Combinatorics of the amplituhedron

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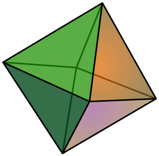
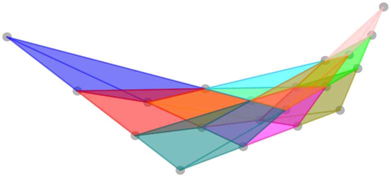
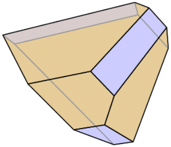
Based on: joint works with Steven Karp, Tomasz Lukowski, Matteo Parisi, Melissa Sherman-Bennett, ...

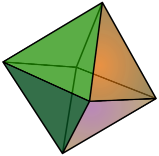


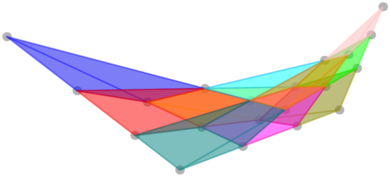


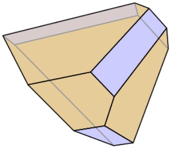


Overview of the talk

- What is the positive Grassmannian?
- What is the amplituhedron?
- We can study it from the point of view of
  - oriented matroids
  - cluster algebras
  - tilings/ triangulations
- Along the way we'll see connections to Eulerian numbers, the positive tropical Grassmannian, and subdivisions of the hypersimplex







The Grassmannian and the matroid stratification

The **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix C.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(C)$ is the minor of the $k \times k$ submatrix of C in column set I.

The *matroid* associated to $C \in Gr_{k,n}$ is $\mathcal{M}(C) := \{I \in \binom{[n]}{k} \mid p_I(C) \neq 0.\}$

Gelfand-Goresky-MacPherson-Serganova '87 introduced the *matroid stratification* of $Gr_{k,n}$.

Given $\mathcal{M} \subset \binom{[n]}{k}$, let $S_{\mathcal{M}} = \{C \in Gr_{k,n} \mid p_I(C) \neq 0 \text{ iff } I \in \mathcal{M}\}$.

If $S_{\mathcal{M}}$ is nonempty, $S_{\mathcal{M}}$ called *matroid stratum*.

Matroid stratification: $Gr_{k,n} = \sqcup_{\mathcal{M}} S_{\mathcal{M}}$.

The Grassmannian and the matroid stratification

Let $\{e_1, \ldots, e_n\}$ be basis of $\mathbb{R}^n$; for $I \subset [n]$, let $e_I := \sum_{i \in I} e_i$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(C) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(C)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(C)|^2} \subset \mathbb{R}^n.$$

Recall: *matroid* assoc to $C \in Gr_{k,n}$ is $\mathcal{M}(C) := \{I \in \binom{[n]}{k} \mid p_I(C) \neq 0.\}$
Can associate two objects to $\mathcal{M}$:

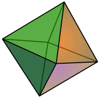
- matroid stratum $S_{\mathcal{M}} := \{C \in Gr_{k,n} \mid p_I(C) \neq 0 \text{ iff } I \in \mathcal{M}\}$.
- matroid polytope $\Gamma_{\mathcal{M}} = \mathrm{Conv}\{e_I \mid I \in \mathcal{M}\}$.

GGMS used the Convexity Theorem to show that the moment map image of a matroid stratum is the corresponding matroid polytope:

$$\overline{\mu(S_{\mathcal{M}})} = \Gamma_{\mathcal{M}}.$$

The Grassmannian and the matroid stratification

Example: if $\mathcal{M} = \binom{[n]}{k}$, then $\overline{\mu(S_{\mathcal{M}})} = \text{Conv}\{e_I \ : \ I = \binom{[n]}{k}\} \subset \mathbb{R}^n$.



This polytope is the *hypersimplex.*

Matroid polytopes have beautiful properties.
Gelfand-Goresky-MacPherson-Serganova '87 characterized them as 01 polytopes such that every edge is parallel to $e_i - e_j$ for some i, j.

However, the topology of matroid strata is terrible – Mnev's *universality theorem* (1987): "The topology of the matroid stratum $S_{\mathcal{M}}$ can be as complicated as that of any algebraic variety."

What is the positive Grassmannian?

Background: Lusztig's total positivity for G/P 1994, Rietsch 1997, Postnikov's 2006 preprint on the *totally non-negative* (TNN) or "positive" Grassmannian.

Let $Gr_{k,n}^{\geq 0}$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $p_I \geq 0$ for all I.

Inspired by matroid stratification, one can partition $Gr_{k,n}^{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{C \in Gr_{k,n}^{\geq 0} \mid p_I(C) > 0 \text{ iff } I \in \mathcal{M}\}$.

In contrast to terrible topology of matroid strata ...

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. So we have *positroid cell decomposition*

$$Gr_{k,n}^{\geq 0} = \sqcup S_{\mathcal{M}}^{tnn}.$$

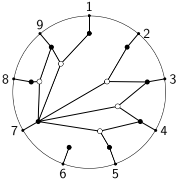
Cells of the positive Grassmannian

- Recall: *matroid* assoc to $C \in Gr_{k,n}$ is $\mathcal{M} = \mathcal{M}(C) := \{I \in \binom{[n]}{k} \mid p_I(C) \neq 0.\}$
- And the *matroid polytope* is $\Gamma_{\mathcal{M}} = \text{Conv}\{e_I \mid I \in \mathcal{M}.\}$
- If $C \in Gr_{k,n}^{\geq 0}$, call $\mathcal{M}(C)$ a *positroid* and $\Gamma_{\mathcal{M}}$ a *positroid polytope*; combinatorial structure of $\Gamma_{\mathcal{M}}$ studied in Ardila-Rincon-W.

Theorem (Postnikov)

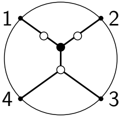
The positroid cells of $Gr_{k,n}^{\geq 0}$ are in bijection with:

- *decorated permutations* π on $[n]$ with k antiexcedances
- equivalence classes of *planar bicolored (plabic) graphs*

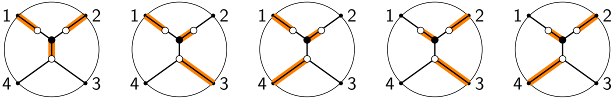


How to read off a positroid cell from a plabic graph

- Positroid cells $\leftrightarrow$ *plabic graphs*, planar bicolored graphs embedded in disk with boundary vertices labeled $1, 2, \ldots, n$ and internal vertices colored black or white.



- WLOG we assume graph G is bipartite and that every boundary vertex is incident to a white vertex.
- Let $\mathcal{M}(G) := \{\partial(P) \mid P$ is an **almost perfect matching** of $G\}$.



  E.g. for graph above, get $\mathcal{M}(G) = \{12, 13, 14, 23, 24\}$.
- Theorem (Postnikov): $\mathcal{M}(G)$ is the set of nonzero Plücker coordinates of a positroid cell, and all cells obtained this way.

What is the amplituhedron?

- Introduced by Arkani-Hamed and Trnka in 2013.
- The amplituhedron is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$:

Fix n, k, m with $k + m \leq n$.
Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to $\mathrm{span}(CZ)$.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}(Z)$ depends on Z but many combin. properties appear not to.
- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" is supposed to compute scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory;
  the *BCFW recurrence* for scattering amplitudes can be reformulated as giving a "triangulation" of the $m = 4$ amplituhedron.

What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $m = n - k$, $\mathcal{A}_{n,k,m} = Gr_{k,n}^{\geq 0}$.

What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$:
  E.g. if $m = 2$, let $Z_1, \ldots, Z_n$ denote rows of $Z \in \mathrm{Mat}_{n,3}^{>0}$.
  Positivity implies they represent vertices of convex polytope in $\mathbb{P}^2$.
  Image of $Gr_{1,3}^{\geq 0}$ under $\tilde{Z}$ gives entire polytope.

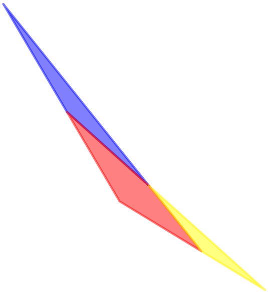
What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$

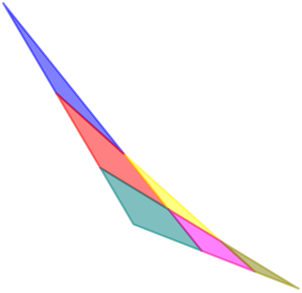
Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

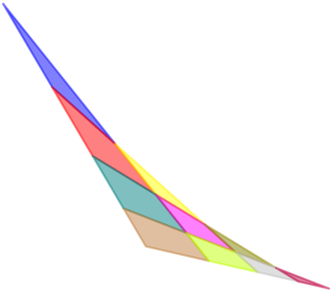
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)



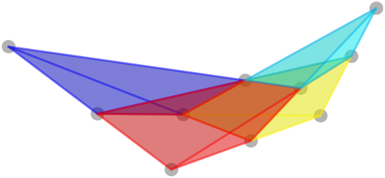
$\mathcal{A}_{4,2,1}$



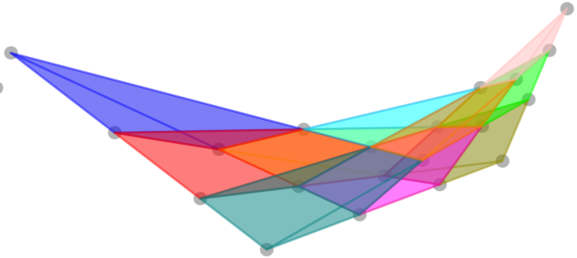
$\mathcal{A}_{5,2,1}$



$\mathcal{A}_{6,2,1}$



$\mathcal{A}_{5,3,1}$



$\mathcal{A}_{6,3,1}$

I. The amplituhedron and oriented matroids

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Need some good coordinates to use for $\mathcal{A}_{n,k,m}$;
ideally, want to describe amplituhedron directly inside $Gr_{k,k+m}$.
Let $Z_1, \dots, Z_n$ be rows of Z. Let $Y \in Gr_{k,k+m}$ (viewed as matrix).
Given $I = \{i_1 < \cdots < i_m\} \subset [n]$, let

$$\langle YZ_I \rangle = \langle YZ_{i_1} \dots Z_{i_m} \rangle := \det \begin{bmatrix} - & Y & - \\ - & Z_{i_1} & - \\ & \vdots & \\ - & Z_{i_m} & - \end{bmatrix}$$

Call it *twistor coordinate* $\langle YZ_I \rangle$ (Arkani-Hamed–Thomas–Trnka)
Rk: $Y \in Gr_{k,k+m}$ determined by twistor coords; $\langle YZ_I \rangle = p_I(Y^{\perp} Z^t)$.

I. The amplituhedron and oriented matroids

Fix n, k, m with $k+m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
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Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Let $Z_1, \dots, Z_n$ be rows of Z. Let $Y \in Gr_{k,k+m}$ (viewed as matrix).
For $I = \{i_1 < \dots < i_m\} \subset [n]$, let

$$\langle YZ_I \rangle = \langle YZ_{i_1} \dots Z_{i_m} \rangle := \det \begin{bmatrix} - & Y & - \\ - & Z_{i_1} & - \\ & \vdots & \\ - & Z_{i_m} & - \end{bmatrix}$$

Inspired by matroid stratification, we define the *amplituhedron sign stratification* – decompose $\mathcal{A}_{n,k,m}(Z)$ into pieces based on the signs of twistor coordinates. (Parisi–Sherman-Bennett–W.; Karp-W.)
Call the top-dimensional pieces *chambers*.

I. The amplituhedron and oriented matroids

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Let $Z_1, \dots, Z_n$ be rows of Z. Let $Y \in Gr_{k,k+m}$ (viewed as matrix).
For $I = \{i_1 < \cdots < i_m\} \subset [n]$, let

$$\langle YZ_I \rangle = \langle YZ_{i_1} \dots Z_{i_m} \rangle$$

Given $\sigma \in \{+, -\}^{\binom{n}{m}}$, define *amplituhedron chamber*

$$\mathcal{A}_{n,k,m}^{\sigma}(Z) := \{Y \in \mathcal{A}_{n,k,m}(Z) \mid \{\mathrm{sign}\langle YZ_I \rangle\}_{I \in \binom{[n]}{m}} = \sigma\}.$$

Which chambers are *realizable*? (not always empty)

I. The amplituhedron and oriented matroids

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
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Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Let $Z_1, \dots, Z_n$ be rows of Z. Let $Y \in Gr_{k,k+m}$.
For $I = \{i_1 < \dots < i_m\} \subset [n]$, let $\langle YZ_I \rangle = \langle YZ_{i_1} \dots Z_{i_m} \rangle$.
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$$\mathcal{A}_{n,k,m}^{\sigma}(Z) := \{Y \in \mathcal{A}_{n,k,m}(Z) \mid \{\mathrm{sign}\langle YZ_I \rangle\}_{I \in \binom{[n]}{m}} = \sigma\}.$$

Say σ is *realizable* if there exists Z such that $\mathcal{A}_{n,k,m}^{\sigma}(Z) \neq \emptyset$.

Theorem (Parisi–Sherman-Bennett–W.)

For $m = 2$, the (realizable) amplituhedron chambers of $\mathcal{A}_{n,k,2}$ are counted by the Eulerian numbers $\{w \in S_{n-1} \mid \mathrm{des}(w) = k\}$.

Note: Volume of hypersimplex $\Delta_{k+1,n}$ is the same Eulerian number.
This is a shadow of strange duality between $\mathcal{A}_{n,k,2}(Z)$ and $\Delta_{k+1,n}$!

II. Cluster algebras and the amplituhedron

Fix n, k, m with $k + m \leq n$.
Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Since $\dim \mathcal{A}_{n,k,m}(Z) = km$, is natural to ask:
When does $\tilde{Z}$ map a km-dimension cell of $Gr_{k,n}^{\geq 0}$ injectively to $\mathcal{A}_{n,k,m}(Z)$?
Say $\tilde{Z}(S_\pi)$ is a *positroid tile* if $\tilde{Z}$ is injective on km-dimensional cell S_π.

Cluster adjacency conjecture (Lukowski-Parisi-Spradlin-Volovich)

Let $\tilde{Z}(S_\pi)$ be positroid tile of $\mathcal{A}_{n,k,2}(Z)$. Then each facet[a] of $\tilde{Z}(S_\pi)$ lies on a hypersurface $\langle YZ_iZ_j\rangle = 0$, and the Plücker coords $\{p_{ij}\}$ corresponding to facets form a collection of compatible cluster variables for $\mathrm{Gr}_{2,n}$.

[a] a facet is a maximal-by-inclusion codimension 1 stratum of the form $\tilde{Z}(S_{\pi'})$ lying in the boundary of $\tilde{Z}(S_\pi)$, such that $S_{\pi'} \subset S_\pi$

II. Cluster algebras and the amplituhedron

Fix n, k, m with $k + m \leq n$.
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Theorem (Parisi–Sherman-Bennett–W.)

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Moreover, if p_{hl} is compatible with $\{p_{ij}\}$, then the twistor coordinate $\langle Y Z_h Z_l \rangle$ has a fixed sign on $\tilde{Z}(S_\pi)$.

[a] a facet is a maximal-by-inclusion codimension 1 stratum of the form $\tilde{Z}(S_{\pi'})$ lying in the boundary of $\tilde{Z}(S_\pi)$, such that $S_{\pi'} \subset S_\pi$

II. Cluster algebras and the amplituhedron

Recall: $\tilde{Z}(S_\pi)$ is a *positroid tile* if $\tilde{Z}$ is injective on km-dimensional cell S_π.

Theorem (Parisi–Sherman-Bennett–W.)

Let $\tilde{Z}(S_\pi)$ be positroid tile of $\mathcal{A}_{n,k,2}(Z)$. Then each facet[a] of $\tilde{Z}(S_\pi)$ lies on a hypersurface $\langle YZ_iZ_j\rangle = 0$, and the Plücker coords $\{p_{ij}\}$ corresponding to facets form a collection of compatible cluster variables for $\mathrm{Gr}_{2,n}$.

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[a] a facet is a maximal-by-inclusion codimension 1 stratum of the form $\tilde{Z}(S_{\pi'})$ lying in the boundary of $\tilde{Z}(S_\pi)$, such that $S_{\pi'} \subset S_\pi$

- To prove theorem, we classified the positroid tiles for $m = 2$.
- Believe there's an analogue of thm for $m > 2$; but classification of tiles for $m > 2$ unknown.
- Theorem suggests there is a cluster algebra structure directly on positroid tiles. We can prove this for $m = 2$.

II. Cluster algebras and the amplituhedron

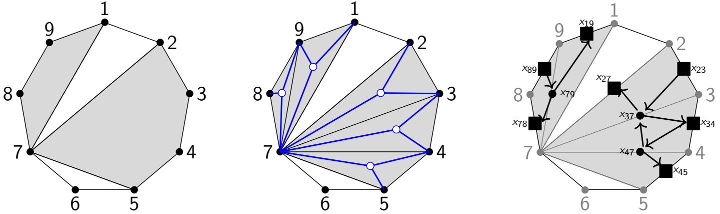
Recall: $\tilde{Z}(S_\pi)$ is a *positroid tile* for $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on km-dimensional cell S_π.

Conj (L-P-S-V); Theorem (P–SB–W)

The positroid tiles for $\mathcal{A}_{n,k,2}(Z)$ are precisely the images of positroid cells whose plabic graphs are constructed as follows:

- Choose a bicolored subdivision of an n-gon consisting of grey polygons which can be triangulated into k triangles.
- Put white vertex in every grey triangle, connected to three vertices.

Get cluster structure with cluster vars $x_{ab} \propto$ twistor coords $\langle YZ_aZ_b \rangle$.



III. Tilings of the amplituhedron

Fix n, k, m with $k + m \leq n$. Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$.
Have $\widetilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

The map $\tilde{Z}$ is surjective onto $\mathcal{A}_{n,k,m}(Z)$ but in general not injective.
Would like to find a km-dimensional cross-section of $Gr_{k,n}^{\geq 0}$ which $\tilde{Z}$ maps injectively onto the amplituhedron.

Let $X = \sqcup_\pi S_\pi$ be a cell complex, and let $\phi : X \to Y$ be a continuous surjective map onto Y, a d-dimensional cell complex (or subset).
Define a *ϕ-induced tiling* of Y to be a collection $\{\overline{\phi(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of d-dimensional cells, such that:

- ϕ is injective on each S_π for $\pi \in \mathcal{C}$ $\;(\overline{\phi(S_\pi)}$ a tile)
- their union equals Y
- their interiors are pairwise disjoint

III. Tilings of the amplituhedron

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}_{n,k+m}^{>0}$.
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Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Let $X = \sqcup_\pi S_\pi$ be a cell complex, and let $\phi : X \to Y$ be a continuous surjective map onto Y, a d-dimensional cell complex (or subset). Define a *ϕ-induced tiling* of Y to be a collection $\{\overline{\phi(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of d-dimensional cells, such that:

- ϕ is injective on each S_π for $\pi \in \mathcal{C}$.
- their union equals Y
- their interiors are pairwise disjoint

Remark: Can drop injectivity requirement – call resulting object a *dissection*. When $\phi : X \to Y$ is affine projection of convex polytopes, we recover Billera-Sturmfels' notion of (tight) ϕ-induced subdivision.

III. Tilings of the amplituhedron

Fix n, k, m with $k + m \leq n$. Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$.
Have $\widetilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map. Then a *$\tilde{Z}$-induced tiling* of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of positroid tiles[a], such that:

- their union equals $\mathcal{A}_{n,k,m}(Z)$
- their interiors are pairwise disjoint

[a] closures of images of km-dimensional cells on which map is injective

$\tilde{Z}$-induced tilings of $\mathcal{A}_{n,k,4}(Z)$ first discussed in Arkani-Hamed–Trnka; conjectured that various collections of BCFW cells give a tiling; recent work of Evan Zohar–Lakrec–Tessler.

III. Tilings of the amplituhedron

$\tilde{Z}$-induced tilings have been studied in special cases. Their cardinalities are interesting!

special case	cardinality of tiling of $\mathcal{A}_{n,k,m}(Z)$	explanation
$m = 0$ or $k = 0$	1	$\mathcal{A}$ is a point
$k + m = n$	1	$\mathcal{A} \cong \mathrm{Gr}_{k,n}^{\geq 0}$
$m = 1$	$\binom{n-1}{k}$	Karp-W.
$m = 2$	$\binom{n-2}{k}$	AH-T-T, Bao-He, P-SB-W
$m = 4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	AH-T, Even-Zohar–Lakrec–Tessler
$k = 1$, m even	$\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$	$\mathcal{A} \cong$ cyclic polytope $C(n, m)$

III. Tilings of the amplituhedron

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map. A *$\tilde{Z}$-induced tiling* of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of positroid tiles such that

- their union equals $\mathcal{A}_{n,k,m}(Z)$
- their interiors are pairwise disjoint

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The cardinality of a $\tilde{Z}$-induced tiling of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.

III. Tilings of the amplituhedron

Remarkably, tilings of $\mathcal{A}_{n,k,2}(Z)$ and of hypersimplex $\Delta_{k+1,n}$ are related!

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,2}(Z) \subset \mathrm{Gr}_{k,k+2}$.
A *$\tilde{Z}$-induced tiling* of $\mathcal{A}_{n,k,2}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of $2k$-dimensional cells, such that:
$\tilde{Z}$ injective on each S_π; union equals $\mathcal{A}_{n,k,2}(Z)$; interiors pairwise disjoint.

Have $Gr_{k+1,n}^{\geq 0} = \sqcup_\pi S_\pi$, and $\mu : Gr_{k+1,n}^{\geq 0} \to \Delta_{k+1,n} \subset \mathbb{R}^n$.
A *μ-induced tiling* of $\Delta_{k+1,n}$ is a collection $\{\overline{\mu(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of $(n-1)$-dimensional cells, such that:
μ injective on each S_π; union equals $\Delta_{k+1,n}$; interiors pairwise disjoint.

Note: each $\overline{\mu(S_\pi)}$ is a positroid polytope.

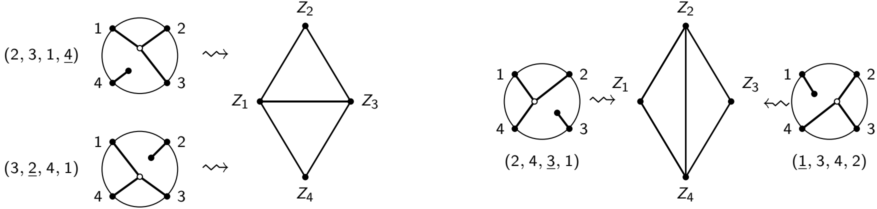
Conj (Lukowski-Parisi-W); Thm (Parisi–Sherman-Bennett–W.)

$\tilde{Z}$-induced tilings of $\mathcal{A}_{n,k,2}$ in bijection with μ-induced tilings of $\Delta_{k+1,n}$.

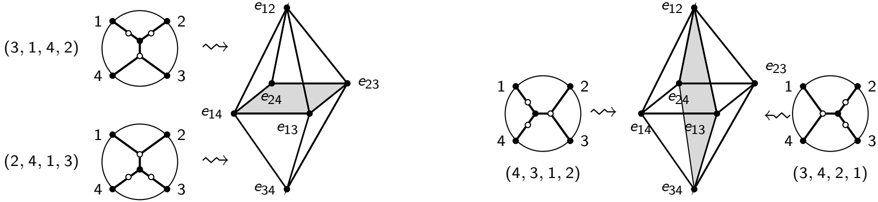
Cor: # of fine subdivisions of $\Delta_{2,n}$ into positroid polytopes is Catalan #.

III. Tilings of the amplituhedron ($k = 1$ and $n = 4$)

There are two tilings of $\mathcal{A}_{4,1,2}(Z)$ (a quadrilateral in $\mathbb{P}^2$):



There are two tilings of hypersimplex $\Delta_{2,4}$ (octahedron):



How can we biject the cells of these tilings? (look for relation between permutations (LPW) and/or plabic graphs)

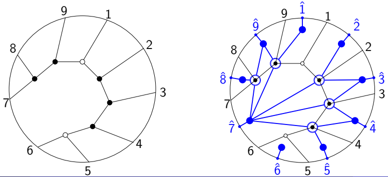
III. Tilings of the amplituhedron ($k = 1$ and $n = 4$)

How can we relate the cells giving tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$?
This needs to map $(n-1)$-dim'l cells of $\mathrm{Gr}_{k+1,n}^{\geq 0}$ to $2k$-dim'l cells of $\mathrm{Gr}_{k,n}^{\geq 0}$.

T-duality or "shift map" on plabic graphs (G-P-W; G; P-SB-W)

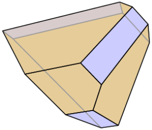
Let G be reduced black-trivalent plabic graph. Define T-dual graph $\hat{G}$ by:

- In each face f of G, place a black vertex $\hat{b}(f)$.
- "On top of" each black vertex b of G, place a white vertex $\hat{w}(b)$;
- For each black vertex b of G incident to face f, add edge $(\hat{w}(b), \hat{b}(f))$;
- Put $\hat{i}$ on the boundary of G between vertices $i-1$ and i and draw an edge from $\hat{i}$ to $\hat{b}(f)$, where f is the adjacent boundary face.



How did we guess connection between $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$?

- With Lukowski–Parisi we realized that the f-vector of the positive tropical Grassmannian Trop $Gr^+_{k+1,n}$ (Speyer-W.) seemed to be enumerating the "good" $\tilde{Z}$-induced subdivisions of $\mathcal{A}_{n,k,2}(Z)$.



- We realized that the f-vector of Trop $Gr^+_{k+1,n}$ was related to positroid subdivisions of the hypersimplex $\Delta_{k+1,n}$;
- and found that the "T-duality" or "shift map" from cells of $Gr^{\geq 0}_{k+1,n}$ to cells of $Gr^{\geq 0}_{k,n}$ seemed to give a bijection between μ-induced (positroid) subdivisions of $\Delta_{k+1,n}$ and $\tilde{Z}$-induced subdivisions of $\mathcal{A}_{n,k,2}(Z)$.

Summary and questions

The amplituhedron includes as special cases

- the positive Grassmannian
- cyclic polytopes
- bounded complex of cyclic hyperplane arrangement

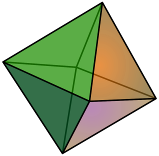
and is closely connected to the hypersimplex.

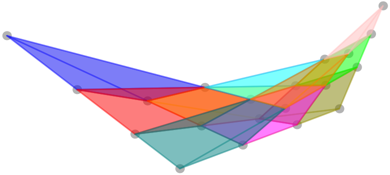
It is useful to study amplituhedron from the point of view of

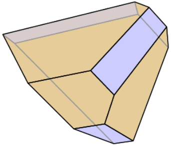
- oriented matroids,
- cluster algebras,
- and tilings.

Lots of open problems!

Thank you for listening!







- "The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron," with Lukowski and Parisi, arXiv:2002.06164
- "The $m = 2$ amplituhedron and the hypersimplex: signs, clusters, triangulations, Eulerian numbers, with Parisi and Sherman-Bennett, arXiv:2104.08254.