**Call5.**

(1) **Q1**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$). If a fraction is negative, **put the negative sign on the numerator** ($\frac{-1}{2}$ but not $\frac{1}{-2}$).

Consider first $\cos(2x)$. Its Taylor expansion is

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{\boxed{\text{a}}^n}{(2n)!} x^{\boxed{\text{b}}n}$$

.

$\boxed{\text{a}}$: $\boxed{-4 \ \checkmark}$ $\boxed{\text{b}}$: $\boxed{2 \ \checkmark}$

It holds that

$$\cos^2 x = \frac{\boxed{\text{c}}}{\boxed{\text{d}}} \cos(2x) + \frac{\boxed{\text{e}}}{\boxed{\text{f}}}.$$

$\boxed{\text{c}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{d}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{e}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{f}}$: $\boxed{2 \ \checkmark}$

From this we obtain the Taylor expansion of $\cos^2 x = \sum_{n=0}^{\infty} a_n x^n$. The first few terms are

$$\cos^2 x = \boxed{\text{g}} + \boxed{\text{h}}x + \boxed{\text{i}}x^2 + \boxed{\text{j}}x^3 + \frac{\boxed{\text{k}}}{\boxed{\text{l}}}x^4 + \boxed{\text{m}}x^5 + \frac{\boxed{\text{n}}}{\boxed{\text{o}}}x^6 + o(x^6).$$

$\boxed{\text{g}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{h}}$: $\boxed{0 \ \checkmark}$ $\boxed{\text{i}}$: $\boxed{-1 \ \checkmark}$ $\boxed{\text{j}}$: $\boxed{0 \ \checkmark}$ $\boxed{\text{k}}$: $\boxed{1 \ \checkmark}$
$\boxed{\text{l}}$: $\boxed{3 \ \checkmark}$ $\boxed{\text{m}}$: $\boxed{0 \ \checkmark}$ $\boxed{\text{n}}$: $\boxed{-2 \ \checkmark}$ $\boxed{\text{o}}$: $\boxed{45 \ \checkmark}$

We can also compute the integral.

$$\int_0^x \cos^2 t dt = \boxed{\text{p}} + \boxed{\text{q}}x + \boxed{\text{r}}x^2 + \frac{\boxed{\text{s}}}{\boxed{\text{t}}}x^3 + \boxed{\text{u}}x^4 + \frac{\boxed{\text{v}}}{\boxed{\text{w}}}x^5 + o(x^5).$$

$\boxed{\text{p}}$: $\boxed{0 \ \checkmark}$ $\boxed{\text{q}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{r}}$: $\boxed{0 \ \checkmark}$ $\boxed{\text{s}}$: $\boxed{-1 \ \checkmark}$ $\boxed{\text{t}}$: $\boxed{3 \ \checkmark}$
$\boxed{\text{u}}$: $\boxed{0 \ \checkmark}$ $\boxed{\text{v}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{w}}$: $\boxed{15 \ \checkmark}$

Choose values of $x$ for which the above series converges.

- $-10$ ✓
- $-1$ ✓
- $-0.1$ ✓
- $0$ ✓
- $0.1$ ✓
- $1$ ✓
- $10$ ✓
- $100$ ✓

For $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $\int_0^{\infty} f(t)dt = \sum_{n=1}^{\infty} \frac{a_n}{n} x^n$. By the ratio test, the series for $\cos^2 x$ and $\sin^2 x$ converges for all $x$. The radius of convergence does not change if it is integrated from the center ($x = 0$ in this case).

(2) **Q1**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$). If a fraction is negative, **put the negative sign on the numerator** ($\frac{-1}{2}$ but not $\frac{1}{-2}$).

Consider first $\cos(2x)$. Its Taylor expansion is

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{\boxed{a}^n}{(2n)!} x^{\boxed{b}n}$$

.

$\boxed{a}$: $\boxed{-4 \ \checkmark}$ $\boxed{b}$: $\boxed{2 \ \checkmark}$

It holds that

$$\sin^2 x = \frac{\boxed{c}}{\boxed{d}} \cos(2x) + \frac{\boxed{e}}{\boxed{f}}$$

$\boxed{c}$: $\boxed{-1 \ \checkmark}$ $\boxed{d}$: $\boxed{2 \ \checkmark}$ $\boxed{e}$: $\boxed{1 \ \checkmark}$ $\boxed{f}$: $\boxed{2 \ \checkmark}$

From this we obtain the Taylor expansion of $\sin^2 x = \sum_{n=0}^{\infty} a_n x^n$. The first few terms are

$$\sin^2 x = \boxed{g} + \boxed{h}x + \boxed{i}x^2 + \boxed{j}x^3 + \frac{\boxed{k}}{\boxed{l}}x^4 + \boxed{m}x^5 + \frac{\boxed{n}}{\boxed{o}}x^6 + o(x^6)$$

$\boxed{g}$: $\boxed{0 \ \checkmark}$ $\boxed{h}$: $\boxed{0 \ \checkmark}$ $\boxed{i}$: $\boxed{1 \ \checkmark}$ $\boxed{j}$: $\boxed{0 \ \checkmark}$ $\boxed{k}$: $\boxed{-1 \ \checkmark}$
$\boxed{l}$: $\boxed{3 \ \checkmark}$ $\boxed{m}$: $\boxed{0 \ \checkmark}$ $\boxed{n}$: $\boxed{2 \ \checkmark}$ $\boxed{o}$: $\boxed{45 \ \checkmark}$

We can also compute the integral.

$$\int_0^x \sin^2 t dt = \boxed{p} + \boxed{q}x + \boxed{r}x^2 + \frac{\boxed{s}}{\boxed{t}}x^3 + \boxed{u}x^4 + \frac{\boxed{v}}{\boxed{w}}x^5 + o(x^5).$$

$\boxed{p}$: $\boxed{0 \ \checkmark}$ $\boxed{q}$: $\boxed{0 \ \checkmark}$ $\boxed{r}$: $\boxed{0 \ \checkmark}$ $\boxed{s}$: $\boxed{1 \ \checkmark}$ $\boxed{t}$: $\boxed{3 \ \checkmark}$
$\boxed{u}$: $\boxed{0 \ \checkmark}$ $\boxed{v}$: $\boxed{-1 \ \checkmark}$ $\boxed{w}$: $\boxed{15 \ \checkmark}$

Choose values of $x$ for which the above series converges.

- $-10$ ✓
- $-1$ ✓
- $-0.1$ ✓

- 0 ✓
- 0.1 ✓
- 1 ✓
- 10 ✓
- 100 ✓

(3) **Q2**

Fill in the blanks with **integers (possibly** 0 **or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$). If a fraction is negative, **put the negative sign on the numerator** ($\frac{-1}{2}$ but not $\frac{1}{-2}$).

Let us find stationary points of the function below, following the suggested steps.

$$f(x, y, z) = x^3 + y^3 + z^3 - x - y - z + xyz.$$

First, we compute the gradient $\nabla f$. We have:

$$\frac{\partial f}{\partial x} = \boxed{\text{a}} x^{\boxed{\text{b}}} + \boxed{\text{c}} + \boxed{\text{d}} yz.$$

$\boxed{\text{a}}$: $\boxed{3 \ \checkmark}$ $\boxed{\text{b}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{c}}$: $\boxed{-1 \ \checkmark}$ $\boxed{\text{d}}$: $\boxed{1 \ \checkmark}$

The equation $\nabla f(x, y, z) = \mathbf{0}$ has many solutions. Let us consider **only those that satisfy** $x = y$. They are $(x, y, z) = (\frac{\boxed{\text{e}}}{\boxed{\text{f}}}, \frac{\boxed{\text{e}}}{\boxed{\text{f}}}, \frac{\boxed{\text{g}}}{\boxed{\text{h}}}), (\sqrt{\frac{\boxed{\text{i}}}{\boxed{\text{j}}}}, \sqrt{\frac{\boxed{\text{i}}}{\boxed{\text{j}}}}, \frac{\boxed{\text{k}}}{\sqrt{\boxed{\text{l}}}}), (-\sqrt{\frac{\boxed{\text{i}}}{\boxed{\text{j}}}}, -\sqrt{\frac{\boxed{\text{i}}}{\boxed{\text{j}}}}, -\frac{\boxed{\text{k}}}{\sqrt{\boxed{\text{l}}}}), (-\frac{\boxed{\text{e}}}{\boxed{\text{f}}}, -\frac{\boxed{\text{e}}}{\boxed{\text{f}}}, -\frac{\boxed{\text{g}}}{\boxed{\text{h}}})$.,

where $\boxed{\text{e}}, \boxed{\text{f}} > 0$:

$\boxed{\text{e}} > 0$: $\boxed{1 \ \checkmark}$ $\boxed{\text{f}} > 0$: $\boxed{2 \ \checkmark}$ $\boxed{\text{g}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{h}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{i}}$: $\boxed{3 \ \checkmark}$ $\boxed{\text{j}}$: $\boxed{7 \ \checkmark}$ $\boxed{\text{k}}$: $\boxed{-2 \ \checkmark}$ $\boxed{\text{l}}$: $\boxed{21 \ \checkmark}$

Consider the first of them $(\frac{\boxed{\text{e}}}{\boxed{\text{f}}}, \frac{\boxed{\text{e}}}{\boxed{\text{f}}}, \frac{\boxed{\text{g}}}{\boxed{\text{h}}})$. The determinant of the Hessian at this point is $\boxed{\text{m}}$.

$\boxed{\text{m}}$: $\boxed{25 \ \checkmark}$

At this point, the function $f(x, y, z)$ takes a

- local minimum ✓
- saddle point
- local maximum

> The equations are $x^2 + yz = 1, y^2 + zx = 1, z^2 + xy = 1$. Assuming $x = y$, we only have to solve $x^2 + xz = 1, z^2 + xz = 1$. Subtracting the sides, we obtain two cases.
> The Hessian determinant can be calculated directly.

(4) **Q2**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$). If a fraction is negative, **put the negative sign on the numerator** ($\frac{-1}{2}$ but not $\frac{1}{-2}$).

Let us find stationary points of the function below, following the suggested steps.

$$f(x, y, z) = 4x^3 + 4y^3 + 4z^3 - x - y - z + 4xyz.$$

First, we compute the gradient $\nabla f$. We have:

$$\frac{\partial f}{\partial x} = \boxed{\text{a}}x^{\boxed{\text{b}}} + \boxed{\text{c}} + \boxed{\text{d}}yz.$$

$\boxed{\text{a}}$: $\boxed{12 \ \checkmark}$ $\boxed{\text{b}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{c}}$: $\boxed{-1 \ \checkmark}$ $\boxed{\text{d}}$: $\boxed{4 \ \checkmark}$

The equation $\nabla f(x, y, z) = \mathbf{0}$ has many solutions. Let us consider **only those that satisfy** $x = y$. They are $(x, y, z) =$

$$\left(\frac{\boxed{\text{e}}}{\boxed{\text{f}}}, \frac{\boxed{\text{e}}}{\boxed{\text{f}}}, \frac{\boxed{\text{g}}}{\boxed{\text{h}}}\right), \left(\frac{\sqrt{\boxed{\text{i}}}}{\boxed{\text{j}}\sqrt{\boxed{\text{k}}}}, \frac{\sqrt{\boxed{\text{i}}}}{\boxed{\text{j}}\sqrt{\boxed{\text{k}}}}, \frac{\boxed{\text{l}}}{\sqrt{\boxed{\text{m}}}}\right), \left(-\frac{\sqrt{\boxed{\text{i}}}}{\boxed{\text{j}}\sqrt{\boxed{\text{k}}}}, -\frac{\sqrt{\boxed{\text{i}}}}{\boxed{\text{j}}\sqrt{\boxed{\text{k}}}}, -\frac{\boxed{\text{l}}}{\sqrt{\boxed{\text{m}}}}\right), \left(-\frac{\boxed{\text{e}}}{\boxed{\text{f}}}, -\frac{\boxed{\text{e}}}{\boxed{\text{f}}}, -\frac{\boxed{\text{g}}}{\boxed{\text{h}}}\right),$$

where $\boxed{\text{e}}, \boxed{\text{f}} > 0$:

$\boxed{\text{e}} > 0$: $\boxed{1 \ \checkmark}$ $\boxed{\text{f}} > 0$: $\boxed{4 \ \checkmark}$ $\boxed{\text{g}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{h}}$: $\boxed{4 \ \checkmark}$ $\boxed{\text{i}}$: $\boxed{3 \ \checkmark}$ $\boxed{\text{j}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{k}}$: $\boxed{7 \ \checkmark}$ $\boxed{\text{l}}$: $\boxed{-1 \ \checkmark}$ $\boxed{\text{m}}$: $\boxed{21 \ \checkmark}$

Consider the first of them $\left(\frac{\boxed{\text{e}}}{\boxed{\text{f}}}, \frac{\boxed{\text{e}}}{\boxed{\text{f}}}, \frac{\boxed{\text{g}}}{\boxed{\text{h}}}\right)$. The determinant of the Hessian at this point is $\frac{\boxed{\text{m}}}{\boxed{\text{n}}}$.

$\boxed{\text{n}}$: $\boxed{200 \ \checkmark}$

At this point, the function $f(x, y, z)$ takes a

- local minimum ✓
- saddle point
- local maximum

(5) **Q3**

Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

**(a)** If $C$ is the path from $(-1,1)$ to $(1,1)$ along the parabola $y = x^2$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} -14 & \checkmark \\ 14 & (50\%) \end{array}}\ /15.$

**(b)** If $C$ is the line segment from $(1,0,2)$ to $(3,4,1)$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} 2xy \\ x^2 + z \\ y \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 40 & \checkmark \\ -40 & (50\%) \end{array}}.$

**(c)** If $C$ is the circle $x^2 + y^2 = 1$ traversed once in a counter clockwise direction and

$$\mathbf{f}(x,y) = \begin{pmatrix} (x+y)/(x^2+y^2) \\ -(x-y)/(x^2+y^2) \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} -2 & \checkmark \\ 2 & (50\%) \end{array}}\ \pi.$

**(a)** Let $\boldsymbol{\alpha}(t) = (t, t^2)$, $t \in [-1, 1]$. Calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} t^2 - 2t^3 \\ t^4 - 2t^3 \end{pmatrix}.$$

Moreover $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = t^2 - 2t^3 + 2t^5 - 4t^4$. Consequently

$$\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \int_{-1}^{1} t^2 - 4t^4\ dt = 2\left[\frac{t^3}{3} - \frac{4t^5}{5}\right]_0^1 = -\frac{14}{15}.$$

**(b)** Let $\boldsymbol{\alpha}(t) = (2t+1, 4t, 2-t)$, $t \in [0, 1]$. Calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix},$$

and

$$\mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} 2(2t+1)(4t) \\ (2t+1)^2 + 2 - t \\ 4t \end{pmatrix} = \begin{pmatrix} 16t^2 + 8t \\ 4t^2 + 3t + 3 \\ 4t \end{pmatrix}.$$

Moreover $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = (32t^2 + 16t) + (16t^2 + 12t + 12) + (-4t) = 48t^2 + 24t + 12$. Consequently

$$\begin{aligned} \int_C \mathbf{f}\ d\boldsymbol{\alpha} &= \int_0^1 48t^2 + 24t + 12\ dt \\ &= \left[\frac{48}{3}t^3 + 12t^2 + 12t\right]_0^1 = \frac{48}{3} + 24. \end{aligned}$$

**(c)** Let $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} \cos t + \sin t \\ -\cos t + \sin t \end{pmatrix}.$$

Moreover $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = -\sin t \cos t - \sin^2 t - \cos^2 t + \sin t \cos t = -1$. Consequently

$$\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \int_0^{2\pi} (-1)\ dt = -2\pi.$$

(6) **Q3**

Fill in the following blanks with the correct **integer**, possibly zero or negative (2 points each).

**(a)** If $C$ is the path from $(-1, 1)$ to $(1, 1)$ along the parabola $y = x^2$ and

$$\mathbf{f}(x, y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} -14 & \checkmark \\ 14 & (50\%) \end{array}}$ /15.

**(b)** If $C$ is the line segment from $(1,0,2)$ to $(3,4,1)$ and

$$\mathbf{f}(x,y) = \begin{pmatrix} xy \\ x^2+z \\ 2y \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 86 & \checkmark \\ -86 & (50\%) \end{array}}$ /3.

**(c)** If $C$ is the circle $x^2+y^2=1$ traversed once in a counter clockwise direction and

$$\mathbf{f}(x,y) = \begin{pmatrix} (x+y)/(x^2+y^2) \\ -(x-y)/(x^2+y^2) \end{pmatrix}$$

is a vector field then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} -2 & \checkmark \\ 2 & (50\%) \end{array}}$ $\pi$.

**(a)** Let $\boldsymbol{\alpha}(t) = (t, t^2)$, $t \in [-1, 1]$. Calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} t^2 - 2t^3 \\ t^4 - 2t^3 \end{pmatrix}.$$

Moreover $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = t^2 - 2t^3 + 2t^5 - 4t^4$. Consequently

$$\int_C \mathbf{f} \; d\boldsymbol{\alpha} = \int_{-1}^{1} t^2 - 4t^4 \; dt = 2\left[\frac{t^3}{3} - \frac{4t^5}{5}\right]_0^1 = -\frac{14}{15}.$$

**(b)** Let $\boldsymbol{\alpha}(t) = (2t+1, 4t, 2-t)$, $t \in [0, 1]$. Calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix},$$

and

$$\mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} (2t+1)(4t) \\ (2t+1)^2 + 2 - t \\ 2(4t) \end{pmatrix} = \begin{pmatrix} 8t^2 + 4t \\ 4t^2 + 3t + 3 \\ 8t \end{pmatrix}.$$

Moreover $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = (16t^2 + 8t) + (16t^2 + 12t + 12) + (-8t) = 32t^2 + 12t + 12$. Consequently

$$\begin{aligned} \int_C \mathbf{f} \; d\boldsymbol{\alpha} &= \int_0^1 32t^2 + 12t + 12 \; dt \\ &= \left[\frac{32}{3}t^3 + 6t^2 + 12t\right]_0^1 = \frac{86}{3}. \end{aligned}$$

**(c)** Let $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} \cos t + \sin t \\ -\cos t + \sin t \end{pmatrix}.$$

Moreover $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = -\sin t \cos t - \sin^2 t - \cos^2 t + \sin t \cos t = -1$. Consequently

$$\int_C \mathbf{f} \; d\boldsymbol{\alpha} = \int_0^{2\pi} (-1) \; dt = -2\pi.$$

(7) **Q4**

Fill in the blanks with the correct **integer**, possibly zero or negative.

Let $D$ be the solid $\{(x, y, z) : x^2 + y^2 + 4z^2 \leq 9, y \geq 0\} \subset \mathbb{R}^3$. In the following we will compute the triple integral $\iiint_D x + y \; dxdydz$. Using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$,

$z = z$, the set $D$ corresponds to

$$E = \{(r, \theta, z) : 0 \leq \theta \leq \pi, 0 \leq r \leq 3, -\varphi(r) \leq z \leq \varphi(r)\}$$

where $\varphi(r) = \frac{1}{2}\sqrt{\boxed{\text{a}} + \boxed{\text{b}}\, r^2}$ with values $\boxed{\text{a}}$: $\boxed{9 \quad \checkmark}$ $\boxed{\text{b}}$: $\boxed{-1 \quad \checkmark}$ (1 point each).

Evaluating the integral we obtain the result $\iiint_D x+y \; dxdydz =$ $\boxed{\begin{array}{ll} 81 & \checkmark \\ -81 & (50\%) \end{array}}$ $\frac{\pi}{8}$ (4 points). *Hint: A substitution like* $u = \sin t$ *can be useful for integrating quantities like* $\sqrt{1-u^2}$. *Double angle formulae:* $2 \sin t \cos t = \sin(2t)$ *and* $1 - \cos(2t) = 2\sin^2 t$.

> Using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the set $D$ corresponds to
>
> $$E = \{(r, \theta, z) : 0 \leq \theta \leq \pi, 0 \leq r \leq 3, -\varphi(r) \leq z \leq \varphi(r)\}$$
>
> where $\varphi(r) = \frac{1}{2}\sqrt{9 - r^2}$. The Jacobian is $J(r, \theta, z) = r$. This means that
>
> $$\iiint_D x+y \; dxdydz = \int_0^\pi \int_0^3 \int_{-\varphi(r)}^{\varphi(r)} r^2(\cos\theta + \sin\theta) \; dzdrd\theta.$$
>
> Since $\int_0^\pi (\cos\theta + \sin\theta) \; d\theta = \int_0^\pi \sin\theta \; d\theta = 2$ and $\int_{-\varphi(r)}^{\varphi(r)} dz = 2\varphi(r) = \sqrt{9 - r^2}$ the integral is equal to
>
> $$2 \int_0^3 r^2 \sqrt{9 - r^2} \; dr.$$
>
> in order to integrate this we make the substitution $r = 3\sin t$ and so the above is equal to
>
> $$2 \int_0^{\pi/2} (3^2 \sin^2 t)(3\cos t)(3\sqrt{1 - \sin^2 t}) \; dt = 162 \int_0^{\pi/2} \sin^2 t \cos^2 t \; dt.$$
>
> Combining the identities $2\sin t \cos t = \sin(2t)$ and $1 - \cos(2t) = 2\sin^2 t$ we obtain the identity $1 - \cos(4t) = 8\sin^2 t \cos^2 t$. Consequently
>
> $$\iiint_D x+y \; dxdydz = \frac{81}{4} \int_0^{\pi/2} 1 - \cos(4t) \; dt = \frac{81}{4}\left(\frac{\pi}{2} - 0\right) = \frac{81}{8}\pi$$
>
> .

(8) **Q4**

Fill in the blanks with the correct **integer**, possibly zero or negative.

Let $D$ be the solid $\{(x,y,z) : x^2+y^2+4z^2 \leq 16, y \geq 0\} \subset \mathbb{R}^3$. In the following we will compute the triple integral $\iiint_D x+y\ dxdydz$. Using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the set $D$ corresponds to

$$E = \{(r,\theta,z) : 0 \leq \theta \leq \pi, 0 \leq r \leq 4, -\varphi(r) \leq z \leq \varphi(r)\}$$

where $\varphi(r) = \frac{1}{2}\sqrt{\boxed{\text{a}} + \boxed{\text{b}}r^2}$ with values $\boxed{\text{a}}$: $\boxed{16\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{-1\ \checkmark}$ (1 point each).

Evaluating the integral we obtain the result $\iiint_D x+y\ dxdydz =$ $\boxed{\begin{array}{ll} 32 & \checkmark \\ -32 & (50\%) \end{array}}$ $\pi$ (4 points). *Hint: A substitution like $u = \sin t$ can be useful for integrating quantities like $\sqrt{1-u^2}$. Double angle formulae: $2\sin t\cos t = \sin(2t)$ and $1-\cos(2t) = 2\sin^2 t$.*

Using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the set $D$ corresponds to

$$E = \{(r,\theta,z) : 0 \leq \theta \leq \pi, 0 \leq r \leq 4, -\varphi(r) \leq z \leq \varphi(r)\}$$

where $\varphi(r) = \frac{1}{2}\sqrt{16-r^2}$. The Jacobian is $J(r,\theta,z) = r$. This means that

$$\iiint_D x+y\ dxdydz = \int_0^\pi \int_0^4 \int_{-\varphi(r)}^{\varphi(r)} r^2(\cos\theta+\sin\theta)\ dzdrd\theta.$$

Since $\int_0^\pi (\cos\theta + \sin\theta)\ d\theta = \int_0^\pi \sin\theta\ d\theta = 2$ and $\int_{-\varphi(r)}^{\varphi(r)} dz = 2\varphi(r) = \sqrt{16-r^2}$ the integral is equal to

$$2\int_0^4 r^2\sqrt{16-r^2}\ dr.$$

in order to integrate this we make the substitution $r = 4\sin t$ and so the above is equal to

$$2\int_0^{\pi/2} (4^2\sin^2 t)(4\cos t)(4\sqrt{1-\sin^2 t})\ dt = 2^9 \int_0^{\pi/2} \sin^2 t\cos^2 t\ dt.$$

Combining the identities $2\sin t\cos t = \sin(2t)$ and $1-\cos(2t) = 2\sin^2 t$ we obtain the identity $1-\cos(4t) = 8\sin^2 t\cos^2 t$. Consequently

$$\iiint_D x+y\ dxdydz = 2^6\int_0^{\pi/2} 1-\cos(4t)\ dt = 2^6\left(\frac{\pi}{2}-0\right) = 32\pi$$

.

(9) **Q5**

Fill in the blanks with the correct **integer**, possibly zero or negative.

Let $T = \{(u,v) : u^2 + v^2 \leq 1\} \subset \mathbb{R}^2$ and let $\mathbf{r} : T \to \mathbb{R}^3$,

$$\mathbf{r}(u,v) = \left( \frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$$

be the representation of the surface $S = \mathbf{r}(T)$.

The surface $S$ is a

| |
|---|
| hemisphere centred at (0,0,0) ✓ |
| cone with point downward |

(2 points).

Calculating the fundamental vector product we find that

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u,v) = \frac{C}{u^2+v^2+1} \begin{pmatrix} 2u \\ 2v \\ u^2+v^2-1 \end{pmatrix}$$

where $C =$

| | |
|---|---|
| $-4$ | ✓ |
| 4 | (50%) |

(2 points).

Consider the vector-field $\mathbf{f}(x,y,z) = \left(\begin{smallmatrix} 0 \\ 0 \\ 1/z \end{smallmatrix}\right)$ and evaluate the surface integral (where $\mathbf{n}$ denotes the normal with positive $z$ component) $\iint_S \mathbf{f} \cdot \mathbf{n}\, dS =$

| | |
|---|---|
| $-4$ | ✓ |
| 4 | (50%) |

$\pi$ (2 points).

The surface $S$ is a hemisphere. In order to see that $S$ is a subset of the unit sphere centred at $(0,0,0)$ we calculate $x^2+y^2+z^2$ and see that

$$\begin{aligned}\left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2 + \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2 \\ = \frac{4u^2+4v^2+(u^2+v^2)^2-2(u^2+v^2)+1}{(u^2+v^2+1)^2} \\ = \frac{(u^2+v^2)^2+2(u^2+v^2)+1}{(u^2+v^2+1)^2} = 1.\end{aligned}$$

We calculate

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u}(u,v) &= \frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} 2(u^2+v^2+1)-(2u)(2u) \\ -(2u)(2v) \\ 2u(u^2+v^2+1)-2u(u^2+v^2-1)\end{pmatrix} \\ &= \frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} 2(-u^2+v^2+1) \\ -4uv \\ 4u\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial v}(u,v) &= \frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} -(2u)(2v) \\ 2(u^2+v^2+1)-(2v)(2v) \\ 2v(u^2+v^2+1)-2v(u^2+v^2-1)\end{pmatrix} \\ &= \frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} -4uv \\ 2(u^2-v^2+1) \\ 4v\end{pmatrix}.\end{aligned}$$

and so

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u,v) = \frac{-4}{u^2+v^2+1}\begin{pmatrix} 2u \\ 2v \\ u^2+v^2-1\end{pmatrix}.$$

We note that this is correctly aligned for the positive $z$-component. We calculate that $\mathbf{f}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)(u,v)$ is equal to

$$\frac{-4}{u^2+v^2+1}\begin{pmatrix} 0 \\ 0 \\ (u^2+v^2+1)/(u^2+v^2-1)\end{pmatrix} \cdot \begin{pmatrix} 2u \\ 2v \\ u^2+v^2-1\end{pmatrix} = -4.$$

Consequently $\iint_S \mathbf{f}\cdot\mathbf{n}\ dS = \iint_T (-4)\ dudv = -4\pi$ since $\iint_T\ dudv = \pi$.

(10) **Q5**
Fill in the blanks with the correct **integer**, possibly zero or negative.

Let $T = \{(u,v) : u^2 + v^2 \leq 1\} \subset \mathbb{R}^2$ and let $\mathbf{r} : T \to \mathbb{R}^3$,

$$\mathbf{r}(u,v) = \left( \frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-(u^2+v^2)}{1+u^2+v^2} \right)$$

be the representation of the surface $S = \mathbf{r}(T)$.

The surface $S$ is a

| |
|---|
| hemisphere centred at (0,0,0) ✓ |
| cone with point upward |

(2 points).

Calculating the fundamental vector product we find that

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u,v) = \frac{C}{1+u^2+v^2} \begin{pmatrix} 2u \\ 2v \\ 1-(u^2+v^2) \end{pmatrix}$$

where $C =$

| | |
|---|---|
| 4 | ✓ |
| $-4$ | (50%) |

(2 points).

Consider the vector-field $\mathbf{f}(x,y,z) = \left( \begin{smallmatrix} 0 \\ 0 \\ 1/z \end{smallmatrix} \right)$ and evaluate the surface integral (where $\mathbf{n}$ denotes the normal with positive $z$ component) $\iint_S \mathbf{f} \cdot \mathbf{n} \, dS =$

| | |
|---|---|
| 4 | ✓ |
| $-4$ | (50%) |

$\pi$ (2 points).

The surface $S$ is a hemisphere. In order to see that $S$ is a subset of the unit sphere centred at $(0,0,0)$ we calculate $x^2+y^2+z^2$ and see that

$$\begin{aligned}\left(\frac{2u}{u^2+v^2+1}\right)^2+\left(\frac{2v}{u^2+v^2+1}\right)^2+\left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2\\ =\frac{4u^2+4v^2+(u^2+v^2)^2-2(u^2+v^2)+1}{(u^2+v^2+1)^2}\\ =\frac{(u^2+v^2)^2+2(u^2+v^2)+1}{(u^2+v^2+1)^2}=1.\end{aligned}$$

We calculate

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u}(u,v)&=\frac{1}{(u^2+v^2+1)^2}\begin{pmatrix}2(u^2+v^2+1)-(2u)(2u)\\ -(2u)(2v)\\ -2u(u^2+v^2+1)+2u(u^2+v^2-1)\end{pmatrix}\\ &=\frac{1}{(u^2+v^2+1)^2}\begin{pmatrix}2(-u^2+v^2+1)\\ -4uv\\ -4u\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial v}(u,v)&=\frac{1}{(u^2+v^2+1)^2}\begin{pmatrix}-(2u)(2v)\\ 2(u^2+v^2+1)-(2v)(2v)\\ -2v(u^2+v^2+1)+2v(u^2+v^2-1)\end{pmatrix}\\ &=\frac{1}{(u^2+v^2+1)^2}\begin{pmatrix}-4uv\\ 2(u^2-v^2+1)\\ -4v\end{pmatrix}.\end{aligned}$$

and so

$$\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}(u,v)=\frac{4}{u^2+v^2+1}\begin{pmatrix}2u\\ 2v\\ 1-(u^2+v^2)\end{pmatrix}.$$

We note that this is correctly aligned for the positive $z$-component. We calculate that $\mathbf{f}(\mathbf{r}(u,v))\cdot\left(\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right)(u,v)$ is equal to

$$\frac{4}{u^2+v^2+1}\begin{pmatrix}0\\ 0\\ (u^2+v^2+1)/(1-u^2-v^2)\end{pmatrix}\cdot\begin{pmatrix}2u\\ 2v\\ 1-(u^2+v^2)\end{pmatrix}=4.$$

Consequently $\iint_S \mathbf{f}\cdot\mathbf{n}\ dS=\iint_T(4)\ dudv=4\pi$ since $\iint_T\ dudv=\pi$.