BSc Engineering Sciences – A. Y. 2019/20
Written exam of the course Mathematical Analysis 2
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Matriculation number: ....

Solve the following problems, motivating in detail the answers.

1. Find the Taylor expansion around the point $x_0 = 0$ of the function

$$f(x) = \frac{1}{x^3 - 2x^2 + x - 2}$$

find its radius of convergence r and study the convegence at $x = \pm r$.

Solution. The function $x^3 - x^2 + 2x - 2$ can be factorized as $(x^2 + 1)(x - 2)$. By putting $f(x) = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$ we obtain

$$
\begin{aligned}
f(x) &= \frac{\frac{1}{5}}{x - 2} + \frac{-\frac{x}{5} - \frac{2}{5}}{x^2 + 1} \\
&= \frac{1}{10} \cdot \frac{1}{\frac{x}{2} - 1} + \frac{1}{5} \cdot \frac{-x - 2}{x^2 + 1}.
\end{aligned}
$$

For $-1 < y < 1$, we have the geometric series $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$. Applying this to $y = \frac{x}{2}, -x^2$ respectively, we obtain

$$
\begin{aligned}
f(x) &= -\frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n + \frac{1}{5}\left(-x \sum_{n=0}^{\infty}(-x^2)^n - 2\sum_{n=0}^{\infty}(-x^2)^n\right) \\
&= \sum_{n=0}^{\infty} -\frac{x^n}{10 \cdot 2^n} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1} - 2(-1)^n x^{2n}}{5}
\end{aligned}
$$

This is a power series expansion of $f(x) = \sum_{m=0}^{\infty} a_m x^m$ with

$$
\begin{cases}
a_{2m} = -\frac{1}{10 \cdot 2^{2m}} - \frac{2(-1)^m}{5}, \\
a_{2m+1} = -\frac{1}{10 \cdot 2^{2m+1}} - \frac{(-1)^{m+1}}{5}.
\end{cases}
$$

As for the radius of convergence, let us divide the series into parts: $f(x) = \sum_{n=0}^{\infty} -\frac{x^n}{10 \cdot 2^n} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{5} - 2\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{5}$. By the ratio test, it is easy to see that the radii of convergence of these series are $2, 1, 1$, respectively. Therefore, the radius of convergence of the whole series is at least 1. On the other hand, if $1 < |x| < 2$, the first series is convergent, while for the latter, terms diverge, hence the series is not convergent. This shows that the radius of convergence is 1.

2. Find all the stationary points of the following scalar field, defined on $\mathbb{R}^3$,

$$f(x, y, z) = (x^2 + 2xy + 2y^2 - 2x - 4y)e^{z^3 - 3z}$$

and classify them into relative minima, maxima and saddle points.

Solution.

For the f given above, it holds that

$$\nabla f(x, y, z)$$
$$= \left((2x + 2y - 2)e^{z^3 - 3z}, (2x + 4y - 4)e^{z^3 - 3z}, (x^2 + 2xy + 2y^2 - 2x - 4y)(3z^2 - 3)e^{z^3 - 3}\right).$$

At stationary points, $\nabla f(x, y, z) = \mathbf{0}$ holds. Namely,

$$(2x + 2y - 2)e^{z^3 - 3z} = 0, (2x + 4y - 4)e^{z^3 - 3z} = 0, (x^2 + 2xy + 2y^2 - 2x - 4y)(3z^2 - 3)e^{z^3 - 3} = 0.$$

As $e^{-z^3 - 3z}$ takes never 0, this is equivalent to

$$2x + 2y - 2 = 0, 2x + 4y - 4 = 0, (x^2 + 2xy + 2y^2 - 2x - 4y)(3z^2 - 3) = 0.$$

From the first two equations, we obtain $x = 0, y = 1$. Substituting this in the last equation, $-6(z^2 - 1) = 0$, namely, $z = \pm 1$. Therefore, the stationary points are $(0, 1, \pm 1)$.

To classify these points, let us compute the Hessian matrix:

$$D_{xx} = 2e^{z^3 - 3z}, D_{yx} = D_{xy} = 2e^{z^3 - 3z}, D_{yy} = 4e^{z^3 - 3z}$$
$$D_{xz} = D_{zx} = (2x + 2y - 2)(3z^2 - 3)e^{z^3 - 3}$$
$$D_{yz} = D_{zy} = (2x + 4y - 4)(3z^2 - 3)e^{z^3 - 3}$$
$$D_{zz} = (x^2 + 2xy + 2y^2 - 2x - 4y)(6z + (3z^2 - 3)^2)e^{z^3 - 3}.$$

(it is a good idea not to expand the formula at this point, because, at the stationary point many terms vanish). At the point $(x, y, z) = (0, 1, 1)$, this becomes

$$e^{-2} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & -12 \end{pmatrix}.$$

$(0, 0, 1)^{\mathrm{t}}$ is an eivenvector with eivenvalue -12. As for the part $\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$, the determinant is positive and the trace is positive, hence there are two positive eigenvalues. Altogether, there are positive and negative eigenvalues, hence the point $(0, 1, 1)$ is a saddle.

At the point $(x, y) = (0, 1, -1)$, this becomes

$$e^{-2} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 12 \end{pmatrix}.$$

In this case, arguing as before, we find that all the eigenvalues are positive, hence $(0, 1, -1)$ is a relative minumum.

3. Consider the following three formulae for vector fields

$$\boldsymbol{f}(x,y) = \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}, \quad \boldsymbol{g}(x,y) = \begin{pmatrix} 1+y+e^{x+y^2} \\ x+2ye^{x+y^2} \end{pmatrix}, \quad \boldsymbol{h}(x,y) = \begin{pmatrix} \frac{1}{y}\cos(\frac{x}{y}) \\ \ln y \cos(\frac{x}{y}) \end{pmatrix}.$$

One of these is a gradient on $\mathbb{R}^2$. Identify which this is and find a scalar field φ such that this particular vector field is equal to $\nabla\varphi$.

Solution. Calculating the difference of derivatives $\partial_y\boldsymbol{f}_1 - \partial_x\boldsymbol{f}_2$ suggests that $\boldsymbol{f}$ might be a gradient but for the fact that the vector field blows up at $(0,0)$. Calculating the difference of derivatives $\partial_y\boldsymbol{h}_1 - \partial_x\boldsymbol{h}_2$ shows that $\boldsymbol{h}$ isn't a gradient. On the other hand $\boldsymbol{g}$ is a gradient. We can find the scalar field by the method of indeterminate integrals:

$$\int 1+y+e^{x+y^2} \, dx = x + xy + e^{x+y^2} + B(y),$$

$$\int x+2ye^{x+y^2} \, dy = xy + e^{x+y^2} + C(x).$$

Consequently we choose $\varphi(x,y) = x + xy + e^{x+y^2}$ and can double check by differentiating.

4. Let $D \subset \mathbb{R}^3$ be the solid pyramid which has 5 planar surfaces and vertices $(-1, 0, 0)$, $(0, -1, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Compute the integral

$$\iiint_D x^2 + y^2 \; dx dy dz.$$

A possible method is: 1. Let $E \subset \mathbb{R}^3$ be the solid bounded by the four surfaces $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$, $\{x + y + z = 1\}$; 2. Calculate $\iiint_E x^2 \; dx dy dz$ and $\iiint_E y^2 \; dx dy dz$ separately in order to compute $\iiint_E x^2 + y^2 \; dx dy dz$; 3. Deduce the full answer by observing the symmetry in this problem.

Solution. D is a pyramid with a square base and E is a quarter of that pyramid. (This integral is the moment of inertia of the solid pyramid.) Let $T = \{(x, y) : x \in [0, 1], 0 \le y \le 1 - x\}$ and so

$$E = \{(x, y, z) : (x, y) \in T, 0 \le z \le 1 - (x + y)\}.$$

This means that

$$\iiint_E x^2 \; dx dy dz = \iint_T \left(\int_0^{1-(x+y)} x^2 \; dz \right) dx dy = \iint_T x^2 - x^3 - x^2 y \; dx dy.$$

In turn

$$\iint_T x^2 - x^3 - x^2 y \; dx dy = \int_0^1 \left(\int_0^{1-x} x^2 - x^3 - x^2 y \; dy \right) dx$$

and

$$\int_0^{1-x} x^2 - x^3 - x^2 y \; dy = \left[x^2 y - x^3 y - \tfrac{1}{2} x^2 y^2 \; dy \right]_0^{1-x} = \tfrac{1}{2} x^2 - x^3 + \tfrac{1}{2} x^4.$$

Finally

$$\int_0^1 \tfrac{1}{2} x^2 - x^3 + \tfrac{1}{2} x^4 dx = \left[\tfrac{1}{6} x^3 - \tfrac{1}{4} x^4 + \tfrac{1}{10} x^5 \right]_0^1 = \frac{1}{60}$$

We have therefore calculated that $\iiint_E x^2 \; dx dy dz = \frac{1}{60}$. Similarly $\iiint_E y^2 \; dx dy dz = \frac{1}{60}$. The rotational symmetry of this problem ($x^2 + y^2$ is simply the square of the distance from the z-axis) means that $\iiint_D x^2 + y^2 \; dx dy dz$ is equal to 4 times $\iiint_E x^2 + y^2 \; dx dy dz$. This means that $\iiint_D x^2 + y^2 \; dx dy dz = \frac{2}{15}$.

5. Let S be the planar triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and consider the vector field $\boldsymbol{f}(x,y,z) = (x^2, y^2, 1)$. Compute $\iint_S \boldsymbol{f} \cdot \mathbf{n} \, dS$, where $\mathbf{n}$ is the unit normal with positive z component.

Solution. As a representation of the surface we choose $T = \{(u,v) : u \in [0,1], 0 \le v \le 1 - u\}$ and

$$\mathbf{r} : (x,y) \mapsto (u, v, 1 - (u+v)).$$

We calculate $\partial_u \mathbf{r} \times \partial_v \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Consequently

$$\iint_S \boldsymbol{f} \cdot \mathbf{n} \, dS = \iint_T \begin{pmatrix} u^2 \\ v^2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \, du \, dv = \int_0^1 \left[\int_0^{1-u} u^2 + v^2 + 1 \, dv \right] du.$$

We calculate

$$\int_0^{1-u} u^2 + v^2 + 1 \, dv = \left[(u^2 + 1)v + \frac{v^3}{3} \right]_0^{1-u} = \frac{4}{3} - 2u + 2u^2 - \frac{4}{3}u^2,$$

and

$$\int_0^1 \frac{4}{3} - 2u + 2u^2 - \frac{4}{3}u^2 \, du = \left[\frac{4}{3}u - u^2 + \frac{2}{3}u^3 - \frac{4}{9}u^3 \right]_0^1 = \frac{5}{9}.$$