Mathematical Analysis 2 – Call 1 – 20/01/2026

Part 1 – 9:30-11:00

Question 1. For each of the following functions, find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (or $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$, etc.).

1. $f(x, y) = e^{xy} \sin(x^2 + y)$
2. $g(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$
3. $h(x, y) = \displaystyle\int_0^{x^2 y} e^{-t^2}\, dt$

Question 2. Find the maximum and minimum values of $f(x, y, z) = xy + xz + yz$ subject to the constraint $x^2 + y^2 + z^2 = 3$.

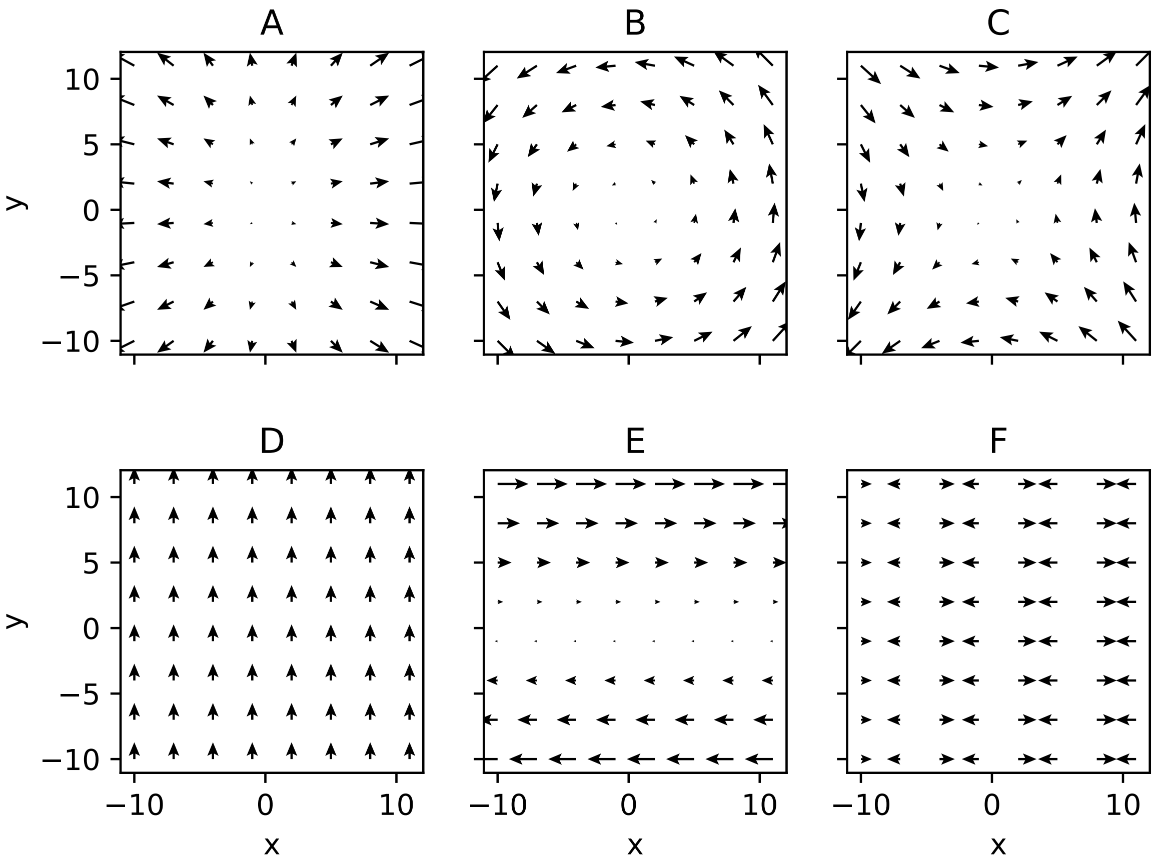
Method: Set up the Lagrange multiplier equations $\nabla f = \lambda \nabla g$. Solve the system of equations to find the critical points (you should obtain two isolated points and a curve). Calculate the value of f at each critical point to determine the maximum and minimum.

Question 3. Consider the following vector fields defined on $\mathbb{R}^2$:

$$\mathbf{F}_1(x, y) = y\mathbf{i} + x\mathbf{j} \qquad \mathbf{F}_2(x, y) = 2\sin(x)\mathbf{i} \qquad \mathbf{F}_3(x, y) = -y\mathbf{i} + x\mathbf{j}$$

$$\mathbf{F}_4(x, y) = \mathbf{j} \qquad \mathbf{F}_5(x, y) = y\mathbf{i} \qquad \mathbf{F}_6(x, y) = 2x\mathbf{i} + y\mathbf{j}$$

1. Match each to one of the plots and briefly explain the logic/calculation for matching each.
2. Calculate the divergence, $\nabla \cdot \mathbf{F}_n(x, y)$ for each of the vector fields.


A
B
C
D
E
F
y
x
10
5
0
−5
−10
−10
0
10

Mathematical Analysis 2 – Call 1 – 20/01/2026

Part 2 – 11:30-13:00

Question 4. Calculate the path integral $\int_C \mathbf{F} \cdot d\boldsymbol{\alpha}$ where $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + (xy)\mathbf{j}$ and C is the path from $(0, 0)$ to $(2, 4)$ along the curve $y = x^2$.

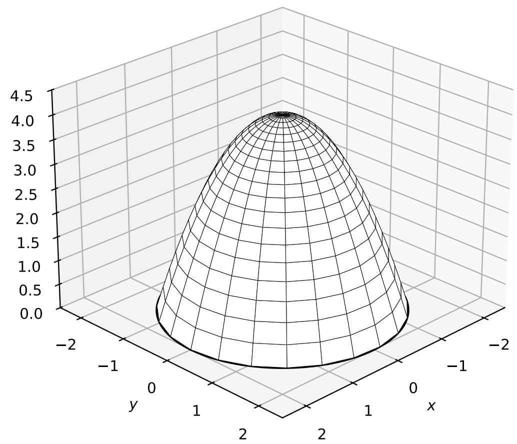
Question 5. Let S be the closed surface formed by the paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$, together with the disk $x^2 + y^2 \leq 4$ in the plane $z = 0$. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $\hat{\mathbf{n}}$ be the outward-pointing unit normal.

Evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$$

by direct calculation. That is, compute the flux through the paraboloid and the disk separately, then add them.

Hint: The paraboloid can be parametrized as $\boldsymbol{\sigma}(r, \theta) = (r\cos\theta, r\sin\theta, 4 - r^2)$ *for* $r \in [0, 2]$, $\theta \in [0, 2\pi]$.



Question 6. Using the same surface S and vector field $\mathbf{F}$ from Question 5, evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$$

using the divergence theorem.

Solutions – Call 1 – 20/01/2026

Question 1 Solution:

1. $f(x,y) = e^{xy}\sin(x^2+y)$
   Using product rule:

$$\begin{aligned}\frac{\partial f}{\partial x} &= ye^{xy}\sin(x^2+y) + e^{xy}\cos(x^2+y)\cdot 2x \\ &= e^{xy}\big(y\sin(x^2+y) + 2x\cos(x^2+y)\big) \\ \frac{\partial f}{\partial y} &= xe^{xy}\sin(x^2+y) + e^{xy}\cos(x^2+y)\cdot 1 \\ &= e^{xy}\big(x\sin(x^2+y) + \cos(x^2+y)\big)\end{aligned}$$

2. $g(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$
   Using quotient rule:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{2x(x^2+y^2)-(x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{2x(x^2+y^2-x^2+y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2} \\ \frac{\partial g}{\partial y} &= \frac{-2y(x^2+y^2)-(x^2-y^2)(2y)}{(x^2+y^2)^2} = \frac{-2y(x^2+y^2+x^2-y^2)}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2}\end{aligned}$$

3. $h(x,y) = \displaystyle\int_0^{x^2y} e^{-t^2}\,dt$
   Using the Fundamental Theorem of Calculus with chain rule:

$$\begin{aligned}\frac{\partial h}{\partial x} &= e^{-(x^2y)^2}\cdot\frac{\partial}{\partial x}(x^2y) = e^{-x^4y^2}\cdot 2xy = 2xy\,e^{-x^4y^2} \\ \frac{\partial h}{\partial y} &= e^{-(x^2y)^2}\cdot\frac{\partial}{\partial y}(x^2y) = e^{-x^4y^2}\cdot x^2 = x^2e^{-x^4y^2}\end{aligned}$$

Question 2 Solution:
We use Lagrange multipliers with $g(x,y,z) = x^2+y^2+z^2-3 = 0$.
We have $\nabla f = (y+z, x+z, x+y)$ and $\nabla g = (2x, 2y, 2z)$.
Setting $\nabla f = \lambda\nabla g$:

$$\begin{aligned} y+z &= 2\lambda x \quad (1) \\ x+z &= 2\lambda y \quad (2) \\ x+y &= 2\lambda z \quad (3)\end{aligned}$$

Adding all three equations: $2(x+y+z) = 2\lambda(x+y+z)$, so $(x+y+z)(1-\lambda) = 0$.
Case 1: $\lambda = 1$. Then from (1): $y+z = 2x$, from (2): $x+z = 2y$, from (3): $x+y = 2z$.

Subtracting (2) from (1): $y - x = 2x - 2y \implies 3y = 3x \implies y = x$. Similarly, subtracting (3) from (2): $z - y = 2y - 2z \implies 3z = 3y \implies z = y$.

So $x = y = z$. With $x^2 + y^2 + z^2 = 3$: $3x^2 = 3 \implies x = \pm 1$.

Points: $(1, 1, 1)$ and $(-1, -1, -1)$, both giving $f = 1 + 1 + 1 = 3$.

Case 2: $x + y + z = 0$. Then $z = -x - y$, and from the constraint:

$$x^2 + y^2 + (x + y)^2 = 3 \implies 2x^2 + 2xy + 2y^2 = 3$$

The value of f in this case:

$$\begin{aligned} f &= xy + xz + yz = xy + x(-x - y) + y(-x - y) \\ &= xy - x^2 - xy - xy - y^2 = -x^2 - xy - y^2 = -\frac{3}{2} \end{aligned}$$

Therefore: **Maximum** $= 3$ at $(1, 1, 1)$ and $(-1, -1, -1)$; **Minimum** $= -\frac{3}{2}$ on the circle $x + y + z = 0$, $x^2 + y^2 + z^2 = 3$.

Question 3 Solution:

1. Matching:
   - $\mathbf{F}_1 = y\mathbf{i} + x\mathbf{j}$ matches **C**: At $(1, 1)$ the vector is $(1, 1)$, at $(1, -1)$ it's $(-1, 1)$. This creates a saddle/hyperbolic pattern.
   - $\mathbf{F}_2 = 2\sin(x)\mathbf{i}$ matches **F**: Purely horizontal vectors that oscillate with x, creating vertical stripes of alternating direction.
   - $\mathbf{F}_3 = -y\mathbf{i} + x\mathbf{j}$ matches **B**: Counter-clockwise rotation (at $(1, 0)$: vector is $(0, 1)$, pointing up).
   - $\mathbf{F}_4 = \mathbf{j}$ matches **D**: Constant upward vectors everywhere.
   - $\mathbf{F}_5 = y\mathbf{i}$ matches **E**: Horizontal shear – vectors point right above x-axis, left below.
   - $\mathbf{F}_6 = 2x\mathbf{i} + y\mathbf{j}$ matches **A**: Stretched radial pattern, horizontal component grows faster than vertical.

2. Divergences:

$$\begin{aligned} \nabla \cdot \mathbf{F}_1(x, y) &= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0 + 0 = 0 \\ \nabla \cdot \mathbf{F}_2(x, y) &= \frac{\partial}{\partial x}(2\sin x) + \frac{\partial}{\partial y}(0) = 2\cos x \\ \nabla \cdot \mathbf{F}_3(x, y) &= \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0 + 0 = 0 \\ \nabla \cdot \mathbf{F}_4(x, y) &= \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(1) = 0 \\ \nabla \cdot \mathbf{F}_5(x, y) &= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(0) = 0 \\ \nabla \cdot \mathbf{F}_6(x, y) &= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(y) = 2 + 1 = 3 \end{aligned}$$

Question 4 Solution:

1. Parametrization: Let $\boldsymbol{\alpha}(t) = (t, t^2)$ for $t \in [0, 2]$.
2. We have $\boldsymbol{\alpha}'(t) = (1, 2t)$.
   $\mathbf{F}(\boldsymbol{\alpha}(t)) = (t^2 + t^2, t \cdot t^2) = (2t^2, t^3)$
3. The path integral:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\boldsymbol{\alpha} &= \int_0^2 \mathbf{F}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)\, dt \\
&= \int_0^2 (2t^2, t^3) \cdot (1, 2t)\, dt \\
&= \int_0^2 (2t^2 + 2t^4)\, dt \\
&= \left[\frac{2t^3}{3} + \frac{2t^5}{5} \right]_0^2 \\
&= \frac{16}{3} + \frac{64}{5} = \frac{80 + 192}{15} = \frac{272}{15}
\end{aligned}$$

Question 5 Solution:

The closed surface S consists of two parts: the paraboloid S_1 and the disk S_2.

Flux through the disk S_2: The disk is $x^2 + y^2 \leq 4$ at $z = 0$. The outward normal points downward: $\hat{\mathbf{n}} = -\mathbf{k}$.

On this surface, $\mathbf{F} = (x, y, 0)$, so $\mathbf{F} \cdot \hat{\mathbf{n}} = 0 \cdot (-1) = 0$.

Therefore, $\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0$.

Flux through the paraboloid S_1: Parametrize using polar coordinates: $\boldsymbol{\sigma}(r, \theta) = (r\cos\theta, r\sin\theta, 4 - r^2)$ for $r \in [0, 2], \theta \in [0, 2\pi]$.

Compute partial derivatives:

$$\begin{aligned}
\frac{\partial \boldsymbol{\sigma}}{\partial r}(r, \theta) &= (\cos\theta, \sin\theta, -2r) \\
\frac{\partial \boldsymbol{\sigma}}{\partial \theta}(r, \theta) &= (-r\sin\theta, r\cos\theta, 0)
\end{aligned}$$

Fundamental vector product:

$$\left(\frac{\partial \boldsymbol{\sigma}}{\partial r} \times \frac{\partial \boldsymbol{\sigma}}{\partial \theta} \right)(r, \theta) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta, 2r^2\sin\theta, r)$$

This points upward/outward (positive z-component), which is correct for the outward normal.

On the paraboloid: $\mathbf{F}(\boldsymbol{\sigma}(r, \theta)) = (r\cos\theta, r\sin\theta, 4 - r^2)$

$$\begin{aligned}
\left(\mathbf{F} \cdot \left(\frac{\partial \boldsymbol{\sigma}}{\partial r} \times \frac{\partial \boldsymbol{\sigma}}{\partial \theta} \right) \right)(r, \theta) &= 2r^3\cos^2\theta + 2r^3\sin^2\theta + r(4 - r^2) \\
&= 2r^3 + 4r - r^3 = r^3 + 4r
\end{aligned}$$

Integrate:

$$\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_0^{2\pi} \int_0^2 (r^3 + 4r)\, dr\, d\theta$$
$$= 2\pi \left[\frac{r^4}{4} + 2r^2 \right]_0^2 = 2\pi(4 + 8) = 24\pi$$

Total flux: $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0 + 24\pi = \boxed{24\pi}$

Question 6 Solution:
By the divergence theorem:

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

Compute the divergence:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

The region V is bounded by the paraboloid $z = 4 - x^2 - y^2$ above and the plane $z = 0$ below, with $x^2 + y^2 \leq 4$.
Using cylindrical coordinates (r, θ, z):

$$\iiint_V 3\, dV = 3 \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\, dz\, dr\, d\theta$$
$$= 3 \int_0^{2\pi} d\theta \int_0^2 r(4 - r^2)\, dr$$
$$= 3 \cdot 2\pi \int_0^2 (4r - r^3)\, dr$$
$$= 6\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2$$
$$= 6\pi\,(8 - 4) = 6\pi \cdot 4 = \boxed{24\pi}$$

This confirms the result from Question 5.