= \int_0^a \int_0^{2\pi} \begin{pmatrix} r^2\sin^2\theta \\ 0 \\ r^2 \end{pmatrix} \cdot \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}\ d\theta dr \\ &= \int_0^a \int_0^{2\pi} (-2r^4\sin^2\theta\cos\theta + r^3)\ d\theta dr. \end{aligned}$$

We calculate that

$$\int_0^a \int_0^{2\pi} r^3\ d\theta dr = 2\pi\left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{2}\pi.$$

On the other hand, using the indefinite integral $\int \sin^2\theta\cos\theta\ d\theta = -\frac{1}{3}\sin^3\theta + C$, we calculate that (or observe the symmetry)

$$\int_0^{2\pi} \sin^2\theta\cos\theta\ d\theta = -\frac{1}{3}\left[\sin^3\theta\right]_0^{2\pi} = 0.$$

This means that

$$\iint_S \mathbf{f} \cdot \mathbf{n}\ dS = \frac{a^4}{2}\pi.$$