Call4.

(1) **Q1**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Find a power series solution of the following differential equation with the initial condition $y(0) = 2, y'(0) = 1$.

$$(1 - x^2)y'' + 2y = 0.$$

By substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, one has

$$\sum_{n=0}^{\infty} \left[(n + \boxed{\text{a}})(n + \boxed{\text{b}})a_{n+2} - (n + \boxed{\text{c}})(n + \boxed{\text{d}})a_n \right] x^n = 0,$$

where $\boxed{\text{a}} > \boxed{\text{b}}$, $\boxed{\text{c}} > \boxed{\text{d}}$. $\boxed{\text{a}}$: $\boxed{2 \checkmark}$ $\boxed{\text{b}}$: $\boxed{1 \checkmark}$ $\boxed{\text{c}}$: $\boxed{1 \checkmark}$ $\boxed{\text{d}}$: $\boxed{-2 \checkmark}$

We have $a_0 = \boxed{\text{e}}$, $a_1 = \boxed{\text{f}}$, $a_2 = \boxed{\text{g}}$, $a_3 = \frac{1}{\boxed{\text{h}}}$.

$\boxed{\text{e}}$: $\boxed{2 \checkmark}$ $\boxed{\text{f}}$: $\boxed{1 \checkmark}$ $\boxed{\text{g}}$: $\boxed{-2 \checkmark}$ $\boxed{\text{h}}$: $\boxed{-3 \checkmark}$

The general coefficients are $a_{2n+1} = \dfrac{\boxed{\text{i}}}{(\boxed{\text{j}}n + \boxed{\text{k}})(\boxed{\text{l}}n + \boxed{\text{m}})}$, where $\boxed{\text{k}} > \boxed{\text{m}}$.

$\boxed{\text{i}}$: $\boxed{-1 \checkmark}$ $\boxed{\text{j}}$: $\boxed{2 \checkmark}$ $\boxed{\text{k}}$: $\boxed{1 \checkmark}$ $\boxed{\text{l}}$: $\boxed{2 \checkmark}$ $\boxed{\text{m}}$: $\boxed{-1 \checkmark}$

The radius of convergence of this series is $\boxed{\text{n}}$.

$\boxed{\text{n}}$: $\boxed{1 \checkmark}$

> Use $y'(x) = \sum_{n=1} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2} n(n-1)a_n x^{n-2}$, and one obtains a recursion relation $a_{n+2} = \frac{n-2}{n+2} a_n$. One also has $a_0 = y(0)$ and $a_1 = y(0)$. The radius of convergence is obtained either by the ratio test or the root test.

(2) **Q1**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Find a power series solution of the following differential equation with the initial condition $y(0) = 1, y'(0) = -\frac{1}{2}$.

$$(1 - x^2)y'' + 2y = 0.$$

By substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, one has

$$\sum_{n=0}^{\infty} \left[(n + \boxed{\text{a}})(n + \boxed{\text{b}})a_{n+2} - (n + \boxed{\text{c}})(n + \boxed{\text{d}})a_n \right] x^n = 0,$$

where $\boxed{\text{a}} > \boxed{\text{b}}, \boxed{\text{c}} > \boxed{\text{d}}$. $\boxed{\text{a}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{b}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{c}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{d}}$: $\boxed{-2 \ \checkmark}$

We have $a_0 = \boxed{\text{e}}, a_1 = \frac{1}{\boxed{\text{f}}}, a_2 = \boxed{\text{g}}, a_3 = \frac{1}{\boxed{\text{h}}}$.

$\boxed{\text{e}}$: $\boxed{\text{i}}$ $\boxed{1 \ \checkmark}$ $\boxed{\text{f}}$: $\boxed{-2 \ \checkmark}$ $\boxed{\text{g}}$: $\boxed{-1 \ \checkmark}$ $\boxed{\text{h}}$: $\boxed{6 \ \checkmark}$

The general coefficients are $a_{2n+1} = \frac{1}{\boxed{\text{i}}(\boxed{\text{j}}n + \boxed{\text{k}})(\boxed{\text{l}}n + \boxed{\text{m}})}$, where $\boxed{\text{k}} > \boxed{\text{m}}$.

$\boxed{\text{i}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{j}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{k}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{l}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{m}}$: $\boxed{-1 \ \checkmark}$

The radius of convergence of this series is $\boxed{\text{n}}$.

$\boxed{\text{n}}$: $\boxed{1 \ \checkmark}$

> Use $y'(x) = \sum_{n=1} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2} n(n-1)a_n x^{n-2}$, and one obtains a recursion relation $a_{n+2} = \frac{n-2}{n+2} a_n$. One also has $a_0 = y(0)$ and $a_1 = y(0)$. The radius of convergence is obtained either by the ratio test or the root test.

(3) **Q2**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Find all stationary points of the function below

$$f(x, y, z, w) = 2x + y + 2z + 2w,$$

under the condition $8 - x^2 - y^2 - z^2 - w^2 = 0, x - y = 0$.

First, we compute the gradient ∇f:

$$\nabla f(x, y, z, w) = \left(\boxed{\text{a}}, \boxed{\text{b}}, \boxed{\text{c}}, \boxed{\text{d}} \right).$$

$\boxed{\text{a}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{b}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{c}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{d}}$: $\boxed{2 \ \checkmark}$

Next, put $g(x, y, z, w) = 8 - x^2 - y^2 - z^2 - w^2$. Compute the gradient ∇g:

$$\nabla g(x, y, z, w) = \left(\boxed{\text{e}} x, \boxed{\text{f}} y, \boxed{\text{g}} z, \boxed{\text{h}} w \right).$$

$\boxed{\text{e}}$: $\boxed{-2 \ \checkmark}$ $\boxed{\text{f}}$: $\boxed{-2 \ \checkmark}$ $\boxed{\text{g}}$: $\boxed{-2 \ \checkmark}$ $\boxed{\text{h}}$: $\boxed{-2 \ \checkmark}$

Put $h(x, y, z, w) = x - y$. Compute thegradient ∇h:

$$\nabla h(x, y, z, w) = \left(\boxed{\text{i}}, \boxed{\text{j}}, 0, 0 \right).$$

$\boxed{\text{i}}$: $\boxed{1 \ \checkmark}$ $\boxed{\text{j}}$: $\boxed{-1 \ \checkmark}$

By Lagrange's multiplier method, introduce $\lambda_1, \lambda_2 \in \mathbb{R}$ and solve the equation $\nabla f(x, y, z, w) = \lambda_1 \nabla g(x, y, z, w) + \lambda_2 \nabla h(x, y, z, w)$. There are two solutions. $(x, y, z, w) = \left(\frac{\boxed{\text{k}}}{\boxed{\text{l}}}, \frac{\boxed{\text{m}}}{\boxed{\text{n}}}, \frac{\boxed{\text{o}}}{\boxed{\text{p}}}, \frac{\boxed{\text{q}}}{\boxed{\text{r}}}\right), \left(-\frac{\boxed{\text{s}}}{\boxed{\text{t}}}, -\frac{\boxed{\text{u}}}{\boxed{\text{v}}}, -\frac{\boxed{\text{w}}}{\boxed{\text{x}}}, -\frac{\boxed{\text{y}}}{\boxed{\text{z}}}\right)$, where $\boxed{\text{k}}, \boxed{\text{l}} > 0$.

$\boxed{\text{k}}$: $\boxed{6 \ \checkmark}$ $\boxed{\text{l}}$: $\boxed{5 \ \checkmark}$ $\boxed{\text{m}}$: $\boxed{6 \ \checkmark}$ $\boxed{\text{n}}$: $\boxed{5 \ \checkmark}$ $\boxed{\text{o}}$: $\boxed{8 \ \checkmark}$ $\boxed{\text{p}}$: $\boxed{5 \ \checkmark}$ $\boxed{\text{q}}$: $\boxed{8 \ \checkmark}$ $\boxed{\text{r}}$: $\boxed{5 \ \checkmark}$ $\boxed{\text{s}}$: $\boxed{6 \ \checkmark}$ $\boxed{\text{t}}$: $\boxed{5 \ \checkmark}$ $\boxed{\text{u}}$: $\boxed{6 \ \checkmark}$ $\boxed{\text{v}}$: $\boxed{5 \ \checkmark}$ $\boxed{\text{w}}$: $\boxed{8 \ \checkmark}$ $\boxed{\text{x}}$: $\boxed{5 \ \checkmark}$ $\boxed{\text{y}}$: $\boxed{8 \ \checkmark}$ $\boxed{\text{z}}$: $\boxed{5 \ \checkmark}$

Choose a correct statement.

- The both solutions are local minuma.
- The first solution is a local minumum and the second one is a saddle point.
- The first solution is a local minumum and the second one is a local maximum.
- The first solution is a saddle point and the second one is a local minumum.
- The first solution is a saddle point and the second one is a local minumum.
- The both solutions are saddle points.
- The first solution is a local maximum and the second one is a local minumum. ✓
- The first solution is a local maximum and the second one is a saddle point.
- The both solutions are local maxima.

By $h(x, y, z, w) = 0$, one has $x = y$ and by the equation of Lagrange's method, it also follows that $1 = \lambda_1 z = \lambda_1 w$, hence $z = w$. Plug this to $g(x, y, z, w) = 0$ to get $x^2 + z^2 = 4$. Again by Lagrange's method for the x and y components, one gets $3 = 4\lambda_1 x$. From this it follows $x = \pm\frac{6}{5}$ and all the rest. Plus the solutions to f to see which is larger.

(4) **Q2**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Find all stationary points of the function below

$$f(x, y, z, w) = -2x - y - 2z - 2w,$$

under the condition $x^2 + y^2 + z^2 + w^2 = 8, -x + y = 0$.

First, we compute the gradient ∇f:

$$\nabla f(x, y, z, w) = \left(\boxed{a}, \boxed{b}, \boxed{c}, \boxed{d}\right).$$

$\boxed{a}$: $\boxed{-2 \ ✓}$ $\boxed{b}$: $\boxed{-1 \ ✓}$ $\boxed{c}$: $\boxed{-2 \ ✓}$ $\boxed{d}$: $\boxed{-2 \ ✓}$

Next, put $g(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 8$. Compute the gradient ∇g:

$$\nabla g(x, y, z, w) = \left(\boxed{e}x, \boxed{f}y, \boxed{g}z, \boxed{h}w\right).$$

$\boxed{e}$: $\boxed{2 \ ✓}$ $\boxed{f}$: $\boxed{2 \ ✓}$ $\boxed{g}$: $\boxed{2 \ ✓}$ $\boxed{h}$: $\boxed{2 \ ✓}$

Put $h(x, y, z, w) = -x + y$. Compute thegradient ∇h:

$$\nabla h(x, y, z, w) = \left(\boxed{i}, \boxed{j}, 0, 0\right).$$

$\boxed{i}$: $\boxed{-1 \ ✓}$ $\boxed{j}$: $\boxed{1 \ ✓}$

By Lagrange's multiplier method, introduce $\lambda_1, \lambda_2 \in \mathbb{R}$ and solve the equation $\nabla f(x, y, z, w) = \lambda_1 \nabla g(x, y, z, w) + \lambda_2 \nabla h(x, y, z, w)$.

There are two solutions. $(x, y, z, w) = \left(\frac{\boxed{k}}{\boxed{l}}, \frac{\boxed{m}}{\boxed{n}}, \frac{\boxed{o}}{\boxed{p}}, \frac{\boxed{q}}{\boxed{r}}\right), \left(-\frac{\boxed{s}}{\boxed{t}}, -\frac{\boxed{u}}{\boxed{v}}, -\frac{\boxed{w}}{\boxed{x}}, -\frac{\boxed{y}}{\boxed{z}}\right)$,

where $\boxed{k}, \boxed{l} > 0$.

$\boxed{k}$: $\boxed{6 \ ✓}$ $\boxed{l}$: $\boxed{5 \ ✓}$ $\boxed{m}$: $\boxed{6 \ ✓}$ $\boxed{n}$: $\boxed{5 \ ✓}$ $\boxed{o}$: $\boxed{8 \ ✓}$ $\boxed{p}$: $\boxed{5 \ ✓}$ $\boxed{q}$: $\boxed{8 \ ✓}$ $\boxed{r}$: $\boxed{5 \ ✓}$ $\boxed{s}$: $\boxed{6 \ ✓}$ $\boxed{t}$: $\boxed{5 \ ✓}$ $\boxed{u}$:

$\boxed{6\ \checkmark}$ $\boxed{\text{v}}$: $\boxed{5\ \checkmark}$ $\boxed{\text{w}}$: $\boxed{8\ \checkmark}$ $\boxed{\text{x}}$: $\boxed{5\ \checkmark}$ $\boxed{\text{y}}$: $\boxed{8\ \checkmark}$ $\boxed{\text{z}}$: $\boxed{5\ \checkmark}$

Choose a correct statement.

- The both solutions are local minuma.
- The first solution is a local minumum and the second one is a saddle point.
- The first solution is a local minumum and the second one is a local maximum. ✓
- The first solution is a saddle point and the second one is a local minumum.
- The first solution is a saddle point and the second one is a local minumum.
- The both solutions are saddle points.
- The first solution is a local maximum and the second one is a local minumum.
- The first solution is a local maximum and the second one is a saddle point.
- The both solutions are local maxima.

> By $h(x, y, z, w) = 0$, one has $x = y$ and by the equation of Lagrange's method, it also follows that $1 = \lambda_1 z = \lambda_1 w$, hence $z = w$. Plug this to $g(x, y, z, w) = 0$ to get $x^2 + z^2 = 4$. Again by Lagrange's method for the x and y components, one gets $3 = 4\lambda_1 x$. From this it follows $x = \pm\frac{6}{5}$ and all the rest. Plus the solutions to f to see which is larger.

(5) **Q3**

Determine which of the following is a parametrization of the path

$$C = \{(x, y) : x^2 + y^2 = 4, x \leq 0, y \geq 0\} \subset \mathbb{R}^2,$$

starting at $(-2, 0)$ and finishing at $(0, 2)$ ($\frac{1}{2}$ point each):

- $(-2\cos t, 2\sin t)$, $t \in [0, \frac{\pi}{2}]$ | is ✓ | is not |
- $(2\sin t, 2\cos t)$, $t \in [0, \frac{\pi}{2}]$ | is | is not ✓ |
- $(\sqrt{4 - t^2}, t)$, $t \in [0, 2]$ | is | is not ✓ |
- $(2\cos t, 2\sin t)$, $t \in [\frac{\pi}{2}, \pi]$ | is | is not ✓ |

- $(2\sin t, 2\cos t)$, $t \in [-\frac{\pi}{2}, 0]$

is	✓
is not	

- $(t, \sqrt{4-t^2})$, $t \in [-2, 0]$

is	✓
is not	

- $(-\sqrt{4-t^2}, t)$, $t \in [0, 2]$

is	✓
is not	

- $(t, t+2)$, $t \in [-2, 0]$

is	
is not	✓

If C is the path above and

$$\mathbf{f}(x,y) = \begin{pmatrix} y^2 \\ x^2 \end{pmatrix}$$

is a vector field on $\mathbf{R}^2$ then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} =$

32	✓
−32	(50%)

/3. Fill in the blank with the correct integer, possibly zero or negative (2 points).

Picking the parametrization $(-2\cos t, 2\sin t)$, $t \in [0, \frac{\pi}{2}]$ we calculate that

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2\sin t \\ 2\cos t \end{pmatrix}$$

and also

$$\begin{pmatrix} 4\sin^2 t \\ 4\cos^2 t \end{pmatrix} \cdot \begin{pmatrix} 2\sin t \\ 2\cos t \end{pmatrix} = 8(\sin^3 t + \cos^3 t).$$

Consequently

$$\int_C \mathbf{f}\ d\boldsymbol{\alpha} = 8 \int_0^{\frac{\pi}{2}} \sin^3 t + \cos^3 t\ dt.$$

Since $\int \cos^3 t\ dt = (\sin 3t + 9\sin t)/12$ and $\int \sin^3 t\ dt = (\cos 3t - 9\cos t)/12$,

$$\begin{aligned} \int_C \mathbf{f}\ d\boldsymbol{\alpha} &= \frac{2}{3}\left[\sin 3t + 9\sin t + \cos 3t - 9\cos t\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{3}(-1+9-1+9) = \frac{32}{3}. \end{aligned}$$

(6) **Q3**
Determine which of the following is a parametrization of the path

$$C = \{(x,y) : x^2 + y^2 = 4, x \le 0, y \ge 0\} \subset \mathbb{R}^2,$$

starting at $(-2,0)$ and finishing at $(0,2)$ ($\frac{1}{2}$ point each):

- $(2\sin t, 2\cos t)$, $t \in [0, \frac{\pi}{2}]$ $\boxed{\begin{array}{l} \text{is} \\ \hline \text{is not } \checkmark \end{array}}$
- $(-2\cos t, 2\sin t)$, $t \in [0, \frac{\pi}{2}]$ $\boxed{\begin{array}{l} \text{is } \checkmark \\ \hline \text{is not} \end{array}}$
- $(2\sin t, 2\cos t)$, $t \in [-\frac{\pi}{2}, 0]$ $\boxed{\begin{array}{l} \text{is } \checkmark \\ \hline \text{is not} \end{array}}$
- $(t, \sqrt{4-t^2})$, $t \in [-2, 0]$ $\boxed{\begin{array}{l} \text{is } \checkmark \\ \hline \text{is not} \end{array}}$
- $(\sqrt{4-t^2}, t)$, $t \in [0, 2]$ $\boxed{\begin{array}{l} \text{is} \\ \hline \text{is not } \checkmark \end{array}}$
- $(2\cos t, 2\sin t)$, $t \in [\frac{\pi}{2}, \pi]$ $\boxed{\begin{array}{l} \text{is} \\ \hline \text{is not } \checkmark \end{array}}$
- $(t, t+2)$, $t \in [-2, 0]$ $\boxed{\begin{array}{l} \text{is} \\ \hline \text{is not } \checkmark \end{array}}$
- $(-\sqrt{4-t^2}, t)$, $t \in [0, 2]$ $\boxed{\begin{array}{l} \text{is } \checkmark \\ \hline \text{is not} \end{array}}$

If C is the path above and

$$\mathbf{f}(x,y) = \begin{pmatrix} y^2 \\ x^2 \end{pmatrix}$$

is a vector field on $\mathbf{R}^2$ then $\int_C \mathbf{f}\ d\boldsymbol{\alpha} = \boxed{\begin{array}{ll} 32 & \checkmark \\ -32 & (50\%) \end{array}}$ /3. Fill in the blank with the correct integer, possibly zero or negative (2 points).

Picking the parametrization $(-2\cos t, 2\sin t)$, $t \in [0, \frac{\pi}{2}]$ we calculate that

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2\sin t \\ 2\cos t \end{pmatrix}$$

and also

$$\begin{pmatrix} 4\sin^2 t \\ 4\cos^2 t \end{pmatrix} \cdot \begin{pmatrix} 2\sin t \\ 2\cos t \end{pmatrix} = 8(\sin^3 t + \cos^3 t).$$

Consequently

$$\int_C \mathbf{f}\ d\boldsymbol{\alpha} = 8\int_0^{\frac{\pi}{2}} \sin^3 t + \cos^3 t\ dt.$$

Since $\int \cos^3 t\ dt = (\sin 3t + 9\sin t)/12$ and $\int \sin^3 t\ dt = (\cos 3t - 9\cos t)/12$,

$$\begin{aligned}\int_C \mathbf{f}\ d\boldsymbol{\alpha} &= \frac{2}{3}\left[\sin 3t + 9\sin t + \cos 3t - 9\cos t\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{3}(-1+9-1+9) = \frac{32}{3}.\end{aligned}$$

(7) **Q4**

We wish to evaluate the integral

$$I = \iiint_V 2x + 3y + 4z\ dxdydz$$

where the integral is over the half ellipsoid

$$V = \left\{(x,y,z) : \frac{x^2}{4} + y^2 + z^2 \leq 1, y \geq 0\right\} \subset \mathbb{R}^3.$$

We choose a change of coordinates $x = r\cos\theta$, $y = \frac{1}{2}r\sin\theta$, $z = z$ under which V is sent to

$$W = \left\{(r,\theta,z) : 0 \leq \theta \leq \boxed{\text{a}}\pi, \boxed{\text{b}} \leq r \leq \boxed{\text{c}}, |z| \leq \sqrt{\boxed{\text{d}} - \frac{r^2}{\boxed{\text{e}}}}\right\}$$

and the Jacobian is $J(r,\theta,z) = \frac{r}{\boxed{\text{f}}}$.

Fill in the following blanks with the correct integers, possibly zero or negative ($\frac{1}{2}$ point each): $\boxed{\text{a}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{0\ \checkmark}$ $\boxed{\text{c}}$: $\boxed{2\ \checkmark}$ $\boxed{\text{d}}$: $\boxed{1\ \checkmark}$ $\boxed{\text{e}}$: $\boxed{4\ \checkmark}$ $\boxed{\text{f}}$: $\boxed{2\ \checkmark}$

Fill in the following blank with the correct integer, possibly zero or negative (3 points). Evaluating the integral we obtain the final result $I = \boxed{3\ \checkmark}\ \frac{\pi}{2}$.

We calculate the Jacobian determinant

$$J(r,\theta,z)=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \frac{1}{2}\sin\theta & \frac{1}{2}r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=\frac{r}{2}.$$

Observing the symmetry of the problem in x and z,

$$I=\iiint_V 2x+3y+4z\ dxdydz=\iiint_V 3y\ dxdydz.$$

Using the change of variables

$$I=3\int_0^\pi\int_0^2\int_{-\sqrt{1-\frac{r^2}{4}}}^{\sqrt{1-\frac{r^2}{4}}}\left(\frac{r}{2}\right)\left(\frac{r}{2}\sin\theta\right)\ dzdrd\theta$$

. Since $\int_0^\pi\sin\theta\ d\theta=2$ and $\int_{-\sqrt{1-\frac{r^2}{4}}}^{\sqrt{1-\frac{r^2}{4}}}\ dz=2\sqrt{1-\frac{r^2}{4}}$,

$$I=3\int_0^2 r^2\sqrt{1-\tfrac{r^2}{4}}\ dr.$$

It is convenient to change variables letting $r=2\sin t$ and hence

$$I=3\int_0^{\frac{\pi}{2}}8\sin^2 t\cos^2 t\ dt.$$

Using the double angle formulae $\sin 2t=2\sin t\cos t$ and $\cos 2t=1-2\sin^2 t$ we know that $8\sin^2 t\cos^2 t=1-\cos 4t$ and so

$$I=3\int_0^{\frac{\pi}{2}}1-\cos 4t\ dt=\frac{3}{2}\pi.$$

(8) **Q4**

We wish to evaluate the integral

$$I=\iiint_V 4x+5y+6z\ dxdydz$$

where the integral is over the half ellipsoid

$$V=\left\{(x,y,z):\frac{x^2}{4}+y^2+z^2\le 1, y\ge 0\right\}\subset\mathbb{R}^3.$$

We choose a change of coordinates $x=r\cos\theta$, $y=\frac{1}{2}r\sin\theta$, $z=z$ under which V is sent to

$$W=\left\{(r,\theta,z):0\le\theta\le\boxed{\text{a}}\pi,\boxed{\text{b}}\le r\le\boxed{\text{c}},|z|\le\sqrt{\boxed{\text{d}}-\frac{r^2}{\boxed{\text{e}}}}\right\}$$

and the Jacobian is $J(r,\theta,z) = \frac{r}{\boxed{\text{f}}}$.

Fill in the following blanks with the correct integers, possibly zero or negative ($\frac{1}{2}$ point each): $\boxed{\text{a}}$: $\boxed{1 \quad \checkmark}$ $\boxed{\text{b}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{c}}$: $\boxed{2 \quad \checkmark}$ $\boxed{\text{d}}$: $\boxed{1 \quad \checkmark}$ $\boxed{\text{e}}$: $\boxed{4 \quad \checkmark}$ $\boxed{\text{f}}$: $\boxed{2 \quad \checkmark}$

Fill in the following blank with the correct integer, possibly zero or negative (3 points). Evaluating the integral we obtain the final result $I = \boxed{5 \quad \checkmark}\ \frac{\pi}{2}$.

We calculate the Jacobian determinant

$$J(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \frac{1}{2}\sin\theta & \frac{1}{2}r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{r}{2}.$$

Observing the symmetry of the problem in x and z,

$$I = \iiint_V 4x + 5y + 6z \; dxdydz = \iiint_V 5y \; dxdydz.$$

Using the change of variables

$$I = 5\int_0^{\pi}\int_0^2\int_{-\sqrt{1-\frac{r^2}{4}}}^{\sqrt{1-\frac{r^2}{4}}} \left(\frac{r}{2}\right)\left(\frac{r}{2}\sin\theta\right)\; dzdrd\theta$$

. Since $\int_0^{\pi}\sin\theta\; d\theta = 2$ and $\int_{-\sqrt{1-\frac{r^2}{4}}}^{\sqrt{1-\frac{r^2}{4}}}\; dz = 2\sqrt{1-\frac{r^2}{4}}$,

$$I = 5\int_0^2 r^2\sqrt{1-\tfrac{r^2}{4}}\; dr.$$

It is convenient to change variables letting $r = 2\sin t$ and hence

$$I = 5\int_0^{\frac{\pi}{2}} 8\sin^2 t\cos^2 t\; dt.$$

Using the double angle formulae $\sin 2t = 2\sin t\cos t$ and $\cos 2t = 1-2\sin^2 t$ we know that $8\sin^2 t\cos^2 t = 1-\cos 4t$ and so

$$I = 5\int_0^{\frac{\pi}{2}} 1-\cos 4t\; dt = \frac{5}{2}\pi.$$

(9) **Q5**

Consider the surface $S = \{(x, y, z) : x^2 + y^2 - z^2 = 1, 0 \leq z \leq 2\sqrt{2}\} \subset \mathbb{R}^3$ and the vector-field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} xz \\ yz \\ 0 \end{pmatrix}.$$

We parametrize S by $\mathbf{r}(u, v) = (v \cos u, v \sin u, \sqrt{v^2 - \boxed{?}})$ where $0 \leq u \leq \boxed{2 \ \checkmark}\,\pi$, $\boxed{1 \ \checkmark} \leq v \leq \boxed{3 \ \checkmark}$ and where $\boxed{?}$ should be $\boxed{1 \ \checkmark}$ ($\frac{1}{2}$ point each). Calculate the fundamental vector product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u, v)$ and hence evaluate the surface integral $\iint_S \mathbf{f} \cdot \mathbf{n} \; dS = \boxed{\begin{matrix} 40 & \checkmark \\ -40 & (50\%) \end{matrix}}\ \pi$ (4 points) where $\mathbf{n}$ is the unit normal vector with negative z-component. Fill in the blanks with the correct integers, possibly zero or negative.

We calculate

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) = \begin{pmatrix} -v \sin u \\ v \cos u \\ 0 \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial v}(u, v) = \begin{pmatrix} \cos u \\ \sin u \\ v(v^2 - 1)^{-\frac{1}{2}} \end{pmatrix}$$

and so

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u, v) = \begin{pmatrix} v^2 \cos u (v^2 - 1)^{-\frac{1}{2}} \\ v^2 \sin u (v^2 - 1)^{-\frac{1}{2}} \\ -v \end{pmatrix}.$$

At this point we note that this is correctly aligned for the negative z-component. Moreover

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u, v) \cdot \mathbf{f}(\mathbf{r}(u, v)) = v^3.$$

This means that

$$\begin{aligned} \iint_S \mathbf{f} \cdot \mathbf{n} \; dS &= \int_0^{2\pi} du \int_1^3 v^3 \; dv \\ &= 2\pi \int_1^3 v^3 \; dv = \frac{\pi}{2}(3^4 - 1) = 40\pi. \end{aligned}$$

(10) **Q5**

Consider the surface $S = \{(x, y, z) : x^2 + y^2 - z^2 = 1, 0 \leq z \leq 2\sqrt{6}\} \subset \mathbb{R}^3$ and the vector-field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} xz \\ yz \\ 0 \end{pmatrix}.$$

We parametrize S by $\mathbf{r}(u, v) = (v \cos u, v \sin u, \sqrt{v^2 - \boxed{?}})$ where $0 \leq u \leq \boxed{2 \quad \checkmark}\ \pi$, $\boxed{1 \quad \checkmark} \leq v \leq \boxed{5 \quad \checkmark}$ and where $\boxed{?}$ should be $\boxed{1 \quad \checkmark}$ ($\frac{1}{2}$ point each). Calculate the fundamental vector product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u, v)$ and hence evaluate the surface integral $\iint_S \mathbf{f} \cdot \mathbf{n}\ dS = \boxed{\begin{array}{ll} 312 & \checkmark \\ -312 & (50\%) \end{array}}\ \pi$ (4 points) where $\mathbf{n}$ is the unit normal vector with negative z-component. Fill in the blanks with the correct integers, possibly zero or negative.

We calculate

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) = \begin{pmatrix} -v \sin u \\ v \cos u \\ 0 \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial v}(u, v) = \begin{pmatrix} \cos u \\ \sin u \\ v(v^2 - 1)^{-\frac{1}{2}} \end{pmatrix}$$

and so

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u, v) = \begin{pmatrix} v^2 \cos u (v^2 - 1)^{-\frac{1}{2}} \\ v^2 \sin u (v^2 - 1)^{-\frac{1}{2}} \\ -v \end{pmatrix}.$$

At this point we note that this is correctly aligned for the negative z-component. Moreover

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u, v) \cdot \mathbf{f}(\mathbf{r}(u, v)) = v^3.$$

This means that

$$\begin{aligned} \iint_S \mathbf{f} \cdot \mathbf{n}\ dS &= \int_0^{2\pi} du \int_1^3 v^3\ dv \\ &= 2\pi \int_1^5 v^3\ dv = \frac{\pi}{2}(5^4 - 1) = 312\pi. \end{aligned}$$