Signal Compression and Reconstruction on Chain Complexes

Discrete Morse Theoretic Methods

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Authors

Figure: The EPFL-DMT Crew: K.M, Celia Hacker and Stefania Ebli

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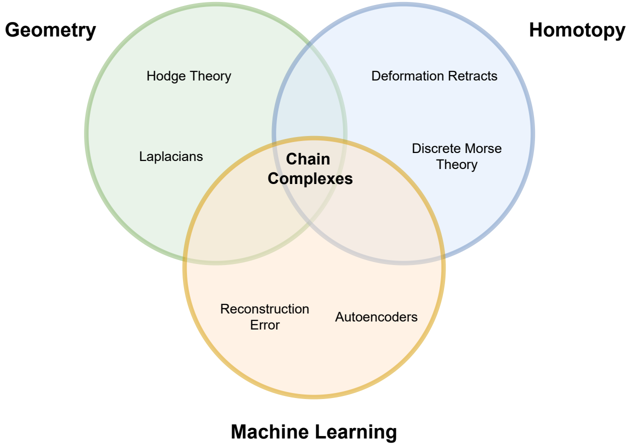
Venn Diagram



Remark: All chain complexes will be finite dimensional over $\mathbb{R}$.

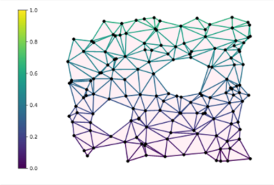
Cellular Signal Processing

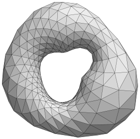
Signals on CW Complex $\Leftrightarrow$ Cellular (co)chains.

Example: Cellular (co)chains

X CW complex,

$$C_n(X) = \{f : n\text{-cells} \to \mathbb{R}\}$$





Chain Complex $(\mathbf{C}, \partial)$

$$\ldots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X)$$

Cellular Signal Processing

Signals on Cellular Complex/Sheaf $\Leftrightarrow$ Cellular (co)chains.

Example: Cellular Sheaves, Simplicial Neural Networks (Ebli. et al 2020, Bodnar et al 2021)

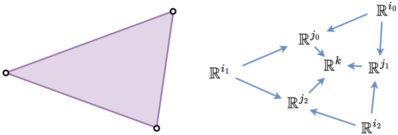


Figure: Cellular Sheaf X

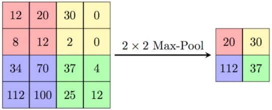
Cellular Sheaf Cochains $\Leftrightarrow$ Vector Representations of Cells

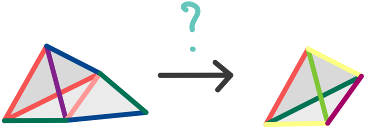
$$C^n(X;\mathbb{R}) = \bigoplus_{\dim \sigma = n} \mathbb{R}^{i_\sigma}$$

Motivation

Goal

Develop **pooling layers** for simplicial/cellular neural networks, cochain data in general.





Key Point

- Set of pixels $\sim$ (co)chain complex $(\mathbf{C}, \partial)$
- Pixel values $\sim$ specific (co)chain.

Motivation

What kind of maps should we use?

Deformation Retracts

A *deformation retract* of chain complexes $(\mathbf{C}, \partial)$ and $(\mathbf{D}, \partial')$

$$
\begin{tikzcd}
\mathbf{D} \arrow[r, shift right, "\Phi"', leftarrow] \arrow[r, shift left, "\Psi"] & \mathbf{C} \arrow[loop right, "h"]
\end{tikzcd}
$$

such that $\Phi\Psi = \mathrm{Id}_{\mathbf{D}}$ and $\partial h + h\partial = \mathrm{Id}_{\mathbf{C}} - \Psi\Phi$

Properties

- Preserves (co)homology.
- Degree-wise linear projection maps $(\Phi_n\Psi_n)^2 = \Phi_n\Psi_n$.
- Reduces dimensionality.

Reconstruction Error

Question

How to evaluate which deformation retracts are good?

Approach

Use concept of reconstruction error minimization

$$\arg\min_{\Phi,\Psi} \|s - \Phi\Psi(s)\|$$

Data in Large Space $\xrightarrow{\psi}$ Data in Latent Space $\xrightarrow{\phi}$ Data in Large Space

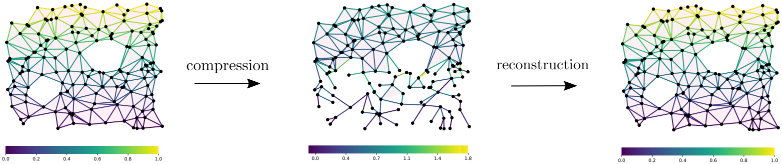
Examples

- (PCA) Ψ, Φ Orthogonal Projections
- (Autoencoders) Ψ, Φ Multi-layer Perceptrons
- (Topological Autoencoders) Michael Moor, Max Horn, Bastian Rieck, and Karsten Borgwardt, Topological autoencoders, 2021

Reconstruction Error

Approach

Use concept of reconstruction error for (co)chain complexes.



Reconstruction Error of Deformed Signal

$$\|s - \Phi\Psi(s)\| = \|\partial h(s) + h\partial(s)\|$$

Key Point

Understanding Reconstruction Error $\sim$ Understanding h

Reconstruction Error

Key Questions

- How to generate deformation retracts that minimize reconstruction error?
- What parts of the signal
  - Successfully reconstructed?
  - Poorly reconstructed?

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Discrete Morse Theory: CW Complexes

Discrete Morse Theory

DMT $\Rightarrow$ generating deformation retracts.

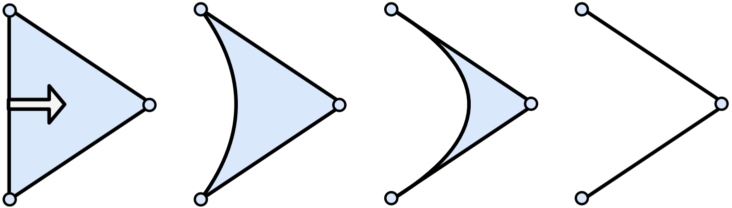


Figure: (Whitehead) A simple homotopy collapse induced by a pairing.

Discrete Morse Theory: CW Complexes

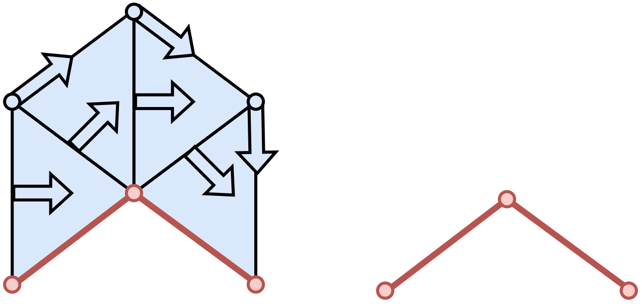


Figure: A discrete Morse matching on a CW complex.

Forman (1998)

Uses simple homotopy to define notions of discrete vector fields and **discrete Morse theory**.

Algebraic Discrete Morse Theory

Recall

Our data: Real-valued (co)chains

Skoldberg (2005)

Invents algebraic discrete Morse theory for based (co)chain complexes.

Consequences

Do everything at algebraic/cochain level.

- Easier to generate matchings.
- Linear algebra.
- Liberate definition of a cell.

Algebraic Discrete Morse Theory

Based Chain Complex

A **based chain complex** $(\mathbf{C}, I)$ is a chain complex $(\mathbf{C}, \partial)$ with direct sum decompositions

$$\mathbf{C}_n = \bigoplus_{\alpha \in I_n} C_\alpha$$

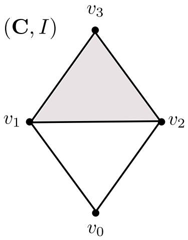
indexed by a finite set $I = \sqcup_n I_n$.

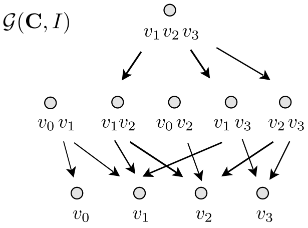
Topological world: n-cells $\Leftrightarrow$ *Algebraic World*: Elements of I_n

Examples

- Simplicial/Cellular Cochains; standard basis.
- Cellular sheaf; stalks.

"Hasse Diagrams" of Based Chain Complexes





Graph of a Complex

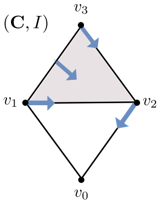
For a based chain complex $(\mathbf{C}, I)$ the graph of complex $\mathcal{G}(\mathbf{C}, I)$ consists of

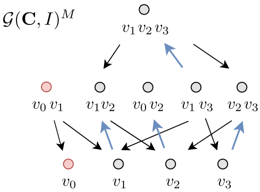
- Vertices $\alpha \in I$
- Directed edges $\alpha \to \beta$ whenever

$$\partial_{\alpha,\beta} : C_\alpha \hookrightarrow \mathbf{C} \xrightarrow{\partial} \mathbf{C} \twoheadrightarrow C_\beta$$

  is non-zero.

Algebraic Morse Matchings





Algebraic Morse Matching

A subset $M \subseteq \mathcal{E}(\mathbf{C}, I)$ such that

1. (Partition) Each $\alpha \in I$ adjacent at most one edge.
2. (Invertibility) $\partial_{\alpha,\beta}$ is an isomorphism for each $(\alpha, \beta) \in M$.
3. (Acyclicity) There are no cyclic, directed paths in $\mathcal{G}(\mathbf{C}, I)^M$.

Critical Cells: $M^0 \Leftrightarrow$ Cells adjacent to no edge in M

Main Theorem: Algebraic DMT

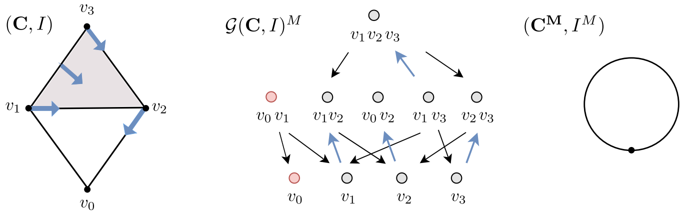


Figure: Morse deformation retract of a complex.

Theorem (Sköldberg, 2005, [1])

Let $(\mathbf{C}, I)$ be a based chain complex, and M a Morse matching. Then there is a deformation retract

$$\mathbf{C}^M \underset{\Phi^M}{\overset{\Psi^M}{\leftrightarrows}} \mathbf{C} \circlearrowleft h$$

Main Theorem: Algebraic DMT

Theorem (Sköldberg, 2005, [1])

Let $(\mathbf{C}, I)$ be a based chain complex, and M a Morse matching. Then there is a deformation retract

$$\begin{tikzcd} \mathbf{C}^M \arrow[r, shift right, "\Phi^M"'] & \mathbf{C} \arrow[l, shift right, "\Psi^M"'] \arrow[loop right, "h"] \end{tikzcd}$$

Key Points

- The maps Φ^M, Ψ^M and h have explicit formulae.
- Straight-forward to generate matchings over $\mathbb{R}$.

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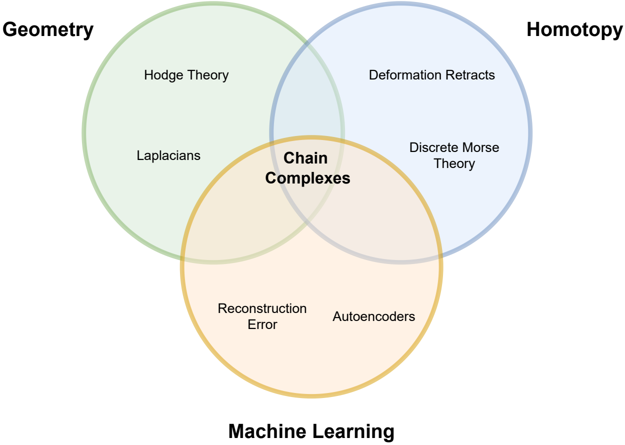
Venn Diagram



Inner products

Let **C** be a (co)chain complex of real inner product spaces, degree-wise finite dimensional.

Consequences

- Metric on (co)chains $f, g \in \mathbf{C}_n$:

$$\|f - g\| := \langle f - g, f - g \rangle^{1/2}$$

- Adjoint maps:

$$\partial_n : C_n \to C_{n-1} \;\Rightarrow\; \partial_n^{\dagger} : C_{n-1} \to C_n$$

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Combinatorial Laplacian

Geometry

Inner Products $\Rightarrow$ Laplacian

$$C_{n+1} \underset{\partial_{n+1}^\dagger}{\overset{\partial_{n+1}}{\rightleftarrows}} C_n \underset{\partial_n^\dagger}{\overset{\partial_n}{\rightleftarrows}} C_{n-1}$$

Combinatorial Laplacian

The **combinatorial Laplacian is the operator**

$$\Delta_n = \partial_n^\dagger \partial_n + \partial_{n+1}\partial_{n+1}^\dagger : C_n \to C_n$$

Remark

NOT FUNCTORIAL!

Combinatorial Laplacian

Amazing Facts (Eckmann, 1940s)

- (Harmonics)

$$H_n(\mathbf{C}) \cong \mathrm{Ker}\Delta_n$$

- (Hodge Decomposition)

$$C_n \cong \mathrm{Im}\partial_{n+1} \oplus \mathrm{Ker}\Delta_n \oplus \mathrm{Im}\partial_n^{\dagger}$$

- (Fourier Basis)

Eigenvectors of Δ $\sim$ Fourier basis for cochains

Remark

- Eigenvalue = 'Frequency' of eigenvector.
- Eigenvectors graded by Hodge decomposition.

Hodge Matching

Connection

$$\text{Hodge Theory} \bigcap \text{DMT} = \text{Partial Matching}$$

Eigen-pairing

SVD of $\partial_n \Rightarrow$ Partial pairing of eigenvectors.

Shared Eigenvalues

$$\begin{bmatrix} \lambda_1^{1/2} & & & & & & & \\ & \lambda_2^{1/2} & & & & & & \\ & & \lambda_3^{1/2} & & & & & \\ & & & \cdots & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \cdots & \\ & & & & & & & \end{bmatrix}$$

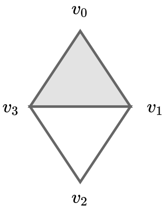
$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n$$

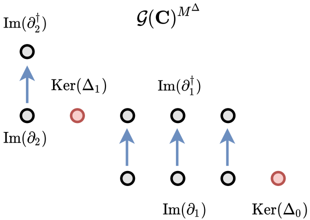
Δ_{n+1}-eigenbasis $\qquad$ Δ_n-eigenbasis

Hodge Matching

Connection

$$\text{Hodge Theory} \bigcap \text{DMT} = \text{Partial Matching}$$

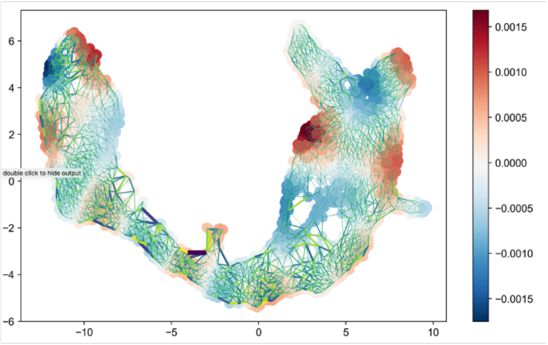




Remark

$$H(\mathbf{C}) \cong (\mathrm{Ker}\Delta, 0) \underset{\Phi^M}{\overset{\Psi^M}{\rightleftarrows}} (\mathbf{C}, \partial) \circlearrowleft h$$

Hodge Matching Example



Example Pairing

There exists eigenvectors $\Delta_0 v = \lambda v$ and $\Delta_1 w = \lambda w$ where

$$\partial_0^\dagger v = \sqrt{\lambda} w$$

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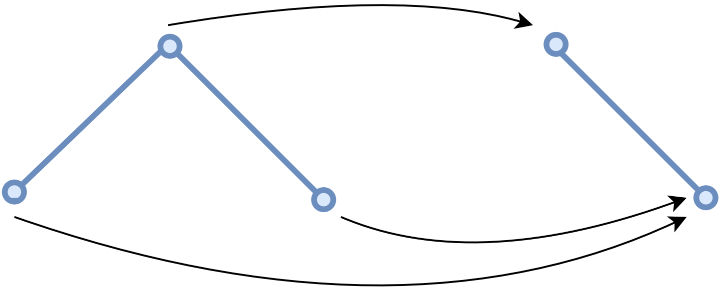
Morsification and Reconstruction

Question

Which deformation retracts arise from Morse matchings?



Morsification and Reconstruction

We say that two deformation retracts

$$\mathbf{D} \underset{\Phi}{\overset{\Psi}{\leftrightarrows}} \mathbf{C} \circlearrowright h \qquad \text{and} \qquad \mathbf{D}' \underset{\Phi'}{\overset{\Psi'}{\leftrightarrows}} \mathbf{C}' \circlearrowright h'$$

are *equivalent* if there exist isomorphisms of chain complexes, $f : \mathbf{D} \to \mathbf{D}'$ and $g : \mathbf{C} \to \mathbf{C}'$ such that the diagrams

$$\begin{tikzcd} \mathbf{D} \arrow[d, "f"', "\cong"] & \mathbf{C} \arrow[l, "\Psi"'] \arrow[d, "g", "\cong"'] \\ \mathbf{D}' & \mathbf{C}' \arrow[l, "\Psi'"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{D} \arrow[r, "\Phi"] \arrow[d, "f"', "\cong"] & \mathbf{C} \arrow[d, "g", "\cong"'] \\ \mathbf{D}' \arrow[r, "\Phi'"'] & \mathbf{C}' \end{tikzcd}$$

commute.

Remark: when $\mathbf{C} = \mathbf{C}'$, reconstruction error is the same.

Morsification and Reconstruction

Theorem (Morsification, (Ebli, Hacker, M. [2]))

Any deformation retract

$$\mathbf{D} \underset{\Phi}{\overset{\Psi}{\leftrightarrows}} \mathbf{C}$$

of finite-type chain complexes of real inner product spaces

- *there is a canonical basis of* $\mathbf{C}$ *and a Morse matching such that*
- *the associated deformation retract is 'equivalent'.*

So What?

- Understanding deformation retracts $\sim$ Understanding Morse retracts.
- Provides an explicit, nice homotopy h.
- Helps understand reconstruction error.

(n,n-1)-free

Question Which deformation retracts have zero reconstruction error on specific Hodge components

$$\mathbf{C}_n = \mathrm{Im}\partial_{n+1} \oplus \mathrm{Ker}\Delta_n \oplus \mathrm{Im}\partial_n^\dagger$$

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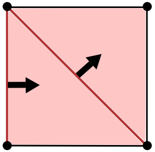
(n,n-1)-free

Question Which deformation retracts have zero reconstruction error on specific Hodge components

$$\mathbf{C}_n = \mathrm{Im}\partial_{n+1} \oplus \mathrm{Ker}\Delta_n \oplus \mathrm{Im}\partial_n^{\dagger}$$

$(n, n-1)$-free

A Morse matching is $(\mathbf{n}, \mathbf{n}-\mathbf{1})$**-free** if no $(n-1)$-cell is paired with a n-cell.



$(1, 0)$-free matching

Signal preservation on the Hodge decomposition

Let $(\mathbf{C}, I)$ be a based chain complex with matching M.

Theorem (Ebli, Hacker, M., 2022, [2])

*(**Cocycle Reconstruction**) Any signal $s \in \mathbf{C}_n$ and its reconstruction $\Phi\Psi s$ encode the same cocycle information:*

$$\mathrm{Proj}_{Ker\Delta_n \oplus Im\partial_n^\dagger}(\Phi\Psi s - s) = 0 \text{ for all } s \in \mathbf{C}_n.$$

$$\Leftrightarrow$$

M is $(n, n-1)$-free

Remarks

- Works in any base I.
- $\mathrm{Ker}\Delta_n \oplus \mathrm{Im}\partial_n^\dagger = \mathrm{Ker}\partial_{n+1}^\dagger$

Signal preservation on the Hodge decomposition

Let $(\mathbf{C}, I)$ be a based real chain complex, inner products with matching M.

Theorem (Ebli, Hacker, M., 2022, [2])

*(**Cycle Reconstruction**) Any signal $s \in \mathbf{C}_{n-1}$ and its reconstruction $\Phi^\dagger\Psi^\dagger s$ encode the same cycle information:*

$$\mathrm{Proj}_{Ker\Delta_n \oplus Im\partial_n}(\Psi^\dagger\Phi^\dagger s - s) = 0 \textit{ for all } s \in \mathbf{C}_{n-1}.$$

$$\Leftrightarrow$$

$$M \textit{ is } (n, n-1)\textit{-free}$$

Remarks

- Works in any base I.
- $\mathrm{Ker}\Delta_{n-1} \oplus \mathrm{Im}\partial_n = \mathrm{Ker}\partial_{n-1}$

Sparsification Theorem

Intuition: the **signals' reconstruction** concentrates only **on critical cells**

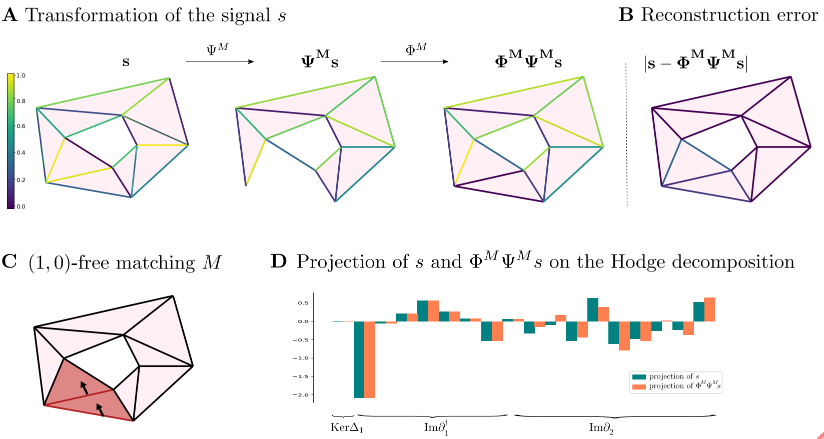
Theorem (Ebli, Hacker, M., 2022, [2])

Let M be an $(n, n-1)$-free Morse matching of a based chain complex $(\mathbf{C}, I)$. Then

.

$$\Phi_n^M \Psi_n^M(s) \in \bigoplus_{\alpha \in M^0 \cap I_n} C_\alpha \text{ for all } s \in \mathbf{C}_n$$

Example



A Transformation of the signal s
B Reconstruction error
$\mathbf{s}$
Ψ^M
$\mathbf{\Psi^M s}$
Φ^M
$\mathbf{\Phi^M \Psi^M s}$
$|\mathbf{s} - \mathbf{\Phi^M \Psi^M s}|$
1.0
0.8
0.6
0.4
0.2
0.0
C (1, 0)-free matching M
D Projection of s and $\Phi^M \Psi^M s$ on the Hodge decomposition
0.5
0.0
−0.5
−1.0
−1.5
−2.0
projection of s
projection of $\Phi^M \Psi^M s$
$\mathrm{Ker}\Delta_1$
$\mathrm{Im}\partial_1^\dagger$
$\mathrm{Im}\partial_2$

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Compression Algorithm

Goal

Find DMT matching that minimizes reconstruction error.

Iterative Approximate Approach

- Find an $(n+1, n)$-pair that minimizes reconstruction error.
- Repeat.

Remark

- Construction is linear complexity in dim $\mathbf{C}_n$ if complex is 'sparse'.

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Compression Algorithm Convergence

Iterative Approximate Approach

- Find all $(n, n-1)$-free pairs that minimizes reconstruction error.
- Repeat.

Thm (Ebli, Hacker, M)

Let (C, ∂) be a chain complex of real inner product spaces. For all $s \in C_n$, the algorithm converges to the chain complex $(\mathbf{C}', \partial')$ where

- $\mathbf{C}'_n \cong H_n(\mathbf{C}) \oplus \mathrm{Im}\partial_n^\dagger$
- $\mathbf{C}'_{n-1} \cong H_{n-1}(\mathbf{C}) \oplus \mathrm{Im}\partial_n$
- $\mathbf{C}'_k \cong H_k(\mathbf{C})$ for all other k.

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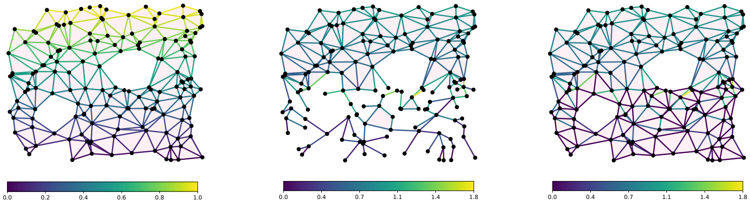
Experimental Results

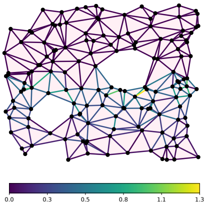
A Sequence of optimal pairings and reconstruction of the signal s

$\mathbf{s}$ = height function $\qquad \mathbf{\Psi}^{\underline{\mathbf{M}}}\mathbf{s} \qquad \mathbf{\Phi}^{\underline{\mathbf{M}}}\mathbf{\Psi}^{\underline{\mathbf{M}}}\mathbf{s}$

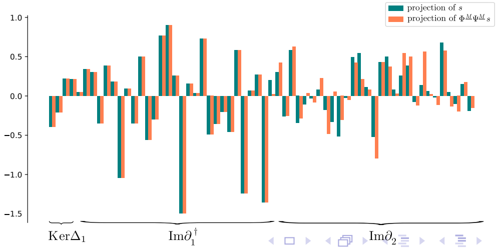


B Reconstruction error

$|\mathbf{s} - \mathbf{\Phi}^{\underline{\mathbf{M}}}\mathbf{\Psi}^{\underline{\mathbf{M}}}\mathbf{s}|$

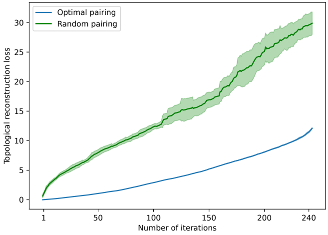


C Projection of s and $\Phi^{\underline{M}}\Psi^{\underline{M}}s$ on the Hodge basis

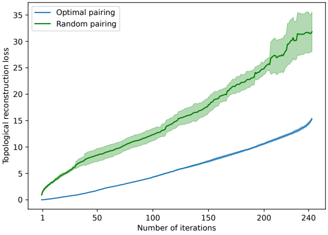


Experimental Results

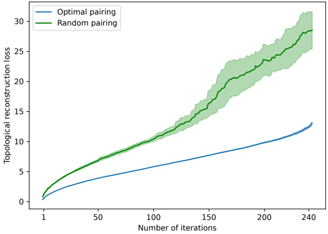
A Signal sampled from a uniform distribution



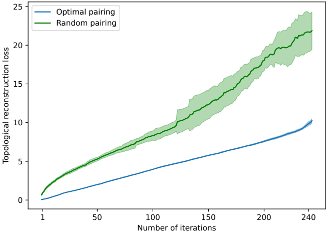
B Signal given by the height function



C Signal sampled from a normal distribution



D Signal given by the distance from the center



Paper: *Morse Theoretic Signal Compression and Reconstruction on Chain Complexes*, arXiv:2203.08571

Code: github.com/stefaniaebli/dmt-signal-processing

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Take-home Message

Theoretical

- Chain complexes of finite dimensional inner product spaces admit a canonical Morse matching based on the Hodge decomposition.
- Any deformation retract of fin dim inner product complexes is canonically equivalent to a Morse retraction

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Take-home Message

Practical

- (Sparsification) The reconstruction of $(n, n-1)$-free matchings is supported on the critical cells.
- (Reconstruction) The reconstruction error of an $(n, n-1)$-free matching is supported on $\mathrm{Im}\partial_{n+1}$ for all signals.

Take-home Message

Algorithmic

- Sequential collapse algorithm converges.
- Outperforms the random collapse baseline algorithm.

Thank you!

Questions?

References I

E. Sköldberg, "Algebraic Morse theory and homological perturbation theory," *Algebra Discrete Math.*, vol. 26, pp. 124–129, 2018.

S. Ebli, C. Hacker, and K. Maggs, "Morse theoretic signal compression and reconstruction on chain complexes," *ArXiv*, vol. abs/2203.08571, 2022.

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