Call3.

(1) **Q1**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

First, put $g(y) = \arctan(2y)$. Then

$$g'(y) = \frac{\boxed{\text{a}}}{\boxed{\text{b}}\,y^{\boxed{\text{c}}} + 1}.$$

$\boxed{\text{a}}$: $\boxed{2\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{4\ \checkmark}$ $\boxed{\text{c}}$: $\boxed{2\ \checkmark}$

> Use $(\arctan z) = \frac{1}{z^2+1}$ and the chain rule.

The function $g'(y)$ has the following expansion around $y = 0$.

$$g'(y) = \boxed{\text{a}} \sum_{n=0}^{\infty} (\boxed{\text{d}})^n y^{\boxed{\text{e}}n}.$$

$\boxed{\text{d}}$: $\boxed{-4\ \checkmark}$ $\boxed{\text{e}}$: $\boxed{2\ \checkmark}$

> Use the geometric series $\frac{1}{z+1} = \sum_{n=0}^{\infty}(-1)^n z^n$.

From this we obtain

$$g(y) = \boxed{\text{a}} \sum_{n=0}^{\infty} \frac{1}{\boxed{\text{f}}n + \boxed{\text{g}}} (\boxed{\text{d}})^n y^{\boxed{\text{e}}n+1}.$$

$\boxed{\text{f}}$: $\boxed{2\ \checkmark}$ $\boxed{\text{g}}$: $\boxed{1\ \checkmark}$

> If $h'(y) = \sum_{n=0}^{\infty} b_n y^n$, then $h(y) = \sum_{n=0} \frac{b_n}{n+1} y^{n+1} + C$, and in this case $C = 0$ because $\arctan 0 = 0$.

Find the Taylor expansion of the function below around $x = 1$, following the suggested steps.

$$f(x) = x \arctan(2(x-1)).$$

First, we have

$$\arctan(2(x-1)) = \boxed{\text{a}} \sum_{n=0}^{\infty} \frac{1}{\boxed{\text{f}}n + \boxed{\text{g}}} (\boxed{\text{d}})^n (x-1)^{\boxed{\text{e}}n+1}.$$

Next, using $f(x) = ((x-1)+1)\arctan(2(x-1)) = \sum_{n=0}^{\infty} a_n(x-1)^n$, we can find its Taylor expansion around $x = 1$. It holds that $a_3 = -\frac{\boxed{\text{h}}}{\boxed{\text{i}}}$

$\boxed{\text{h}}$: $\boxed{8\ \checkmark}$ $\boxed{\text{i}}$: $\boxed{3\ \checkmark}$

> The expansion $\arctan((x-1))$ has only the odd terms $(x-1)^{2n+1}$, and when multiplied by $(x-1)$, they become even, so they do not contribute to a_3, hence one just has to take the coefficient of $(x-1)^3$ in the expansion of $\arctan(2(x-1))$.

The radius of convergence of this expansion is $\frac{1}{\boxed{\text{j}}}$.

$\boxed{\text{j}}$: $\boxed{2\ \checkmark}$

> Use the ration test.

(2) **Q1**
Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction appears, write the simplified form (for example, $\frac{1}{2}$ is accepted but not $\frac{2}{4}$).

First, put $g(y) = \arctan(3y)$. Then

$$g'(y) = \frac{\boxed{\text{a}}}{\boxed{\text{b}}y^{\boxed{\text{c}}} + 1}.$$

$\boxed{\text{a}}$: $\boxed{3\ \checkmark}$ $\boxed{\text{b}}$: $\boxed{9\ \checkmark}$ $\boxed{\text{c}}$: $\boxed{2\ \checkmark}$

> Use $(\arctan z) = \frac{1}{z^2+1}$ and the chain rule.

The function $g'(y)$ has the following expansion.

$$g'(y) = \boxed{\text{a}} \sum_{n=0}^{\infty} (\boxed{\text{d}})^n y^{\boxed{\text{e}}n}.$$

$\boxed{\text{d}}$: $\boxed{-9\ \checkmark}$ $\boxed{\text{e}}$: $\boxed{2\ \checkmark}$

> Use the geometric series $\frac{1}{z+1} = \sum_{n=0}^{\infty}(-1)^n z^n$.

From this we obtain

$$g(y) = \boxed{\text{a}} \sum_{n=0}^{\infty} \frac{1}{\boxed{\text{f}}n + \boxed{\text{g}}} (\boxed{\text{d}})^n y^{\boxed{\text{e}}n+1}.$$

$\boxed{\text{f}}$: $\boxed{2 \ \checkmark}$ $\boxed{\text{g}}$: $\boxed{1 \ \checkmark}$

> If $h'(y) = \sum_{n=0}^{\infty} b_n y^n$, then $h(y) = \sum_{n=0} \frac{b_n}{n+1} y^{n+1} + C$, and in this case $C = 0$ because $\arctan 0 = 0$.

Find the Taylor expansion of the function below around $x = 1$, following the suggested steps.

$$f(x) = x \arctan(3(x-1)).$$

First, we have

$$\arctan(3(x-1)) = \boxed{\text{a}} \sum_{n=0}^{\infty} \frac{1}{\boxed{\text{f}}n + \boxed{\text{g}}} (\boxed{\text{d}})^n (x-1)^{\boxed{\text{e}}n+1}.$$

Next, using $f(x) = ((x-1)+1)\arctan(3(x-1)) = \sum_{n=0}^{\infty} a_n (x-1)^n$ we can find its Taylor expansion around $x = 1$. It holds that $a_1 = \boxed{\text{h}}, a_3 = \boxed{\text{i}}$.

$\boxed{\text{h}}$: $\boxed{3 \ \checkmark}$ $\boxed{\text{i}}$: $\boxed{-9 \ \checkmark}$

> The expansion $\arctan(3(x-1))$ has only the odd terms $(x-1)^{2n+1}$, and when multiplied by $(x-1)$, they become even, so they do not contribute to a_1, a_3, hence one just has to take the coefficient of $(x-1), (x-1)^3$ in the expansion of $\arctan(3(x-1))$.

The radius of convergence of this expansion is $\frac{1}{\boxed{\text{j}}}$.

$\boxed{\text{j}}$: $\boxed{3 \ \checkmark}$

> Use the ratio test.

(3) **Q1**

Fill in the blanks with **integers (possibly** 0 **or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

First, put $g(y) = \arctan(4y)$. Then

$$g'(y) = \frac{\boxed{\text{a}}}{\boxed{\text{b}}y^{\boxed{\text{c}}} + 1}.$$

$\boxed{\text{a}}$: $\boxed{4 \quad \checkmark}$ $\boxed{\text{b}}$: $\boxed{16 \quad \checkmark}$ $\boxed{\text{c}}$: $\boxed{2 \quad \checkmark}$

> Use $(\arctan z) = \frac{1}{z^2+1}$ and the chain rule.

The function $g'(y)$ has the following expansion.

$$g'(y) = \boxed{\text{a}} \sum_{n=0}^{\infty} (\boxed{\text{d}})^n y^{\boxed{\text{e}}n}.$$

$\boxed{\text{d}}$: $\boxed{-16 \quad \checkmark}$ $\boxed{\text{e}}$: $\boxed{2 \quad \checkmark}$

> Use the geometric series $\frac{1}{z+1} = \sum_{n=0}^{\infty}(-1)^n z^n$.

From this we obtain

$$g(y) = \boxed{\text{a}} \sum_{n=0}^{\infty} \frac{1}{\boxed{\text{f}}n + \boxed{\text{g}}} (\boxed{\text{d}})^n y^{\boxed{\text{e}}n+1}.$$

$\boxed{\text{f}}$: $\boxed{2 \quad \checkmark}$ $\boxed{\text{g}}$: $\boxed{1 \quad \checkmark}$

> If $h'(y) = \sum_{n=0}^{\infty} b_n y^n$, then $h(y) = \sum_{n=0} \frac{b_n}{n+1} y^{n+1} + C$, and in this case $C = 0$ because $\arctan 0 = 0$.

Find the Taylor expansion of the function below around $x = 1$, following the suggested steps.

$$f(x) = x \arctan(4(x-1)^2).$$

First, we have

$$\arctan(4(x-1)) = \boxed{\text{a}} \sum_{n=0}^{\infty} \frac{1}{\boxed{\text{f}}n + \boxed{\text{g}}} (\boxed{\text{d}})^n (x-1)^{\boxed{\text{e}}n+1}.$$

Next, using $f(x) = ((x-1)+1)\arctan(4(x-1)) = \sum_{n=0}^{\infty} a_n (x-1)^n$, we can find its Taylor expansion around $x = 1$. It holds that $a_3 = -\frac{\boxed{\text{h}}}{\boxed{\text{i}}}$.

$\boxed{\text{h}}$: $\boxed{64 \quad \checkmark}$ $\boxed{\text{i}}$: $\boxed{3 \quad \checkmark}$

The expansion $\arctan(4(x-1))$ has only the odd terms $(x-1)^{2n+1}$, and when multiplied by $(x-1)$, they become even, so they do not contribute to a_3, hence one just has to take the coefficient of $(x-1)^3$ in the expansion of $\arctan(4(x-1))$.

The radius of convergence of this expansion is $\frac{1}{\boxed{\text{j}}}$.

$\boxed{\text{j}}$: $\boxed{4 \quad \checkmark}$

Use the ratio test.

(4) **Q2**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Find all stationary points of the function below, following the suggested steps.

$$f(x,y) = x^4 - 4xy + 2y^2.$$

First, we compute the gradient ∇f:

$$\nabla f(x,y) = \begin{pmatrix} \boxed{\text{a}}x^{\boxed{\text{b}}} + \boxed{\text{c}}y) \\ \boxed{\text{d}}x + \boxed{\text{e}}y) \end{pmatrix}.$$

$\boxed{\text{a}}$: $\boxed{4 \quad \checkmark}$ $\boxed{\text{b}}$: $\boxed{3 \quad \checkmark}$ $\boxed{\text{c}}$: $\boxed{-4 \quad \checkmark}$ $\boxed{\text{d}}$: $\boxed{-4 \quad \checkmark}$ $\boxed{\text{e}}$: $\boxed{4 \quad \checkmark}$

Use $(z^n)' = nz^{n-1}$.

The equation $\nabla f(x,y) = \mathbf{0}$ has three solutions. They are $(x,y) = (\boxed{\text{f}}, \boxed{\text{g}}), (\boxed{\text{h}}, \boxed{\text{i}}), (\boxed{\text{j}}, \boxed{\text{k}})$, where $\boxed{\text{f}} > 0, \boxed{\text{h}} < 0$.

$\boxed{\text{f}}$: $\boxed{1 \quad \checkmark}$ $\boxed{\text{g}}$: $\boxed{1 \quad \checkmark}$ $\boxed{\text{h}}$: $\boxed{-1 \quad \checkmark}$ $\boxed{\text{i}}$: $\boxed{-1 \quad \checkmark}$ $\boxed{\text{j}}$: $\boxed{0 \quad \checkmark}$ $\boxed{\text{k}}$: $\boxed{0 \quad \checkmark}$

By taking the difference of two equations, eliminate y.

Consider the first of them $(\boxed{\text{f}}, \boxed{\text{g}})$. The determinant of the Hessian at this point is

- positive ✓
- 0

- negative

At this point, the function $f(x,y)$ takes a
- local minimum ✓
- saddle point
- local maximum

Consider the solution ($\boxed{\text{h}}$, $\boxed{\text{i}}$). The determinant of the Hessian at this point is
- positive ✓
- 0
- negative

At this point, the function $f(x,y)$ takes a
- local minimum ✓
- saddle point
- local maximum

(5) **Q2**

Fill in the blanks with **integers (possibly 0 or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Find all stationary points of the function below, following the suggested steps.

$$f(x,y) = x^4 + 4xy + 2y^2.$$

First, we compute the gradient ∇f:

$$\nabla f(x,y) = \begin{pmatrix} \boxed{\text{a}}x^{\boxed{\text{b}}} + \boxed{\text{c}}y) \\ \boxed{\text{d}}x + \boxed{\text{e}}y) \end{pmatrix}.$$

$\boxed{\text{a}}$: $\boxed{4\ ✓}$ $\boxed{\text{b}}$: $\boxed{3\ ✓}$ $\boxed{\text{c}}$: $\boxed{4\ ✓}$ $\boxed{\text{d}}$: $\boxed{4\ ✓}$ $\boxed{\text{e}}$: $\boxed{4\ ✓}$

> Use $(z^n)' = nz^{n-1}$.

The equation $\nabla f(x,y) = \mathbf{0}$ has three solutions. They are $(x,y) = (\boxed{\text{f}}, \boxed{\text{g}}), (\boxed{\text{h}}, \boxed{\text{i}}), (\boxed{\text{j}}, \boxed{\text{k}})$, where $\boxed{\text{f}} > 0, \boxed{\text{h}} < 0$.

$\boxed{\text{f}}$: $\boxed{1\ ✓}$ $\boxed{\text{g}}$: $\boxed{-1\ ✓}$ $\boxed{\text{h}}$: $\boxed{-1\ ✓}$ $\boxed{\text{i}}$: $\boxed{1\ ✓}$ $\boxed{\text{j}}$: $\boxed{0\ ✓}$ $\boxed{\text{k}}$: $\boxed{0\ ✓}$

> By taking the difference of two equations, eliminate y.

Consider the first of them ($\boxed{\text{f}}$, $\boxed{\text{g}}$). The determinant of the Hessian at this point is

- positive ✓
- 0
- negative

At this point, the function $f(x, y)$ takes a

- local minimum ✓
- saddle point
- local maximum

Consider the solution ($\boxed{\text{j}}$, $\boxed{\text{k}}$). The determinant of the Hessian at this point is

- positive
- 0
- negative ✓

At this point, the function $f(x, y)$ takes a

- local minimum
- saddle point ✓
- local maximum

(6) **Q2**

Fill in the blanks with **integers (possibly** 0 **or negative)**, unless otherwise specified. If a fraction or a root appears, write the simplified form (for example, $\frac{1}{2}$ and $2\sqrt{2}$ are accepted but not $\frac{2}{4}$ and $\sqrt{8}$).

Find all stationary points of the function below, following the suggested steps.

$$f(x, y) = 2x^4 - 4xy + y^2.$$

First, we compute the gradient ∇f:

$$\nabla f(x, y) = \begin{pmatrix} \boxed{\text{a}}x^{\boxed{\text{b}}} + \boxed{\text{c}}y) \\ \boxed{\text{d}}x + \boxed{\text{e}}y) \end{pmatrix}.$$

$\boxed{\text{a}}$: $\boxed{8\ ✓}$ $\boxed{\text{b}}$: $\boxed{3\ ✓}$ $\boxed{\text{c}}$: $\boxed{-4\ ✓}$ $\boxed{\text{d}}$: $\boxed{-4\ ✓}$ $\boxed{\text{e}}$: $\boxed{2\ ✓}$

> Use $(z^n)' = nz^{n-1}$.

The equation $\nabla f(x, y) = \mathbf{0}$ has three solutions. They are $(x, y) = (\boxed{\text{f}}, \boxed{\text{g}}), (\boxed{\text{h}}, \boxed{\text{i}}), (\boxed{\text{j}}, \boxed{\text{k}})$, where $\boxed{\text{f}} > 0, \boxed{\text{h}} < 0$.

$\boxed{\text{f}}$: $\boxed{1\ ✓}$ $\boxed{\text{g}}$: $\boxed{2\ ✓}$ $\boxed{\text{h}}$: $\boxed{-1\ ✓}$ $\boxed{\text{i}}$: $\boxed{-2\ ✓}$ $\boxed{\text{j}}$: $\boxed{0\ ✓}$ $\boxed{\text{k}}$: $\boxed{0\ ✓}$

> By taking the difference of two equations, eliminate y.

Consider the last solution ($\boxed{\text{j}}$, $\boxed{\text{k}}$). The determinant of the Hessian at this point is

- positive
- 0
- negative ✓

At this point, the function $f(x, y)$ takes a

- local minimum
- saddle point ✓
- local maximum

Consider the first solution ($\boxed{\text{f}}$, $\boxed{\text{g}}$). The determinant of the Hessian at this point is

- positive ✓
- 0
- negative

At this point, the function $f(x, y)$ takes a

- local minimum ✓
- saddle point
- local maximum

(7) **Q3**

A vector-field $\mathbf{f}$ is said to be a *gradient* on a domain D if there exists a function φ such that $\mathbf{f} = \nabla\varphi$ on D. (Choose the correct option in each of the four places.) The vector-field

$$\mathbf{f}(x, y) = \begin{pmatrix} y^2(\cos x - x \sin x) \\ 2xy \cos x \end{pmatrix}$$

is ✓	a gradient on $\mathbb{R}^2$.
is not	

> If $\varphi(x, y) = xy^2 \cos x$ then $\mathbf{f} = \nabla\varphi$.

Whereas the vector-field

$$\mathbf{g}(x, y) = \begin{pmatrix} y^2 \\ -x^2 \end{pmatrix}$$

is	a gradient on $\mathbb{R}^2$.
is not ✓	

> Let $g_1(x, y) = y^2$, $g_2(x, y) = -x^2$ and observe that $\frac{\partial g_2}{\partial x} = -2x$ but $\frac{\partial g_1}{\partial y} = 2y$.

On the other hand the vector-field

$$\mathbf{h}(x,y) = \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}$$

is ✓	
is not	

a gradient on the domain $\{(x,y) : |y| > 0\}$ and

is	
is not ✓	

a gradient on the annular domain $\{(x,y) : 1 \leq x^2+y^2 \leq 4\}$.

> Calculate that $\frac{\partial h_2}{\partial x} = \frac{y^2-x^2}{x^2+y^2} = \frac{\partial h_1}{\partial y}$ and observe that $\{(x,y) : |y| > 0\}$ is a simply connected domain. For the second part consider the path $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ and observe that $\int \mathbf{h} \; d\boldsymbol{\alpha} = 2\pi \neq 0$.

Let $\boldsymbol{\alpha}$ denote the anticlockwise path with four straight segments and vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Calculate the line integral $\int \mathbf{g} \; d\boldsymbol{\alpha} =$

-2	✓
2	(50%)

where $\mathbf{g}$ is the vector-field defined above (fill in the blank with the correct integer, possibly zero or negative).

> By Green's theorem $\int \mathbf{g} \; d\boldsymbol{\alpha} = \int_0^1 \int_0^1 \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \; dxdy$. This integral is equal to $-2 \int_0^1 \int_0^1 x + y \; dxdy = -2(\int_0^1 x \; dx + \int_0^1 y \; dy) = -2$.

(8) **Q3**

A vector-field $\mathbf{f}$ is said to be a *gradient* on a domain D if there exists a function φ such that $\mathbf{f} = \nabla\varphi$ on D. (Choose the correct option in each of the four places.) The vector-field

$$\mathbf{f}(x,y) = \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}$$

is	
is not ✓	

a gradient on $\mathbb{R}^2$.

> Let $f_1(x,y) = -y^2$, $f_2(x,y) = x^2$ and observe that $\frac{\partial f_2}{\partial x} = 2x$ but $\frac{\partial f_1}{\partial y} = -2y$.

Whereas the vector-field

$$\mathbf{g}(x,y) = \begin{pmatrix} y^2(\cos x - x\sin x) \\ 2xy\cos x \end{pmatrix}$$

is ✓	a gradient on $\mathbb{R}^2$.
is not	

> If $\varphi(x,y) = xy^2 \cos x$ then $\mathbf{g} = \nabla\varphi$.

On the other hand the vector-field

$$\mathbf{h}(x,y) = \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}$$

is ✓	a gradient on the domain $\{(x,y) : \lvert y\rvert > 0\}$ and	is	
is not		is not ✓	

a gradient on the annular domain $\{(x,y) : 1 \leq x^2+y^2 \leq 4\}$.

> Calculate that $\frac{\partial h_2}{\partial x} = \frac{y^2-x^2}{x^2+y^2} = \frac{\partial h_1}{\partial y}$ and observe that $\{(x,y) : \lvert y\rvert > 0\}$ is a simply connected domain. For the second part consider the path $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ and observe that $\int \mathbf{h}\ d\boldsymbol{\alpha} = 2\pi \neq 0$.

Let $\boldsymbol{\alpha}$ denote the anticlockwise path with four straight segments and vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Calculate the line integral $\int \mathbf{f}\ d\boldsymbol{\alpha} =$

2	✓
−2	(50%)

where $\mathbf{f}$ is the vector-field defined above (fill in the blank with the correct integer, possibly zero or negative).

> By Green's theorem $\int \mathbf{f}\ d\boldsymbol{\alpha} = \int_0^1 \int_0^1 \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\ dxdy$. This integral is equal to $2\int_0^1 \int_0^1 x + y\ dxdy = 2(\int_0^1 x\ dx + \int_0^1 y\ dy) = 2$.

(9) **Q4**

Let V be the three-dimensional object formed as the union of the two cylinders $V_1 = \{(x,y,z) : \lvert x\rvert \leq 2, y^2+z^2 \leq 1\}$ and $V_2 = \{(x,y,z) : \lvert z\rvert \leq 2, x^2+y^2 \leq 1\}$. In order to calculate the volume of this cross-shaped object it is convenient to divide it into three *disjoint* pieces. We choose one piece as the cylinder V_1 and the other two remaining pieces are identical shape to each other and are not quite cylinders. One such piece is

$$W = \{(x,y,z) : (x,y) \in S, \sqrt{1-y^2} \leq z \leq 2\} \subset \mathbb{R}^2$$

where $S = \{(x, y) : x^2 + y^2 \leq 1\}$. The volume is

$$\mathrm{Vol}(W) = \iint_S 2 - \sqrt{1 - y^2}\; dxdy.$$

It is convenient to write $S = \{(x, y) : y \in [-1, 1], -\sqrt{\varphi(y)} \leq x \leq \sqrt{\varphi(y)}\}$ where $\varphi(y)$ is equal to

- $1 - y^2$, ✓
- $1 - z^2$,
- $z^2 - 1$.

Evaluate the integral and calculate $\mathrm{Vol}(W) =$ $\boxed{2\ ✓}$ $\pi +$ $\boxed{\begin{array}{ll} -8 & ✓ \\ 8 & (50\%) \end{array}}$ /3. (Hint: a change of variables $y = \sin u$ is sometimes convenient for integrating terms of the from $\sqrt{1 - y^2}$.)

$$\begin{aligned}
\mathrm{Vol}(W) &= \int_{-1}^{1} (2\sqrt{1-y^2})(2 - \sqrt{1-y^2})\; dy \\
&= 8\int_0^1 \sqrt{1-y^2}\; dy - 4\int_0^1\; dy + 4\int_0^1 y^2\; dy \\
&= 8\frac{\pi}{4} - 4 + \frac{4}{3} = 2\pi - \frac{8}{3}.
\end{aligned}$$

Now combine all the pieces and calculate the total volume $\mathrm{Vol}(V) =$ $\boxed{8\ ✓}$ $\pi +$ $\boxed{\begin{array}{ll} -16 & ✓ \\ 16 & (50\%) \end{array}}$ /3. (Fill in the blanks with the correct integers, possibly zero or negative.)

$\mathrm{Vol}(V_1) = 4\pi$ and so $\mathrm{Vol}(V) = 4\pi + 2(2\pi - \frac{8}{3})$.

(10) **Q4**

Let V be the three-dimensional object formed as the union of the two cylinders $V_1 = \{(x, y, z) : |x| \leq 3, y^2 + z^2 \leq 1\}$ and $V_2 = \{(x, y, z) : |z| \leq 3, x^2 + y^2 \leq 1\}$. In order to calculate the volume of this cross-shaped object it is convenient to divide it into three *disjoint* pieces. We choose one piece as the cylinder V_1 and the other two remaining pieces are identical shape to each other and are not quite cylinders. One such piece is

$$W = \{(x, y, z) : (x, y) \in S, \sqrt{1 - y^2} \leq z \leq 3\} \subset \mathbb{R}^2$$

where $S = \{(x,y) : x^2 + y^2 \leq 1\}$. The volume is

$$\mathrm{Vol}(W) = \iint_S 3 - \sqrt{1-y^2}\ dxdy.$$

It is convenient to write $S = \{(x,y) : y \in [-1,1], -\sqrt{\varphi(y)} \leq x \leq \sqrt{\varphi(y)}\}$ where $\varphi(y)$ is equal to

- $1 - y^2$, ✓
- $1 - z^2$,
- $z^2 - 1$.

Evaluate the integral and calculate $\mathrm{Vol}(W) =$ $\boxed{3 \ ✓}$ $\pi +$ $\boxed{\begin{array}{ll} -8 & ✓ \\ 8 & (50\%) \end{array}}$ /3. (Hint: a change of variables $y = \sin u$ is sometimes convenient for integrating terms of the from $\sqrt{1-y^2}$.)

$$\begin{aligned}\mathrm{Vol}(W) &= \int_{-1}^{1} (2\sqrt{1-y^2})(3 - \sqrt{1-y^2})\ dy \\ &= 12\int_0^1 \sqrt{1-y^2}\ dy - 4\int_0^1\ dy + 4\int_0^1 y^2\ dy \\ &= 12\frac{\pi}{4} - 4 + \frac{4}{3} = 3\pi - \frac{8}{3}.\end{aligned}$$

Now combine all the pieces and calculate the total volume $\mathrm{Vol}(V) =$ $\boxed{12 \ ✓}$ $\pi +$ $\boxed{\begin{array}{ll} -16 & ✓ \\ 16 & (50\%) \end{array}}$ /3. (Fill in the blanks with the correct integers, possibly zero or negative.)

$\mathrm{Vol}(V_1) = 6\pi$ and so $\mathrm{Vol}(V) = 6\pi + 2(3\pi - \frac{8}{3})$.

(11) **Q5**

Consider the the sphere $S = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$ and the vector-field

$$\mathbf{f}(x,y,z) = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}.$$

We wish to compute $\iint_S \mathbf{f} \cdot \mathbf{n}\ dS$ where $\mathbf{n}$ is the outgoing unit normal on S. If we let $V = \{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}$ then, by Gauss, we know that

$$\iint_S \mathbf{f} \cdot \mathbf{n}\ dS = \iiint_V \nabla \cdot \mathbf{f}\ dV$$

($\nabla \cdot \mathbf{f}$ denotes the divergence of $\mathbf{f}$). In this case $\nabla \cdot \mathbf{f}$ is equal to

- $3(x^2 + y^2 + z^2)$, ✓
- $x^3 + y^3 + z^3$,
- $(x^4 + y^4 + z^4)/4$.

To evaluate the three-dimensional integral use spherical coordinates $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r\cos\theta$. The Jacobian $J(r, \theta, \varphi)$ is equal to

- $r \cos\theta$,
- $r^2 \sin\theta$, ✓
- $r^3 \tan\varphi$.

Consequently

$$\iiint_V \nabla \cdot \mathbf{f}\ dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \boxed{?}\ dr d\theta d\varphi$$

where the blank $\boxed{?}$ should be

- $3r^4 \sin\theta$, ✓
- $2r^4 \sin\theta$,
- $r^3 \sin\theta$.

Evaluate the integral and hence calculate $\iint_S \mathbf{f} \cdot \mathbf{n}\ dS =$ $\boxed{12\ \checkmark}\ \frac{\pi}{5} + \boxed{0\ \checkmark}$ (fill in the blanks with the correct integers, possibly zero or negative).

$$\begin{aligned}
\iiint_V \nabla \cdot \mathbf{f}\ dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 3r^4 \sin\theta\ dr d\theta d\varphi \\
&= 2\pi \left(\int_0^{\pi} \sin\theta\ d\theta \right) \left(\int_0^1 3r^4\ dr \right) \\
&= 2\pi \cdot 2 \cdot \frac{3}{5} = \frac{12}{5}\pi.
\end{aligned}$$

(12) **Q5**

Consider the the sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1/4\}$ and the vector-field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}.$$

We wish to compute $\iint_S \mathbf{f} \cdot \mathbf{n}\ dS$ where $\mathbf{n}$ is the outgoing unit normal on S. If we let $V = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1/4\}$ then, by Gauss, we know that

$$\iint_S \mathbf{f} \cdot \mathbf{n}\ dS = \iiint_V \nabla \cdot \mathbf{f}\ dV$$

($\nabla \cdot \mathbf{f}$ denotes the divergence of $\mathbf{f}$). In this case $\nabla \cdot \mathbf{f}$ is equal to

- $3(x^2 + y^2 + z^2)$, ✓
- $x^3 + y^3 + z^3$,
- $(x^4 + y^4 + z^4)/4$.

To evaluate the three-dimensional integral use spherical coordinates $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r\cos\theta$. The Jacobian $J(r, \theta, \varphi)$ is equal to

- $r \cos\theta$,
- $r^2 \sin\theta$, ✓
- $r^3 \tan\varphi$.

Consequently

$$\iiint_V \nabla \cdot \mathbf{f}\ dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{1/2} \boxed{?}\ dr d\theta d\varphi$$

where the blank $\boxed{?}$ should be

- $3r^4 \sin\theta$, ✓
- $2r^4 \sin\theta$,
- $r^3 \sin\theta$.

Evaluate the integral and hence calculate $\iint_S \mathbf{f} \cdot \mathbf{n}\ dS =$ $\boxed{3 \ \checkmark}\ \frac{\pi}{40} + \boxed{0 \ \checkmark}$ (fill in the blanks with the correct integers, possibly zero or negative).

$$\begin{aligned}
\iiint_V \nabla \cdot \mathbf{f}\ dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{1/2} 3r^4 \sin\theta\ dr d\theta d\varphi \\
&= 2\pi \left(\int_0^{\pi} \sin\theta\ d\theta \right) \left(\int_0^{1/2} 3r^4\ dr \right) \\
&= 2\pi \cdot 2 \cdot \frac{3}{5 \cdot 2^5} = \frac{3}{40}\pi.
\end{aligned}$$