Persistence Steenrod modules

Topology of Data in Roma Tor Vergata

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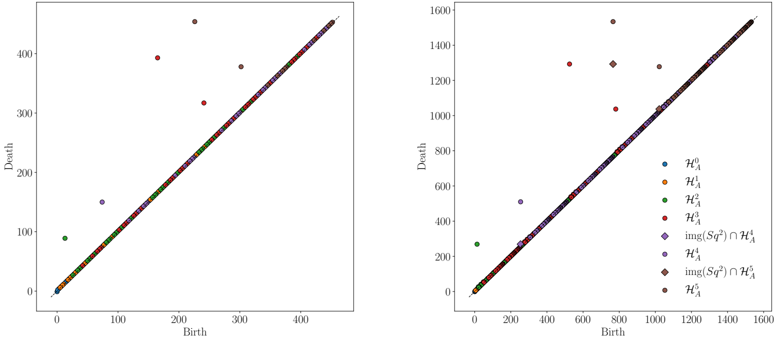

(a) $\mathrm{C}\,\Sigma(S^2 \vee S^4)$ (b) $\mathrm{C}\,\Sigma\,\mathbb{C}\mathrm{P}^2$

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To construct invariants of spaces up to some notion of equivalence.

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Cohomology via chain complex vs maps to Eilenberg-Maclane spaces.

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Break spaces into contractible combinatorial pieces:

Simplices, cubes, ...

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Our goals (loosely stated)

Understand the diagonal map of these standard complexes better to present effective/local computations of finer invariants in cohomology.

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Defined by dualizing an **explicit** chain approximation to the diagonal

$$\mathrm{C}(\mathbb{\Delta}^n) \to \mathrm{C}(\mathbb{\Delta}^n) \otimes \mathrm{C}(\mathbb{\Delta}^n)$$

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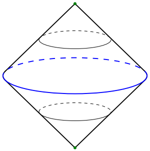

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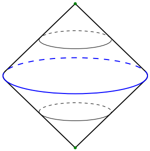

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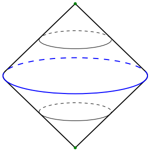

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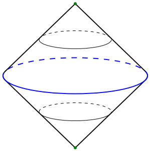

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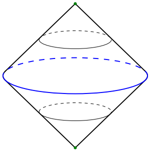

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Question: Can it be described **explicitly** at the chain level?

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Unlike the diagonal of spaces, chain approxs to it are **not** invariant under

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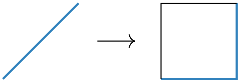

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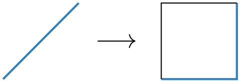

To correct homotopically the breaking of this symmetry, Steenrod introduced **explicit** maps

$$\Delta_i \colon \mathrm{C}(\mathbb{\Delta}^n) \to \mathrm{C}(\mathbb{\Delta}^n)^{\otimes 2} \quad \text{satisfying} \quad \partial \Delta_i = \big(1 \pm T\big)\Delta_{i-1},$$

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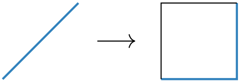

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These define the Steenrod squares as

$$\begin{aligned} \mathrm{Sq}^k \colon H^\bullet(X; \mathbb{F}_2) &\to H^\bullet(X; \mathbb{F}_2) \\ [\alpha] &\mapsto \big[(\alpha \otimes \alpha)\Delta_i(-)\big] \end{aligned}$$

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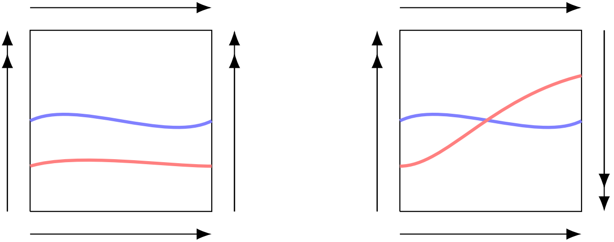

(a) Torus T (b) Klein Bottle K

(a) $\operatorname{rank}\left(\mathrm{Sq}^1\colon H^1(\mathrm{T};\mathbb{F}_2) \to H^2(\mathrm{T};\mathbb{F}_2)\right) = 0$,

(b) $\operatorname{rank}\left(\mathrm{Sq}^1\colon H^1(\mathrm{K};\mathbb{F}_2) \to H^2(\mathrm{K};\mathbb{F}_2)\right) = 1$.

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Definition (Med.)

For a basis element $x \in \mathrm{C}_m(\mathbb{\Delta}^n, \mathbb{F}_2)$

$$\Delta_i(x) \ = \sum_{U \in \mathrm{P}^n_{m-i}} d_{U^0}(x) \otimes d_{U^1}(x)$$

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Example:

$$\Delta_0[0,1,2] = \Big(d_{12} \otimes \mathsf{id} + d_2 \otimes d_0 + \mathsf{id} \otimes d_{01}\Big)[0,1,2]^{\otimes 2}$$
$$= [0] \otimes [0,1,2] + [0,1] \otimes [1,2] + [0,1,2] \otimes [2].$$

Fast computation of Steenrod squares

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Comparing with SAGE: (algorithm based on EZ-AW contraction)

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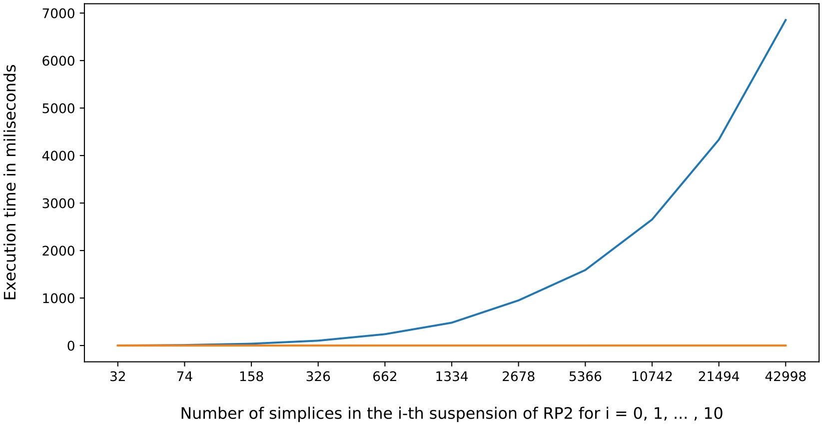
Comparing with SAGE: (algorithm based on EZ-AW contraction)

Sq^1 on $\Sigma^i \mathbb{R}P^2$ (i^{th} suspension of the real projective plane)



Steenrod squares for simplicial complexes

Input: $A = \{a_1, \ldots, a_m\} \subseteq X_n$
$B = \emptyset$
forall a_i and a_j with $i < j$ **do**
  $a_{ij} = a_i \cup a_j$
  if $a_{ij} \in X_{n+k}$ **then**
    $\overline{a}_i = a_i \setminus a_j$; $\overline{a}_j = a_j \setminus a_i$; $\overline{a}_{ij} = \overline{a}_i \cup \overline{a}_j$
    $index \colon \overline{a}_{ij} \to \{0, 1\}$
    forall $v \in \overline{a}_{ij}$ **do**
      $p =$ position of v in a_{ij} ; $\overline{p} =$ position of v in $\overline{a}_{ij}$
      $index(v) = p + \overline{p}$ residue mod 2
    end
    if $index(\overline{a}_i) \triangle index(\overline{a}_j) = \{0, 1\}$ **then**
      $B = B \triangle \{a_{ij}\}$
    end
  end
end
Output: B

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Theorem (Ling Zhou–Med.–Mémoli)
These barcodes are stable.

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Filtrations of the cone on the suspension of $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$.

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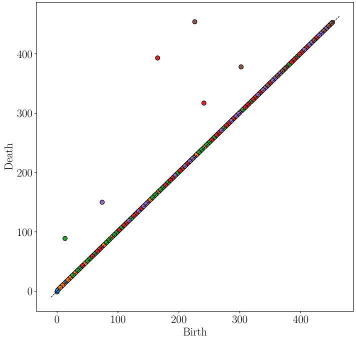

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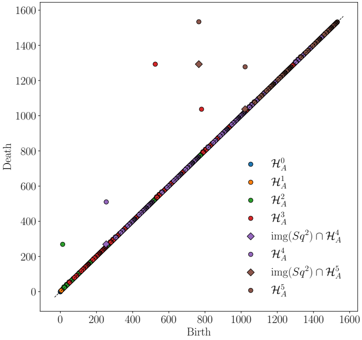

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3 Using that R is made of generating coboundaries, apply a reduction algorithm to Q^k with respect to R recording the rank of $Q^k_{\leq j}$.

Third step


```
Input: R, Q^k
Alive = {0, ..., m}, Barcode = ∅
for j = 0, ..., m do
    R_{≤j} | Q^k_{≤j} = Reduce (R_{≤j} | Q^k_{≤j})
    for i = 0, ..., j do
        if i ∈ Alive and Q^k_i = 0 then
            remove i from Alive
            if i < j then
                add [m − j, m − i] to Barcode
            end
        end
    end
end
for i ∈ Alive do
    add [−1, m − i] to Barcode
end
Output: Barcode
```

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It can easily installed via

```
python -m pip install -U steenroder
```

and we accept contributions at

https://github.com/Steenroder/steenroder

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Points in $\mathbb{R}^{24}$ (positions of 8 carbons in $\mathbb{R}^3$)

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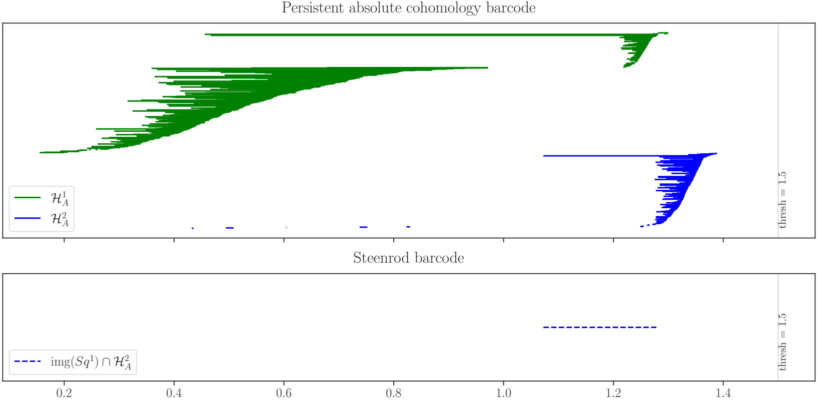
Computing Sq^1 barcode of a "smooth component" of this point cloud



Consistent with a Klein bottle component.

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from the symmetry of diagonal $X \to X \times \cdots \times X$.

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Example
Using the computer algebra system `ComCH` we have $\Delta_{3,2}[0,1,2] =$

```
- [0,1][0,1,2][0,1] + [0,1,2][0,2][0,1] + [0,2][0,2][0,1,2]
- [0,1,2][0,1,2][1] - [0,2][0,1,2][1,2] + [0,1,2][1,2][1,2]
- [0,1][1,2][0,1,2] - [0,1,2][2][0,1,2] - [0][0,1,2][0,1,2]
```

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What other data sets exhibit non-trivial Steenrod barcodes?

What are they telling us about the data set, domain specifically?

Thank you!

1. Medina-Mardones, Anibal M. "New formulas for cup-i products and fast computation of Steenrod squares." Computational Geometry 109 (2023).

2. Umberto Lupo, Anibal M. Medina-Mardones, and Guillaume Tauzin. "Persistence Steenrod modules." Journal of Applied and Computational Topology (2022).