Call6.

(1) **Q1**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$). If a fraction is negative, **put the negative sign on the numerator** ($\frac{-1}{2}$ but not $\frac{1}{-2}$).

Find a power series solution of the following differential equation.

$$(1-x)^2 y'' - 2y = 0.$$

By substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, one has

$$\sum_{n=0}^{\infty} \left[(n+\boxed{\text{a}})(n+\boxed{\text{b}})a_{n+2} + \boxed{\text{c}}n(n+\boxed{\text{d}})a_{n+1} + (n+\boxed{\text{e}})(n+\boxed{\text{f}})a_n \right] x^n = 0,$$

where $\boxed{\text{a}} > \boxed{\text{b}}$, $\boxed{\text{e}} > \boxed{\text{f}}$. $\boxed{\text{a}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{b}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{c}}$: $\boxed{-2 \ \checkmark}$
$\boxed{\text{d}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{e}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{f}}$: $\boxed{-2 \ \checkmark}$

From this we obtain $(n+\boxed{\text{a}})(n+\boxed{\text{b}})a_{n+2} + \boxed{\text{c}}n(n+\boxed{\text{d}})a_{n+1} + (n+\boxed{\text{e}})(n+\boxed{\text{f}})a_n = 0$. This is equivalent to

$$(n+1)[(n+\boxed{\text{g}})(a_{n+2}-a_{n+1}) - (n+\boxed{\text{h}})(a_{n+1}-a_n)] = 0.$$

$\boxed{\text{g}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{h}}$: $\boxed{-2 \ \checkmark}$

Let us put $b_n = a_{n+1} - a_n$. Then we have $b_{n+1} = \frac{n+\boxed{\text{h}}}{n+\boxed{\text{g}}} b_n$.

If $y(0) = 1, y'(0) = -2$, then we have $b_0 = \boxed{\text{i}}$, $b_1 = \boxed{\text{j}}$, $b_2 = \boxed{\text{k}}$ and $a_2 = \boxed{\text{l}}$, $a_3 = \boxed{\text{m}}$, $a_4 = \boxed{\text{n}}$.

$\boxed{\text{i}}$: $\boxed{-3 \ \checkmark}$ $\boxed{\text{j}}$: $\boxed{3 \ \checkmark}$ $\boxed{\text{k}}$: $\boxed{-1 \ \checkmark}$ $\boxed{\text{l}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{m}}$: $\boxed{0 \ \checkmark}$
$\boxed{\text{n}}$: $\boxed{0 \ \checkmark}$

In this case, the series $\sum_{n=0}^{\infty}$ converges for x:

- -10 ✓
- -1 ✓
- -0.1 ✓
- 0 ✓
- 0.1 ✓
- 1 ✓
- 10 ✓
- 100 ✓

If $y(0) = 1, y'(0) = 1$, similarly as above, $a_2 = \boxed{\text{o}}$, $a_3 = \boxed{\text{p}}$, $a_4 = \boxed{\text{q}}$. $\boxed{\text{o}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{p}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{q}}$: $\boxed{1 \ \checkmark}$

In this case, the series $\sum_{n=0}^{\infty}$ converges for x:

- -10
- -1
- -0.1 ✓
- 0 ✓
- 0.1 ✓
- 1
- 10
- 100

Use $y'(x) = \sum_{n=1} na_n x^{n-1}$ and $y''(x) = \sum_{n=2} n(n-1)a_n x^{n-2}$, and one obtains the equation

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2n(n+1)a_{n+1} + (n+1)(n-2)a_n]x^n = 0,$$

and this must hold for all x, so it follows that

$$(n+2)(n+1)a_{n+2} - 2n(n+1)a_{n+1} + (n+1)(n-2)a_n = 0$$

The factor $n+1$ is common and nonzero, so this is equivalent to

$$\begin{aligned} 0 &= (n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n \\ &= (n+2)(a_{n+2} - a_{n+1}) - (n-2)(a_{n+1} - a_n). \end{aligned}$$

By putting $b_n = a_{n+1} - a_n$, we have a relation for b_n:

$$b_{n+1} = \frac{n-2}{n+2} b_n.$$

From the initial condition, we get the values of a_0, a_1 and hence of b_0. We can determine recursively the values of b_n, and also of a_n.

For the case $y(0) = 1, y'(0) = -2$, one has $b_0 = -3, b_1 = 3, b_2 = -1, b_n = 0$ for $n \geq 3$. Accordingly, $a_0 = 1, a_1 = -2, a_2 = 1, a_n = 0$ for $n \geq 3$. This is a polynomial and the radius of convergence is ∞.

For the case $y(0) = 1, y'(0) = 1$, one has $b_0 = 0$ and $b_n = 0$ for all n. Accordingly, $a_n = 1$ for all n. This has the radius of convergence is 1, and for $x = \pm 1$ the series does not converge.

(2) **Q1**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$). If a fraction is negative, **put the negative sign on the numerator** ($\frac{-1}{2}$ but not $\frac{1}{-2}$).

Find a power series solution of the following differential equation.

$$(2-x)^2y'' - 2y = 0.$$

By substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, one has

$$\sum_{n=0}^{\infty} \left[4(n+\boxed{\text{a}})(n+\boxed{\text{b}})a_{n+2} + \boxed{\text{c}}n(n+\boxed{\text{d}})a_{n+1} + (n+\boxed{\text{e}})(n+\boxed{\text{f}})a_n\right] x^n = 0,$$

where $\boxed{\text{a}} > \boxed{\text{b}}$, $\boxed{\text{e}} > \boxed{\text{f}}$. $\boxed{\text{a}}$: $\boxed{2\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{c}}$: $\boxed{-4\ \checkmark}$ $\boxed{\text{d}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{e}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{f}}$: $\boxed{-2\ \checkmark}$

From this we obtain $(n+\boxed{\text{a}})(n+\boxed{\text{b}})a_{n+2}+\boxed{\text{c}}n(n+\boxed{\text{d}})a_{n+1}+(n+\boxed{\text{e}})(n+\boxed{\text{f}})a_n = 0$. This is equivalent to

$$(n+1)[2(n+\boxed{\text{g}})(\boxed{\text{h}}a_{n+2} - a_{n+1}) - (n-2)(\boxed{\text{h}}a_{n+1} - a_n)] = 0.$$

$\boxed{\text{g}}$: $\boxed{2\ \checkmark}$ $\boxed{\text{h}}$: $\boxed{2\ \checkmark}$

Let us put $b_n = \boxed{\text{h}}a_{n+1} - a_n$. Then we have $b_{n+1} = \frac{n-2}{2(n+\boxed{\text{g}})}b_n$.

If $y(0) = 4, y'(0) = -4$, then we have $b_0 = \boxed{\text{i}}$, $b_1 = \boxed{\text{j}}$, $b_2 = \boxed{\text{k}}$ and $a_2 = \boxed{\text{l}}$, $a_3 = \boxed{\text{m}}$, $a_4 = \boxed{\text{n}}$.

$\boxed{\text{i}}$: $\boxed{-12\ \checkmark}$ $\boxed{\text{j}}$: $\boxed{6\ \checkmark}$ $\boxed{\text{k}}$: $\boxed{-1\ \checkmark}$ $\boxed{\text{l}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{m}}$: $\boxed{0\ \checkmark}$ $\boxed{\text{n}}$: $\boxed{0\ \checkmark}$

In this case, the series $\sum_{n=0}^{\infty}$ converges for x:

- -10 ✓
- -1 ✓
- -0.1 ✓
- 0 ✓
- 0.1 ✓
- 1 ✓
- 10 ✓
- 100 ✓

If $y(0) = \frac{1}{2}, y'(0) = \frac{1}{4}$, similarly as above, $a_2 = \frac{1}{\boxed{\text{o}}}$, $a_3 = \frac{1}{\boxed{\text{p}}}$, $a_4 = \frac{1}{\boxed{\text{q}}}$. $\boxed{\text{o}}$: $\boxed{8\ \checkmark}$ $\boxed{\text{p}}$: $\boxed{16\ \checkmark}$ $\boxed{\text{q}}$: $\boxed{32\ \checkmark}$

In this case, the series $\sum_{n=0}^{\infty}$ converges for x:

- -10

- -1 ✓
- -0.1 ✓
- 0 ✓
- 0.1 ✓
- 1 ✓
- 10
- 100

> This can be solved in a very similar way to the above, with $b_n = 2a_{n+1} - a_n$.

(3) **Q2**

Fill in the blanks with **integers (possibly** 0 **or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Let us find the points that are nearest and fartherest from $(0,0,0)$ on the line defined by

$$x + y = 1, x - z = 1.$$

By applying Lagrange's multiplier method, we find stationary points of a function $f(x,y,z)$ under the condition $g(x,y,z) = x+y-1 = 0$ and $h(x,y,z) = x - z - 1 = 0$.

Choose functions $f(x,y,z)$ which are appropriate for this purpose.

- $x + y + z$
- $x^2 + y^2 + z^2$ ✓
- $x^3 + y^3 + z^3$
- $(x^2 + y^2 + z^2)^{\frac{1}{2}}$ ✓
- $(x^3 + y^3 + z^3)^{\frac{1}{3}}$
- $\exp(x^2 + y^2 + z^2)$ ✓
- $\sin(x^2 + y^2 + z^2)$

Compute the gradient ∇g:

$$\nabla g(x,y,z) = (\boxed{\text{a}}, \boxed{\text{b}}, \boxed{\text{c}}).$$

$\boxed{\text{a}}$: $\boxed{1 ✓}$ $\boxed{\text{b}}$: $\boxed{1 ✓}$ $\boxed{\text{c}}$: $\boxed{0 ✓}$

$$\nabla h(x,y,z) = (\boxed{\text{a}}, \boxed{\text{b}}, \boxed{\text{c}}).$$

$\boxed{\text{a}}$: $\boxed{1 ✓}$ $\boxed{\text{b}}$: $\boxed{0 ✓}$ $\boxed{\text{c}}$: $\boxed{-1 ✓}$

Choose an appropriate f. By Lagrange's multiplier method, introduce $\lambda_1, \lambda_2 \in \mathbb{R}$ and solve the equation $\nabla f(x,y,z) =$

$\lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z)$. There is one solution $(x,y,z) = \left(\frac{\boxed{\text{d}}}{\boxed{\text{e}}}, \frac{\boxed{\text{f}}}{\boxed{\text{g}}}, \frac{\boxed{\text{h}}}{\boxed{\text{i}}}\right)$.

$\boxed{\text{d}}$: $\boxed{2\ \checkmark}$ $\boxed{\text{e}}$: $\boxed{3\ \checkmark}$ $\boxed{\text{f}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{g}}$: $\boxed{3\ \checkmark}$ $\boxed{\text{h}}$: $\boxed{-1\ \checkmark}$ $\boxed{\text{i}}$: $\boxed{3\ \checkmark}$

Choose a correct statement.

- This solution is a minimum. ✓
- This solution is a maximum.
- This solution is a saddle point.

The question asks to find the point nearest or fartherst from $(0,0,0)$. This is equivalent to minimize or maximize the distance from $(0,0,0)$ to (x,y,z), namely $\sqrt{x^2+y^2+z^2}$. Actually, the minumal and maximal points do not change if one compose it with a monotonic function, so one can also consider $f(x,y,z) = \sqrt{x^2+y^2+z^2}, x^2+y^2+z^2, \exp(x^2+y^2+z^2)$. But sin is not monotonic.

The easiest choice is $f(x,y,z) = x^2+y^2+z^2$. To find an extremal point of f under the condition $g(x,y,z) = h(x,y,z) = 0$, by Lagrange's multiplier method, introduce λ_1, λ_2 and put $\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$. In this case, it gives

$$(2x, 2y, 2z) = (\lambda_1, \lambda_1, 0) + (\lambda_2, 0, -\lambda_2)$$

together with $x+y-1=0, x-z-1=0$.

This solution is a minumum, because the equation $g = h =$ gives a plane, and there is a nearest point from $(0,0,0)$, but no fartherest point or a saddle point.

(4) **Q2**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Let us find the points that are nearest and fartherest from $(0,0,0)$ on the line defined by

$$y+z=1, y-x=1.$$

By applying Lagrange's multiplier method, we find stationary points of a function $f(x,y,z)$ under the condition $g(x,y,z) = y+z-1=0$ and $h(x,y,z) = y-x-1=0$.

Choose functions $f(x,y,z)$ which are appropriate for this purpose.

- $x+y+z$
- $x^2+y^2+z^2$ ✓
- $x^3+y^3+z^3$
- $(x^2+y^2+z^2)^{\frac{1}{2}}$ ✓
- $(x^3+y^3+z^3)^{\frac{1}{3}}$
- $\exp(x^2+y^2+z^2)$ ✓
- $\sin(x^2+y^2+z^2)$

Compute the gradient ∇g:

$$\nabla g(x,y,z) = (\boxed{\text{a}}, \boxed{\text{b}}, \boxed{\text{c}}).$$

$\boxed{\text{a}}$: $\boxed{0\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{c}}$: $\boxed{1\ \checkmark}$

$$\nabla h(x,y,z) = (\boxed{\text{a}}, \boxed{\text{b}}, \boxed{\text{c}}).$$

$\boxed{\text{a}}$: $\boxed{-1\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{c}}$: $\boxed{0\ \checkmark}$

Choose an appropriate f. By Lagrange's multiplier method, introduce $\lambda_1, \lambda_2 \in \mathbb{R}$ and solve the equation $\nabla f(x,y,z) = \lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z)$. There is one solution $(x,y,z) = (\frac{\boxed{\text{d}}}{\boxed{\text{e}}}, \frac{\boxed{\text{f}}}{\boxed{\text{g}}}, \frac{\boxed{\text{h}}}{\boxed{\text{i}}})$.

$\boxed{\text{d}}$: $\boxed{-1\ \checkmark}$ $\boxed{\text{e}}$: $\boxed{3\ \checkmark}$ $\boxed{\text{f}}$: $\boxed{2\ \checkmark}$ $\boxed{\text{g}}$: $\boxed{3\ \checkmark}$ $\boxed{\text{h}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{i}}$: $\boxed{3\ \checkmark}$

Choose a correct statement.

- This solution is a minimum. ✓
- This solution is a maximum.
- This solution is a saddle point.

(5) **Q2**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Let us find the points that are nearest and fartherest from $(0,0,0)$ on the line defined by

$$z+x=1, z-y=1.$$

By applying Lagrange's multiplier method, we find stationary points of a function $f(x,y,z)$ under the condition $g(x,y,z) = z+x-1 = 0$ and $h(x,y,z) = z-y-1 = 0$.

Choose functions $f(x,y,z)$ which are appropriate for this purpose.

- $x+y+z$
- $x^2+y^2+z^2$ ✓
- $x^3+y^3+z^3$
- $(x^2+y^2+z^2)^{\frac{1}{2}}$ ✓
- $(x^3+y^3+z^3)^{\frac{1}{3}}$
- $\exp(x^2+y^2+z^2)$ ✓
- $\sin(x^2+y^2+z^2)$

Compute the gradient ∇g:

$$\nabla g(x,y,z) = (\boxed{\mathrm{a}}, \boxed{\mathrm{b}}, \boxed{\mathrm{c}}).$$

$\boxed{\mathrm{a}}$: $\boxed{1\ ✓}$ $\boxed{\mathrm{b}}$: $\boxed{0\ ✓}$ $\boxed{\mathrm{c}}$: $\boxed{1\ ✓}$

$$\nabla h(x,y,z) = (\boxed{\mathrm{a}}, \boxed{\mathrm{b}}, \boxed{\mathrm{c}}).$$

$\boxed{\mathrm{a}}$: $\boxed{0\ ✓}$ $\boxed{\mathrm{b}}$: $\boxed{-1\ ✓}$ $\boxed{\mathrm{c}}$: $\boxed{1\ ✓}$

Choose an appropriate f. By Lagrange's multiplier method, introduce $\lambda_1, \lambda_2 \in \mathbb{R}$ and solve the equation $\nabla f(x,y,z) = \lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z)$. There is one solution $(x,y,z) = (\frac{\boxed{\mathrm{d}}}{\boxed{\mathrm{e}}}, \frac{\boxed{\mathrm{f}}}{\boxed{\mathrm{g}}}, \frac{\boxed{\mathrm{h}}}{\boxed{\mathrm{i}}})$.

$\boxed{\mathrm{d}}$: $\boxed{1\ ✓}$ $\boxed{\mathrm{e}}$: $\boxed{3\ ✓}$ $\boxed{\mathrm{f}}$: $\boxed{-1\ ✓}$ $\boxed{\mathrm{g}}$: $\boxed{3\ ✓}$ $\boxed{\mathrm{h}}$: $\boxed{2\ ✓}$ $\boxed{\mathrm{i}}$: $\boxed{3\ ✓}$

Choose a correct statement.

- This solution is a maximum.
- This solution is a minimum. ✓
- This solution is a saddle point.

(6) **Q3**

Determine which of the following is a parametrization of the path

$$C = \{(x,y) : x^2 + (y-2)^2 = 4, y \leq 2\} \subset \mathbb{R}^2,$$

starting at $(-2,2)$ and finishing at $(2,2)$ ($\frac{1}{2}$ point each):

- $(2\cos t, 2(1+\sin t))$, $t \in [-\pi, 0]$ | is ✓ | is not |
- $(2\sin t, 2\cos t)$, $t \in [-\pi, 0]$ | is | is not ✓ |
- $(t, 2-\sqrt{4-t^2})$, $t \in [-2, 2]$ | is ✓ | is not |

- $(\frac{8t}{t^2+4}, \frac{4t^2}{t^2+4})$, $t \in [-2, 2]$

is ✓
is not

- $(\frac{4t}{t^2+1}, \frac{4t^2}{t^2+1})$, $t \in [-1, 1]$

is ✓
is not

- $(2t, 2 - 2\sqrt{1-t^2})$, $t \in [-1, 1]$

is ✓
is not

- $(-\sqrt{4-(t-2)^2}, t)$, $t \in [0, 2]$

is
is not ✓

- $(2\cos t, 2(1 + \sin t))$, $t \in [\pi, 2\pi]$

is ✓
is not

If C is the path above and

$$\mathbf{f}(x, y) = \begin{pmatrix} y^2 - 4y + 4 \\ x^2 \end{pmatrix}$$

is a vector field on $\mathbb{R}^2$ calculate $\int_C \mathbf{f}\, d\boldsymbol{\alpha} =$

32	✓
-32	(50%)

/3. Fill in the blank with the correct **integer**, possibly zero or negative (2 points).

(7) **Q3**

Determine which of the following is a parametrization of the path

$$C = \{(x, y) : x^2 + (y-2)^2 = 4, y \leq 2\} \subset \mathbb{R}^2,$$

starting at $(-2, 2)$ and finishing at $(2, 2)$ ($\frac{1}{2}$ point each):

- $(-\sqrt{4-(t-2)^2}, t)$, $t \in [0, 2]$

is
is not ✓

- $(2\cos t, 2(1 + \sin t))$, $t \in [\pi, 2\pi]$

is ✓
is not

- $(t, 2 - \sqrt{4-t^2})$, $t \in [-2, 2]$

is ✓
is not

- $(\frac{8t}{t^2+4}, \frac{4t^2}{t^2+4})$, $t \in [-2, 2]$

is ✓
is not

- $(\frac{4t}{t^2+1}, \frac{4t^2}{t^2+1})$, $t \in [-1, 1]$

is ✓
is not

- $(2t, 2 - 2\sqrt{1-t^2})$, $t \in [-1, 1]$

is ✓
is not

- $(2\cos t, 2(1 + \sin t))$, $t \in [-\pi, 0]$

is ✓
is not

- $(2\sin t, 2\cos t)$, $t \in [-\pi, 0]$

is
is not ✓

If C is the path above and

$$\mathbf{f}(x,y) = \begin{pmatrix} y^2 - 4y + 4 \\ x^2 \end{pmatrix}$$

is a vector field on $\mathbb{R}^2$ calculate $\int_C \mathbf{f}\, d\boldsymbol{\alpha} =$

32	✓
−32	(50%)

/3. Fill in the blank with the correct **integer**, possibly zero or negative (2 points).

(8) **Q3**

Determine which of the following is a parametrization of the path

$$C = \{(x,y) : x^2 + (y-3)^2 = 9, y \leq 3\} \subset \mathbb{R}^2,$$

starting at $(-3,3)$ and finishing at $(3,3)$ ($\frac{1}{2}$ point each):

- $(3\cos t, 3(1+\sin t))$, $t \in [-\pi, 0]$

is ✓
is not

- $(3\sin t, 3\cos t)$, $t \in [-\pi, 0]$

is
is not ✓

- $(t, 3-\sqrt{9-t^2})$, $t \in [-3,3]$

is ✓
is not

- $(\frac{18t}{t^2+9}, \frac{6t^2}{t^2+9})$, $t \in [-3,3]$

is ✓
is not

- $(\frac{6t}{t^2+1}, \frac{6t^2}{t^2+1})$, $t \in [-1,1]$

is ✓
is not

- $(3t, 3-3\sqrt{1-t^2})$, $t \in [-1,1]$

is ✓
is not

- $(-\sqrt{9-(t-3)^2}, t)$, $t \in [0,3]$

is
is not ✓

- $(3\cos t, 3(1+\sin t))$, $t \in [\pi, 2\pi]$

is ✓
is not

If C is the path above and

$$\mathbf{f}(x,y) = \begin{pmatrix} y^2 - 6y + 9 \\ x^2 \end{pmatrix}$$

is a vector field on $\mathbb{R}^2$ calculate $\int_C \mathbf{f}\, d\boldsymbol{\alpha} =$

36	✓
−36	(50%)

. Fill in the blank with the correct **integer**, possibly zero or negative (2 points).

(9) **Q3**
Determine which of the following is a parametrization of the path

$$C = \{(x, y) : x^2 + (y-3)^2 = 9, y \leq 3\} \subset \mathbb{R}^2,$$

starting at $(-3, 3)$ and finishing at $(3, 3)$ ($\frac{1}{2}$ point each):

- $(-\sqrt{9-(t-3)^2}, t)$, $t \in [0, 3]$ | is | is not ✓ |
- $(3\cos t, 3(1+\sin t))$, $t \in [\pi, 2\pi]$ | is ✓ | is not |
- $(t, 3-\sqrt{9-t^2})$, $t \in [-3, 3]$ | is ✓ | is not |
- $(\frac{18t}{t^2+9}, \frac{6t^2}{t^2+9})$, $t \in [-3, 3]$ | is ✓ | is not |
- $(\frac{6t}{t^2+1}, \frac{6t^2}{t^2+1})$, $t \in [-1, 1]$ | is ✓ | is not |
- $(3t, 3-3\sqrt{1-t^2})$, $t \in [-1, 1]$ | is ✓ | is not |
- $(3\cos t, 3(1+\sin t))$, $t \in [-\pi, 0]$ | is ✓ | is not |
- $(3\sin t, 3\cos t)$, $t \in [-\pi, 0]$ | is | is not ✓ |

If C is the path above and

$$\mathbf{f}(x, y) = \begin{pmatrix} y^2 - 6y + 9 \\ x^2 \end{pmatrix}$$

is a vector field on $\mathbb{R}^2$ calculate $\int_C \mathbf{f}\ d\boldsymbol{\alpha} =$ | 36 ✓ | −36 (50%) |. Fill in the blank with the correct **integer**, possibly zero or negative (2 points).

Let $a = 2$ or $a = 3$ depending on the alternative version with curve $C = \{(x, y) : x^2 + (y - a)^2 = a^2, y \leq a\}$.
The formula $\boldsymbol{\alpha}(t) = (a \sin t, a \cos t)$, $t \in [-\pi, 0]$ is not a parametrization of C since $(a \sin t^2) + (a \cos t - a)^2 \neq a^2$.
The formula $\boldsymbol{\alpha}(t) = (-\sqrt{a^2 - (t - a)^2}, t)$, $t \in [0, a]$ is not a parametrization of C since $\boldsymbol{\alpha}(0) = (0, 0)$ but the path is required to start at $(-a, a)$.
Picking the parametrization

$$\boldsymbol{\alpha}(t) = (a \cos t, a(1 + \sin t)), \quad t \in [\pi, 2\pi]$$

we calculate that

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} -a \sin t \\ a \cos t \end{pmatrix},$$

$y^2 - 2ay + a^2 = (y - a)^2 = a^2 \sin^2 t$, and also

$$\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = \begin{pmatrix} a^2 \sin^2 t \\ a^2 \cos^2 t \end{pmatrix} \cdot \begin{pmatrix} -a \sin t \\ a \cos t \end{pmatrix} = a^3(\cos^3 t - \sin^3 t).$$

Consequently

$$\int_C \mathbf{f}\ d\boldsymbol{\alpha} = a^3 \int_\pi^{2\pi} \cos^3 t - \sin^3 t\ dt.$$

To proceed we note the indefinite integrals $\int \cos^3 t\ dt = -\frac{1}{3} \sin^3 t + \sin t + C$ and $\int \sin^3 t\ dt = \frac{1}{3} \cos^3 t - \cos t + C$.
Consequently

$$\begin{aligned} \int_C \mathbf{f}\ d\boldsymbol{\alpha} &= \frac{a^3}{3} \left[- \sin^3 t + 3 \sin t - \cos^3 t + 3 \cos t\right]_\pi^{2\pi} \\ &= \frac{a^3}{3} \left((0 + 0 - 1 + 3) - (0 + 0 + 1 - 3)\right) = \frac{4a^3}{3}. \end{aligned}$$

(10) **Q4**
Fill in the blanks with the correct **integer**, possibly zero or negative.

Let V be the solid bounded above by the sphere $x^2+y^2+z^2=5$ and below by the paraboloid $x^2+y^2=4z$. We can write

$$V=\left\{(x,y,z):(x,y)\in D,\ \frac{x^2+y^2}{4}\leq z\leq\sqrt{5-(x^2+y^2)}\right\}\subset\mathbb{R}^3$$

where $D=\{(x,y):x^2+y^2\leq\boxed{4\ \checkmark}\}\subset\mathbb{R}^2$ (1 point). In order to find the volume of V by evaluating the triple integral $\iiint_V dV$ we change to cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$, $z=z$. The Jacobian $|J(r,\theta,z)|$ is equal to (1 point)

- $r\cos\theta$,
- $r^2\sin\theta$,
- $r\sin\theta$,
- r. ✓

Complete the triple integral and show that the volume of V is equal to ($\boxed{10\ \checkmark}$ $\sqrt{5}+$ $\boxed{-8\ \checkmark}$)$\frac{\pi}{3}$ (2 points each part).

It can be helpful to sketch V. The disk D is the set of (x,y) such that $(x^2+y^2) \leq 4\sqrt{5-(x^2+y^2)}$. This implies that $(x^2+y^2)^2 + 4^2(x^2+y^2) - 4^2 \cdot 5 \leq 0$ and in turn that $(x^2+y^2) \leq 4(\sqrt{4+5}-2) = 4$.
In cylindrical coordinates the solid V corresponds to the set $\tilde{V}$ where,

$$\tilde{V} = \left\{(r,\theta,z) : r \in [0,2], \theta \in [0,2\pi], r^2/4 \leq z \leq \sqrt{5-r^2}\right\},$$

The volume integral is

$$\begin{aligned}\iiint_V dV &= 2\pi \int_0^2 r\left(\sqrt{5-r^2} - \frac{r^2}{4}\right)\, dr \\ &= 2\pi \int_0^2 r(5-r^2)^{\frac{1}{2}}\, dr - \frac{\pi}{2}\int_0^2 r^3\, dr.\end{aligned}$$

Observe the indefinite integrals $\int r(a-r^2)^{\frac{1}{2}}\, dr = -\frac{1}{3}(a-r^2)^{\frac{3}{2}} + C$ and $\int r^3\, dr = \frac{1}{4}r^4 + C$. This means that

$$\begin{aligned}\iiint_V dV &= -\frac{2}{3}\pi\left[(5-r^2)^{\frac{3}{2}}\right]_0^2 - \frac{\pi}{8}\left[r^4\right]_0^2 \\ &= -\frac{2}{3}\pi\left((5-4)^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) - \frac{\pi}{8}16 \\ &= \left(-\frac{2}{3} + \frac{2}{3}\cdot 5\sqrt{5} - 2\right)\pi = \left(10\sqrt{5}-8\right)\frac{\pi}{3}.\end{aligned}$$

(11) **Q4**

Fill in the blanks with the correct **integer**, possibly zero or negative.

Let V be the solid bounded above by the sphere $x^2+y^2+z^2 = 12$ and below by the paraboloid $x^2 + y^2 = 4z$. We can write

$$V = \left\{(x, y, z) : (x, y) \in D, \frac{x^2 + y^2}{4} \leq z \leq \sqrt{12 - (x^2 + y^2)}\right\} \subset \mathbb{R}^3$$

where $D = \{(x, y) : x^2+y^2 \leq \boxed{8 \quad \checkmark}\} \subset \mathbb{R}^2$ (1 point). In order to find the volume of V by evaluating the triple integral $\iiint_V dV$ we change to cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The Jacobian $|J(r, \theta, z)|$ is equal to (1 point)

- $r\cos\theta$,
- $r^2\sin\theta$,
- $r\sin\theta$,
- r. ✓

Complete the triple integral and show that the volume of V is equal to $(\boxed{16 \quad \checkmark}\sqrt{3}+\boxed{-40 \quad \checkmark}/3)\pi$ (2 points each part).

It can be helpful to sketch V. The disk D is the set of (x, y) such that $(x^2 + y^2) \leq 4\sqrt{12 - (x^2 + y^2)}$. This implies that $(x^2 + y^2)^2 + 4^2(x^2 + y^2) - 4^2 \cdot 12 \leq 0$ and in turn that $(x^2 + y^2) \leq 4(\sqrt{4 + 12} - 2) = 8$.
In cylindrical coordinates the solid V corresponds to the set $\tilde{V}$ where,

$$\tilde{V} = \left\{ (r, \theta, z) : r \in [0, 2\sqrt{2}], \theta \in [0, 2\pi], r^2/4 \leq z \leq \sqrt{12 - r^2} \right\},$$

The volume integral is

$$\begin{aligned} \iiint_V dV &= 2\pi \int_0^{2\sqrt{2}} r \left(\sqrt{12 - r^2} - \frac{r^2}{4} \right) \, dr \\ &= 2\pi \int_0^{2\sqrt{2}} r(12 - r^2)^{\frac{1}{2}} \, dr - \frac{\pi}{2} \int_0^{2\sqrt{2}} r^3 \, dr. \end{aligned}$$

Observe the indefinite integrals $\int r(a - r^2)^{\frac{1}{2}} \, dr = -\frac{1}{3}(a - r^2)^{\frac{3}{2}} + C$ and $\int r^3 \, dr = \frac{1}{4} r^4 + C$. This means that

$$\begin{aligned} \iiint_V dV &= -\frac{2}{3}\pi \left[(12 - r^2)^{\frac{3}{2}} \right]_0^{2\sqrt{2}} - \frac{\pi}{8} \left[r^4 \right]_0^{2\sqrt{2}} \\ &= -\frac{2}{3}\pi \left((12 - 8)^{\frac{3}{2}} - 12^{\frac{3}{2}} \right) - \frac{\pi}{8} 64 \\ &= \pi \left(-\frac{2}{3} \cdot 8 + \frac{2}{3} \cdot 3 \cdot 8\sqrt{3} - 8 \right) \\ &= \left(16\sqrt{3} - \frac{40}{3} \right) \pi. \end{aligned}$$

(12) **Q5**

Fill in each blank with the correct **integer**, possibly zero or negative.

Consider the surface $S = \{(x, y, z) : x^2 + y^2 = z, z \leq 9\}$. A possible choice for the parametric form of the surface S is to let $T = \{(r, \theta) : r \in [0, \boxed{3 \ \checkmark}], \theta \in [0, 2\pi]\}$ ($\frac{1}{2}$ point) and

$$\mathbf{r} : (r, \theta) \mapsto \left(r\cos\theta, \boxed{\text{a}}, \boxed{\text{b}}\right).$$

For this parametric representation we calculate that

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \boxed{\text{c}} \\ \boxed{\text{d}} \\ \boxed{\text{e}} \end{pmatrix}.$$

The missing formulae are ($\frac{1}{2}$ point each):

$\boxed{\text{a}}$:
• $r\sin\theta$ ✓ • $r\cos\theta$ • r • r^2 • $r^2\cos\theta$ • $r^2\sin\theta$

$\boxed{\text{b}}$:
• $r\sin\theta$ • $r\cos\theta$ • r • r^2 ✓ • $r^2\cos\theta$ • $r^2\sin\theta$

$\boxed{\text{c}}$:
• $2r\sin\theta$ • r • $2r^2$ • $-r^2\sin\theta$ • $-2r^2\cos\theta$ ✓

$\boxed{\text{d}}$:
• $2r\sin\theta$ • r • r^2 • $-2r^2\sin\theta$ ✓ • $-2r^2\cos\theta$

$\boxed{\text{e}}$:
• $2r\sin\theta$ • r ✓ • r^2 • $-2r^2\sin\theta$ • $-2r^2\cos\theta$

Consider the vector field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix}$$

and let $\mathbf{n}$ be the unit normal to S which has **negative** z-component. The surface integral $\iint_S \mathbf{f} \cdot \mathbf{n} \, dS = \boxed{\begin{array}{ll} 63 & \checkmark \\ -63 & (50\%) \end{array}}$ $\frac{\pi}{2}$ (3 points).

We choose the parametric form of the surface S by letting $T = \{(r,\theta) : r \in [0,3], \theta \in [0,2\pi]\}$ and

$$\mathbf{r} : (r,\theta) \mapsto (r\cos\theta, r\sin\theta, r^2).$$

We calculate

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}.$$

We observe that this corresponds to the opposite normal compared to the one that we want so we will need to add a minus sign.

$$\begin{aligned} \iint_S \mathbf{f} \cdot \mathbf{n} \, dS &= -\int_0^3 \int_0^{2\pi} \begin{pmatrix} r\cos\theta \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix} \, d\theta dr \\ &= \int_0^3 \int_0^{2\pi} 2r^3\cos^2\theta - r \, d\theta dr. \end{aligned}$$

We calculate that

$$\int_0^3 \int_0^{2\pi} (-r) \, d\theta dr = -2\pi \left[\frac{1}{2}r^2\right]_0^3 = -3^2\pi.$$

On the other hand, using the indefinite integral $\int \cos^2\theta \, d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$, we calculate that

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{2\pi} = \pi.$$

This means that

$$\int_0^3 \int_0^{2\pi} 2r^3\cos^2\theta \, d\theta dr = 2\pi \int_0^3 r^3 \, dr = \frac{\pi}{2}\left[r^4\right]_0^3 = \frac{3^4}{2}\pi.$$

Summing together the two parts of the integral we have

$$\iint_S \mathbf{f} \cdot \mathbf{n} \, dS = \left(\frac{3^4}{2} - 3^2\right)\pi = \frac{63}{2}\pi.$$

(13) **Q5**

Fill in each blank with the correct **integer**, possibly zero or negative.

Consider the surface $S = \{(x, y, z) : x^2 + y^2 = z, z \le 16\}$. A possible choice for the parametric form of the surface S is to let $T = \{(r, \theta) : r \in [0, \boxed{4 \quad \checkmark}\,], \theta \in [0, 2\pi]\}$ ($\frac{1}{2}$ point) and

$$\mathbf{r} : (r, \theta) \mapsto \left(r\cos\theta, \boxed{\text{a}}, \boxed{\text{b}}\right).$$

For this parametric representation we calculate that

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \boxed{\text{c}} \\ \boxed{\text{d}} \\ \boxed{\text{e}} \end{pmatrix}.$$

The missing formulae are ($\frac{1}{2}$ point each):

$\boxed{\text{a}}$:
• $r\sin\theta$ ✓ • $r\cos\theta$ • r • r^2 • $r^2\cos\theta$ • $r^2\sin\theta$

$\boxed{\text{b}}$:
• $r\sin\theta$ • $r\cos\theta$ • r • r^2 ✓ • $r^2\cos\theta$ • $r^2\sin\theta$

$\boxed{\text{c}}$:
• $2r\sin\theta$ • r • $2r^2$ • $-r^2\sin\theta$ • $-2r^2\cos\theta$ ✓

$\boxed{\text{d}}$:
• $2r\sin\theta$ • r • r^2 • $-2r^2\sin\theta$ ✓ • $-2r^2\cos\theta$

$\boxed{\text{e}}$:
• $2r\sin\theta$ • r ✓ • r^2 • $-2r^2\sin\theta$ • $-2r^2\cos\theta$

Consider the vector field

$$\mathbf{f}(x, y, z) = \left(\begin{smallmatrix} x \\ 0 \\ 1 \end{smallmatrix}\right)$$

and let $\mathbf{n}$ be the unit normal to S which has **negative** z-component. The surface integral $\iint_S \mathbf{f} \cdot \mathbf{n} \, dS = \boxed{\begin{matrix} 112 & \checkmark \\ -112 & (50\%) \end{matrix}}$ π (3 points).

We choose the parametric form of the surface S by letting $T = \{(r, \theta) : r \in [0, 4], \theta \in [0, 2\pi]\}$ and

$$\mathbf{r} : (r, \theta) \mapsto (r\cos\theta, r\sin\theta, r^2).$$

We calculate

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}.$$

We observe that this corresponds to the opposite normal compared to the one that we want so we will need to add a minus sign.

$$\begin{aligned} \iint_S \mathbf{f} \cdot \mathbf{n} \, dS &= -\int_0^4 \int_0^{2\pi} \begin{pmatrix} r\cos\theta \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix} \, d\theta dr \\ &= \int_0^4 \int_0^{2\pi} 2r^3\cos^2\theta - r \, d\theta dr. \end{aligned}$$

We calculate that

$$\int_0^4 \int_0^{2\pi} (-r) \, d\theta dr = -2\pi \left[\frac{1}{2}r^2\right]_0^4 = -4^2\pi.$$

On the other hand, using the indefinite integral $\int \cos^2\theta \, d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$, we calculate that

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{2\pi} = \pi.$$

This means that

$$\int_0^4 \int_0^{2\pi} 2r^3\cos^2\theta \, d\theta dr = 2\pi \int_0^4 r^3 \, dr = \frac{\pi}{2}\left[r^4\right]_0^4 = \frac{4^4}{2}\pi.$$

Summing together the two parts of the integral we have

$$\iint_S \mathbf{f} \cdot \mathbf{n} \, dS = \left(\frac{4^4}{2} - 4^2\right)\pi = 112\pi.$$