Many new conjectures on Fully-Packed Loop configurations

Andrea Sportiello
work in collaboration with L. Cantini

Algebraic Combinatorics and Mathematical Physics
Università degli Studi di Roma “Tor Vergata”
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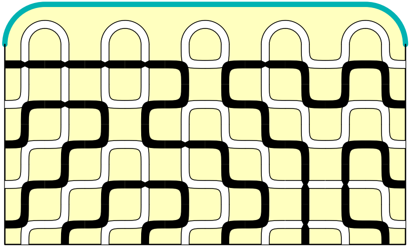
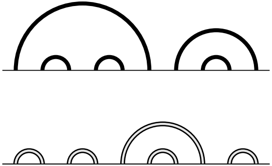




$$\#\{\bigcirc\} + \#\{\circledcirc\} = 2$$

Part I

All you'll need to know today about Schur functions

Simple general facts on algebras

An algebra Λ with linear basis $\{f_\alpha\}$ is determined by its structure constants

$$f_\alpha\, f_\beta = \sum_\gamma c_{\alpha\beta}^{\gamma}\, f_\gamma$$

If we have a scalar product $\langle \cdot | \cdot \rangle$, the dual basis is the set of functions $\{g^\alpha\}$ such that $\langle g^\alpha | f_\beta \rangle = \delta_{\alpha\beta}$. The dual basis comes with its own structure constants

$$g^\alpha\, g^\beta = \sum_\gamma d_\gamma^{\alpha\beta}\, g^\gamma$$

For pairs of indices, define the 'skew' combinations $g^{\gamma/\beta} := \sum_\alpha c_{\alpha\beta}^{\gamma}\, g^\alpha$ and $f_{\gamma/\beta} := \sum_\alpha d_\gamma^{\alpha\beta}\, f_\alpha$. Then it is easily seen that the following properties hold

$$\langle g^{\gamma/\beta} | h \rangle = \langle g^\gamma | h\, f_\beta \rangle \qquad \forall\ h \in \Lambda$$

$$\langle h | f_{\gamma/\beta} \rangle = \langle h\, g^\beta | f_\gamma \rangle \qquad \forall\ h \in \Lambda$$

Simple general facts on algebras

Given an algebra as before, and a linear change of basis $\hat{f}_\alpha = \sum_\beta B_\alpha{}^\beta f_\beta$, call $\bar{B} := B^{-T}$. We must set $\hat{g}^\alpha = \sum_\beta \bar{B}^\alpha{}_\beta\, g^\beta$ in order to have $\langle \hat{g}^\alpha | \hat{f}_\beta \rangle = \delta_{\alpha\beta}$. Then the new structure constants

$$\hat{f}_\alpha\, \hat{f}_\beta = \sum_\gamma \hat{c}^\gamma_{\alpha\beta}\, \hat{f}_\gamma \qquad \hat{g}^\alpha\, \hat{g}^\beta = \sum_\gamma \hat{d}^{\alpha\beta}_\gamma\, \hat{g}^\gamma$$

are given by

$$\hat{c}^\gamma_{\alpha\beta} = \sum_{\alpha'\beta'\gamma'} B_\alpha{}^{\alpha'} B_\beta{}^{\beta'} \bar{B}^\gamma{}_{\gamma'}\, c^{\gamma'}_{\alpha'\beta'}$$

$$\hat{d}^{\alpha\beta}_\gamma = \sum_{\alpha'\beta'\gamma'} \bar{B}^\alpha{}_{\alpha'} \bar{B}^\beta{}_{\beta'} B_\gamma{}^{\gamma'}\, d^{\alpha'\beta'}_{\gamma'}$$

If Λ is a algebra of functions, a Cauchy identity is an identity of the form

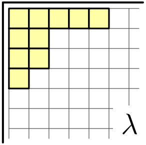
$$\sum_\alpha g^\alpha(x) f_\alpha(y) = Z(x,y)$$

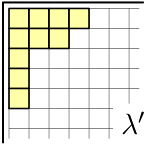
Then, it is easily seen that also $\sum_\alpha \hat{g}^\alpha(x) \hat{f}_\alpha(y) = Z(x,y)$.

Schur Functions

Now, consider the algebra of symmetric polynomials in n variables, Λ_n
Here a good labeling of the basis is given by integer partitions,
also called here Young Diagrams

$$
\begin{aligned}
\lambda &= (5,2,2,1) \\
\ell(\lambda) &= \#\,\textit{parts} = 4 && (\textit{length}) \\
|\lambda| &= \#\,\textit{cells} = 10 && (\textit{size}) \\
\lambda' &= (4,3,1,1,1) && (\textit{transposed i.p.})
\end{aligned}
$$





Standard Young Tableaux $SYT(\lambda)$:

Fillings of λ with the integers $\{1, 2, \dots, |\lambda|\}$, no repetitions, satisfying $\begin{array}{cc} \bullet & < \bullet \\ \wedge & \\ \bullet & \end{array}$

Play a crucial role in the representation theory of the symmetric group S_m, with $m = |\lambda|$

1	2	6	7	10
3	5			
4	8			
9				

Schur Functions

Semi-Standard Young Tableaux $SSYT(\lambda, n)$:

Fillings of λ with the integers $\{1, 2, \dots, n\}$, repetitions allowed, satisfying $\begin{array}{ccc} \bullet & \leq & \bullet \\ \wedge & & \\ \bullet & & \end{array}$

1	1	3	4	4
2	3			
5	6			
6				

Play a crucial role in the representation theory of the general linear group GL_n

Remark 1: $SSYT(\lambda, n) = \varnothing$ if $n < \ell(\lambda)$

Remark 2: $\lim_{n\to\infty} n^{-|\lambda|}|SSYT(\lambda, n)| = |SYT(\lambda)| \,/\, |\lambda|!$
(in a sense, SSYT are richer than SYT)

Schur polynomials are the 'generating functions' of SSYT's:

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in SSYT(\lambda,n)} \prod_{i=1}^{n} x_i^{\#\{i \in T\}}$$

s$(x_1, \dots, x_6) = \cdots + x_1^2 x_2 x_3^2 x_4^2 x_5 x_6^2 + \cdots$

Many beautiful facts about Schur Functions

For $n \in \mathbb{N}$, call $\delta_n = (n-1, n-2, \dots, 1)$

❶ A famous fact (coming from the Weyl character formula) is that the Schur polynomials can be written as the ratio of two determinants

$$s_\lambda(x_1, \dots, x_n) = \frac{1}{\Delta(\vec{x})} \det\left(\left(x_i^{(\lambda+\delta_n)_j}\right)_{i,j=1,\dots,n}\right)$$

$$\Delta(\vec{x}) = \det\left(\left(x_i^{(\delta_n)_j}\right)_{i,j=1,\dots,n}\right) = \prod_{i<j}(x_i - x_j)$$

This is remarkable, as evaluating a Schur polynomial at a given point by the previous formula $s_\lambda(\vec{x}) = \sum_{T \in SSYT(\lambda,n)} \cdots$ has complexity $\sim \exp(|\lambda| \ln n)$, while this formula has just complexity $\sim n^3$

This hints towards the fact that these polynomials may be "partition functions of free-fermionic models"...

Many beautiful facts about Schur Functions

❷ Schur polynomials are symmetric (seen via the Bender–Knuth involution), and homogeneous of degree $|\lambda|$.

Triangularity w.r.t. the monomial basis, in dominance order, implies that they form a basis of the algebras of symmetric polynomials

$$\Lambda_{n,\mathbb{K}}(\vec{x}) = \left[\begin{matrix} \text{algebra of symm.} \\ \text{polyn. in } x_1, \dots, x_n \end{matrix} \right] = span_{\mathbb{K}} \big(s_\lambda(x_1, \dots, x_n) \big)_{\lambda \,:\, \ell(\lambda) \leq n}$$

❸ Call $\begin{cases} e_k(\vec{x}) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \\ h_k(\vec{x}) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k} \end{cases}$

We can write $s_\lambda(x_1, \dots, x_n)$ as polynomials in the $e_k(x_1, \dots, x_n)$'s, or the $h_k(x_1, \dots, x_n)$'s. As soon as $n \geq \ell(\lambda)$, these expressions $s_\lambda = P_\lambda(\{e_k\}) = Q_\lambda(\{h_k\})$ stabilise (i.e., become independent of n)

This allows to define Schur functions, well-defined also for infinite alphabets.

Many beautiful facts about Schur Functions

❹ The expressions of s_λ in terms of e_k's and h_k's are given by the Jacobi–Trudi and dual Jacobi–Trudi formulas

$$\begin{aligned} s_\lambda &= \det\left(\left(h_{\lambda_i+j-i}\right)_{i,j=1,\dots,\ell(\lambda)}\right) \quad (JT) \\ &= \det\left(\left(e_{\lambda'_i+j-i}\right)_{i,j=1,\dots,\lambda_1}\right) \quad (dJT) \end{aligned}$$

Further generalisations: Giambelli identities, Lascoux–Pragacz "ribbon decomposition" formula, and Hamel–Goulden formulas.

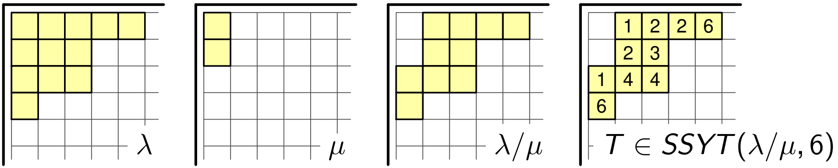
✍ Jang Soo Kim and Meesue Yoo, *Generalized Schur Function Determinants Using the Bazin Identity*, SIAM J. Discrete Math., **35** 2021

❺ One useful class of infinite alphabets is induced by the ('supersymmetry') ω-involution, that exchanges e_k's and h_k's. That is, we have Schur functions (in fact, polynomials) depending on a 'finite supersymmetric alphabet', $s_\lambda(x_1,\dots,x_n|y_1,\dots,y_m)$
It turns out that $s_\lambda(x_1,\dots,x_n|y_1,\dots,y_m) = s_{\lambda'}(y_1,\dots,y_m|x_1,\dots,x_n)$

Many beautiful facts about Schur Functions

❻ The skew Schur polynomials defined as

$$s_{\lambda/\mu}(x_1, \dots, x_n) = \sum_{T \in SSYT(\lambda/\mu, n)} \prod_{i=1}^{n} x_i^{\#\{i \in T\}}$$



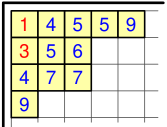
coincide with the ones induced by $\langle h | f_{\lambda/\mu} \rangle = \langle h\, g^{\mu} | f_\lambda \rangle \; \forall h$
in the scalar product $\langle \cdot | \cdot \rangle$ such that the Schur basis is self-dual

$$\langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}$$

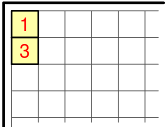
Many beautiful facts about Schur Functions

It follows that

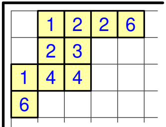
$$s_\lambda(x_1,\dots,x_n,x_{n+1},\dots,x_{n+m}) = \sum_\mu s_\mu(x_1,\dots,x_n)\,s_{\lambda/\mu}(x_{n+1},\dots,x_{n+m})$$



$T_1 \in SSYT(\lambda, n+m)$



$T_2 \in SSYT(\mu, n)$



$T_3 \in SSYT(\lambda/\mu, m)$

(this is evident for finite alphabets, but the formula $s_\lambda(\vec{x}\cup\vec{y}) = \sum_\mu s_\mu(\vec{x})s_{\lambda/\mu}(\vec{y})$ holds also for infinite alphabets)

❼ The structure constants $c^\lambda_{\mu\nu}$ of the algebra $\Lambda = span_{\mathbb{K}}\big(s_\lambda(\vec{x})\big)_\lambda$ are non-negative integers known as Littlewood–Richardson coefficients

$$s_\mu(\vec{x})s_\nu(\vec{x}) = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda(\vec{x}) \qquad c^\lambda_{\mu\nu} \in \mathbb{N}$$

Many beautiful facts about Schur Functions

What we said above implies that the three problems

$$\begin{cases} s_\mu(\vec{x})s_\nu(\vec{x}) &= \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda(\vec{x}) \\ s_{\lambda/\mu}(\vec{x}) &= \sum_\nu c^\lambda_{\mu\nu}\, s_\nu(\vec{x}) \\ s_\lambda(\vec{x},\vec{y}) &= \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(\vec{x})s_\nu(\vec{y}) \end{cases}$$

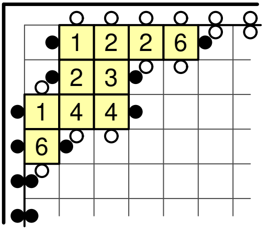
are all solved by the same Littlewood–Richardson coefficients

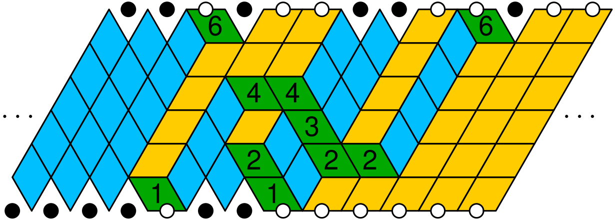
Many other interesting basis of symmetric functions (Hall–Littlewood, Grothendieck, Borodin's 2014 'symmetric rational functions', ...) generalise the Schur case in some sense, but, if we insist on keeping the Hall ($\langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}$) scalar product, self-duality is not present in general

Representation of Schur polynomials as Vertex Models

(Skew-)Schur polynomials can be represented as partition functions of tiling models, namely as free-fermionic $\mathcal{U}_q(\widehat{sl}_2)$ Yang–Baxter integrable Vertex Models with homogeneous vertical spectral parameters, the horizontal ones determine the alphabet

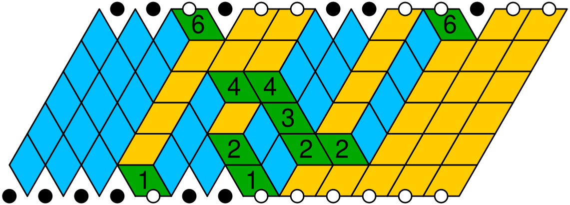
$s_{\lambda/\mu}(x_1, \ldots, x_n)$ is described by an infinite horizontal strip, of height n, where all non-trivial tiles occur within a width $\lambda_1 + \ell(\lambda)$
The partitions λ and μ fix the top and bottom boundary conditions





Representation of Schur polynomials as Vertex Models

Lozenge tilings are nice, but, in order to describe in a symmetric way the 'supersymmetric' (skew-)Schur functions, we shall rather shear the triangular lattice into the square lattice



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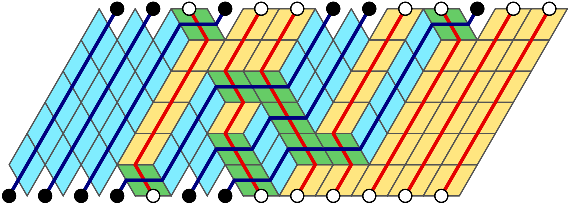
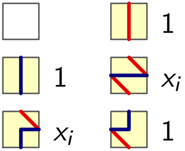
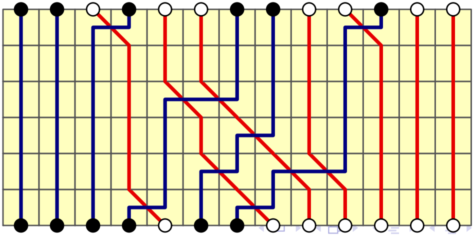


$T(x_i)$:





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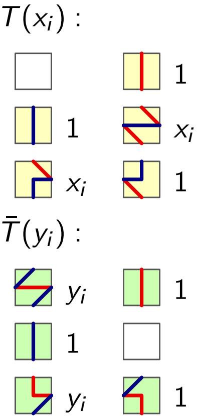
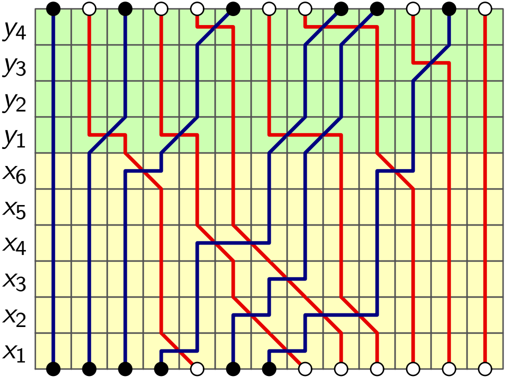


A supersymmetric skew Schur function:



$$s(x_1, \ldots, x_6 | y_1, \ldots, y_4) = \cdots + x_1^2 x_2^3 x_3 x_4^2 x_6^2 \; y_1^4 y_3 y_4^3 + \cdots$$

Representation of Schur polynomials as Vertex Models

The operators $T(x)$ and $\bar{T}(y)$ are 'transfer matrices'.
They act on the Hilbert space indexed by integer partitions, as

$$\langle \mu | T(x) | \lambda \rangle = \begin{cases} x^{|\lambda/\mu|} & \mu \preceq \lambda\,; \lambda/\mu \text{ is a 'horizontal strip' (no } \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\text{)} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \mu | \bar{T}(y) | \lambda \rangle = \begin{cases} y^{|\lambda/\mu|} & \mu \preceq \lambda\,; \lambda/\mu \text{ is a 'vertical strip' (no } \begin{array}{|c|c|}\hline & \\ \hline\end{array}\text{)} \\ 0 & \text{otherwise} \end{cases}$$

$$s_{\lambda/\mu}(x_1, \dots, x_n | y_1, \dots, y_m) = \langle \mu | T(x_1) \cdots T(x_n) \bar{T}(y_1) \cdots \bar{T}(y_m) | \lambda \rangle$$

In particular $\langle \mu | T(x) | \lambda \rangle = \langle \mu' | \bar{T}(x) | \lambda' \rangle$

Of course, by definition of transpose operator,
$\langle \mu | T^{+}(x) | \lambda \rangle = \langle \lambda | T(x) | \mu \rangle$ and $\langle \mu | \bar{T}^{+}(x) | \lambda \rangle = \langle \lambda | \bar{T}(x) | \mu \rangle$

Operators $T(x)$, $\bar{T}(y)$ and their transpose form an interesting algebra

Schur processes

Operators $T(x)$, $\bar{T}(y)$ and their transpose form an interesting algebra

$$T(x)|\varnothing\rangle = \bar{T}(x)|\varnothing\rangle = |\varnothing\rangle \qquad \langle\varnothing|T^+(x) = \langle\varnothing|\bar{T}^+(x) = \langle\varnothing|$$

$$\left[T(x), T(y)\right] = \left[\bar{T}(x), \bar{T}(y)\right] = \left[T(x), \bar{T}(y)\right] = 0$$

$$T(x)T^+(y) = \frac{1}{1-xy}T^+(y)T(x) \qquad \bar{T}(x)\bar{T}^+(y) = \frac{1}{1-xy}\bar{T}^+(y)\bar{T}(x)$$

$$T(x)\bar{T}^+(y) = (1+xy)\bar{T}^+(y)T(x) \qquad \bar{T}(x)T^+(y) = (1+xy)T^+(y)\bar{T}(x)$$

This is proven through the Yang–Baxter equation for the corresponding 'free-fermionic 5-Vertex Model with electric fields'.

Partition functions and correlation functions of several dimer models (lozenges, domino tilings,...) can be calculated in this way

A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16** (2003)

Littlewood–Richardson coefficients as a Vertex Model

Remarkably, also the Littlewood–Richardson coefficients are described by an integrable Vertex Model, this time of square-triangle tilings, with underlying $\mathcal{U}_q(\widehat{sl}_3)$ symmetry.

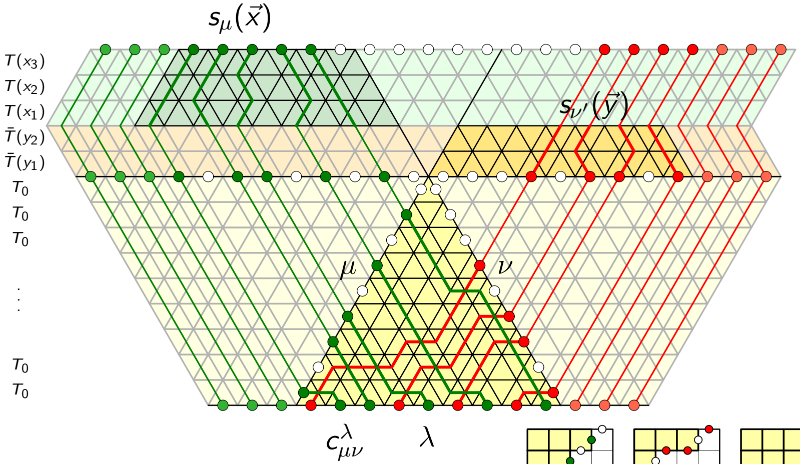
A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003); P. Zinn-Justin, *Littlewood–Richardson Coefficients and Integrable Tilings*, EJC **16** (2009)

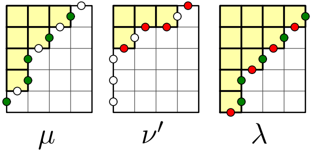
The key idea is to express the two sides of the coproduct identity $s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(\vec{x}) s_{\nu'}(\vec{y})$ as partition functions in a rank-2 model (i.e., with particles of three colours)

The three Schur terms, $s_\lambda(\vec{x}|\vec{y})$, $s_\mu(\vec{x})$ and $s_{\nu'}(\vec{y})$, are realised within the three possible embeddings of $\widehat{sl}_2$ in $\widehat{sl}_3$ that is, the three choices of two colours among three

The identity is a consequence of commutation of transfer matrices, which in turns comes from the Yang–Baxter equation of the rank-2 model

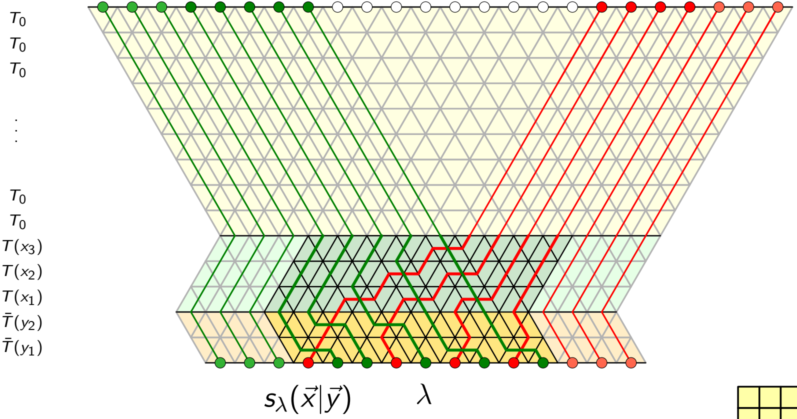
Littlewood–Richardson coefficients as a Vertex Model

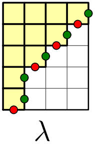




$$s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(\vec{x}) s_{\nu'}(\vec{y})$$

Littlewood–Richardson coefficients as a Vertex Model





$$s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu,\nu} c^{\lambda}_{\mu\nu}\, s_\mu(\vec{x}) s_{\nu'}(\vec{y})$$

Part II

All you'll need to know today about the Razumov–Stroganov conjecture

The many Razumov–Stroganov conjectures

There exists a whole class of Razumov–Stroganov conjectures

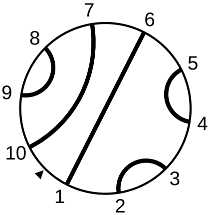
A.V. Razumov and Yu.G. Stroganov, *Combinatorial nature of ground state vector of $O(1)$ loop model*, Theor. Math. Phys. **138** (2004); —, *$O(1)$ loop model with different boundary conditions and symmetry classes of alternating-sign matrices*, Theor. Math. Phys. **142** (2005); J. de Gier, *Loops, matchings and alternating-sign matrices*, Discr. Math. **298** (2005); S. Mitra, B. Nienhuis, J. de Gier and M.T. Batchelor, *Exact expressions for correlations in the ground state of the dense $O(1)$ loop model*, JSTAT (2004); J. de Gier and V. Rittenberg, *Refined Razumov–Stroganov conjectures for open boundaries*, JSTAT (2004); Ph. Duchon, *On the link pattern distribution of quater-turn symmetric FPL configurations*, Proc. of FPSAC 2008

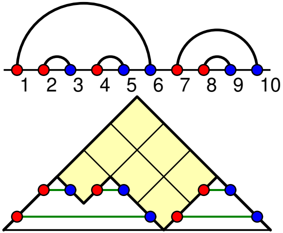
Formulated in the early 2000's, they relate the probabilities of some connectivity patterns in two different integrable models: the $O(1)$ Dense Loop Model and the Fully-Packed Loop Model

A nice fact is that they can be formulated in purely combinatorial way, despite the fact that they are related to the "physics" of the XXZ Quantum Spin Chain of the 6-Vertex Model

Link patterns

A link pattern $\pi \in LP(2n)$ is a pairing of $\{1, 2, \dots, 2n\}$ having no pairs (a, c), (b, d) such that $a < b < c < d$ (i.e., the drawing consists of n non-crossing arcs).





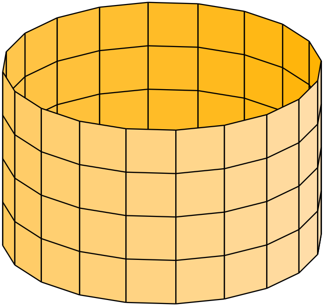
They are $C_n = \frac{1}{n+1}\binom{2n}{n}$ (the n-th *Catalan number*), and are in easy bijection with Dyck Paths of length $2n$ that is, integer partitions $\lambda \preceq \delta_n$

$$\pi = \big((1,6),(2,3),(4,5),(7,10),(8,9)\big) \qquad \lambda(\pi) = (3,3,1) \preceq \delta_5$$

$O(1)$ Dense Loop Model / XXZ $\Delta = -\frac{1}{2}$ spin chain

Consider dense loop configurations on a semi-infinite cylinder
i.e. tilings of $\{1, \dots, 2n\} \times \mathbb{N}$ with the two tiles , 
(with the uniform measure)

Link patterns are naturally associated to these configurations
(despite the fact that they are infinite!)



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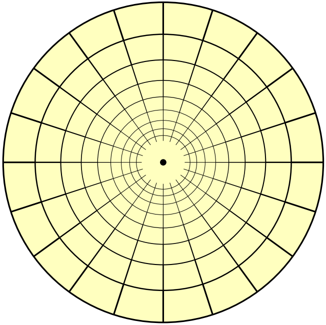

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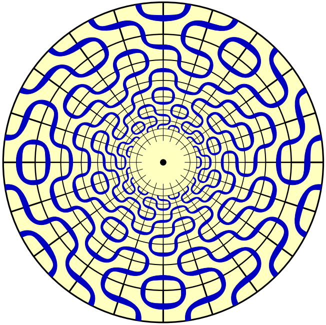
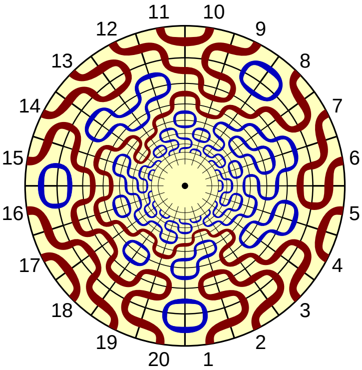
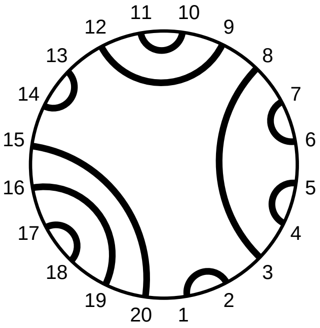


$O(1)$ Dense Loop Model / XXZ $\Delta = -\frac{1}{2}$ spin chain

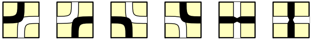
Consider dense loop configurations on a semi-infinite cylinder
i.e. tilings of $\{1, \dots, 2n\} \times \mathbb{N}$ with the two tiles  , 
(with the uniform measure)

Link patterns are naturally associated to these configurations
(despite the fact that they are infinite!)

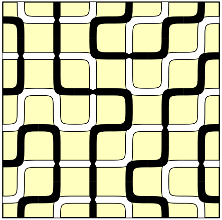




Fully-Packed Loops

Fully-Packed Loop configurations are tilings of the $n \times n$ square using the six tiles  and with black/white alternating boundary conditions

Again, a link pattern π is naturally associated, according to the connectivities among the black terminations on the boundary



Note that, by now, we ignore the link pattern associated to white, and the potential presence of loops

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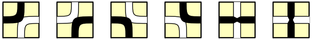
Fully-Packed Loop configurations are tilings of the $n \times n$ square using the six tiles  and with black/white alternating boundary conditions

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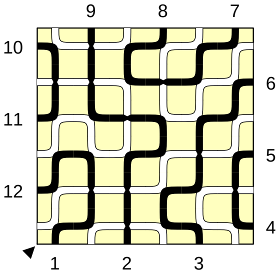

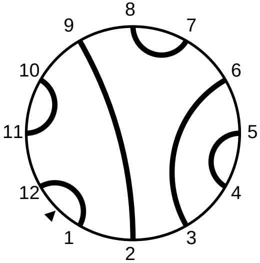

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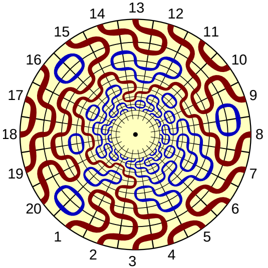
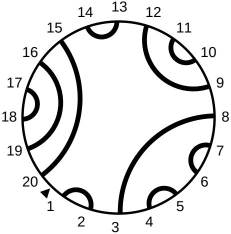
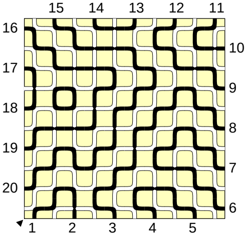
The dihedral Razumov–Stroganov correspondence







$\tilde{\Psi}_n(\pi)$: probability of π
in the $O(1)$ Dense Loop Model
in the $\{1, \dots, 2n\} \times \mathbb{N}$ cylinder

$\Psi_n(\pi)$: probability of π
for FPL with uniform measure
in the $n \times n$ square

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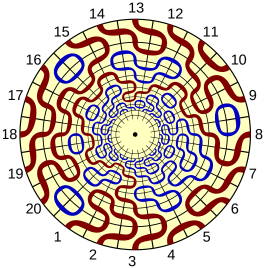

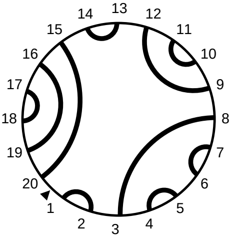

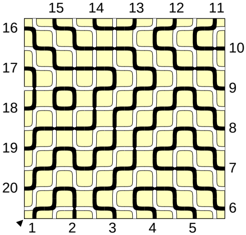

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Razumov–Stroganov correspondence

(conjecture: Razumov and Stroganov, 2001a for the $n \times n$ square;
proof: AS and Cantini, 2010, for all the 'dihedral domains')

$$\tilde{\Psi}_n(\pi) = \Psi_n(\pi)$$

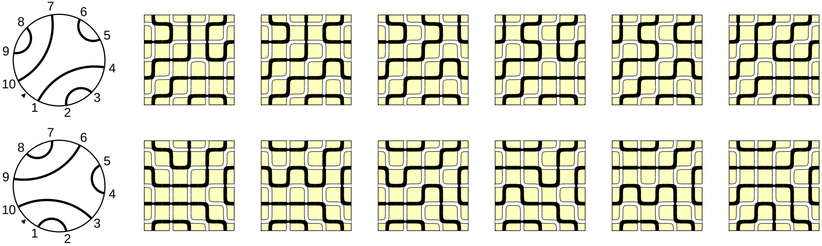
Dihedral symmetry of FPL

A corollary of the Razumov–Stroganov correspondence...
(...that was known *before* the Razumov–Stroganov conjecture)

call R the operator that rotates a link pattern by one position

Dihedral symmetry of FPL (proof: Wieland, 2000)

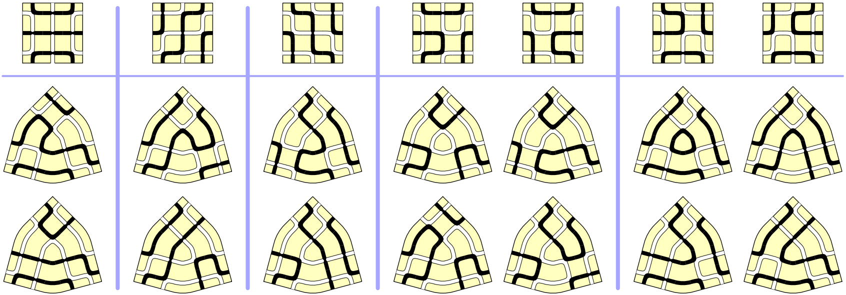
$$\Psi_n(\pi) = \Psi_n(R\pi)$$



Domains with dihedral Razumov–Stroganov correspondence

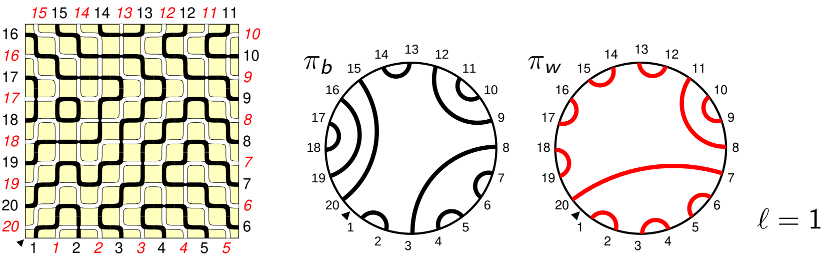
In the case of the dihedral Razumov–Stroganov correspondence, Wieland gyration (and its generalisations) has been a crucial ingredient.

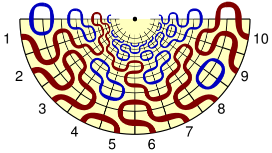
Not surprisingly, understanding the most general family of domains for which the correspondence holds has been inspiring

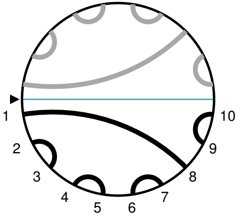


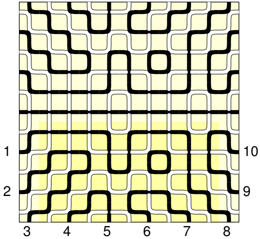
Remark: What is natural to consider in Wieland gyration lemma is the triple (π_b, π_w, ℓ) for the black and white link patterns, and the total number of loops (black+white)

However, we have no candidate replacing the $O(1)$ Dense Loop Model in a black+white version of the Razumov–Stroganov conjecture! (. . . no, the Rotor Model doesn't seem to work. . .)









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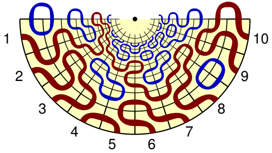
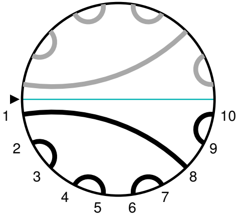
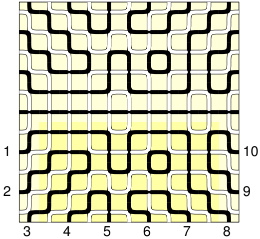
for vertically-symmetric FPL
with uniform measure in the
$(2n+1) \times (2n+1)$ square

A Vertical Razumov–Stroganov Conjecture







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Vertical Razumov–Stroganov conjecture
(Razumov and Stroganov, 2001b for the square of side $2n+1$)

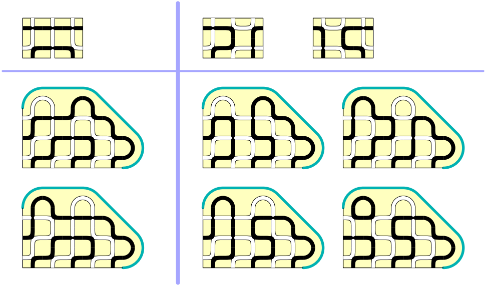
$$\tilde{\Psi}_n(\pi) = \Psi_n(\pi)$$

Domains with Vertical Razumov–Stroganov correspondence

The Vertical Razumov–Stroganov conjectures are a whole second family
They involve FPL with some version of reflecting wall and the $O(1)$ Dense Loop Model on a strip with a boundary.

Our proof methods do not seem to work for any of the Vertical Razumov–Stroganov conjectures, which are all open at present.

But at least we think we know the precise list of domains with Vertical RS



$3 + x + 7y + 2xy + 4y^2 + xy^2$ $\quad$ $6 + 2x + 14y + 4xy + 8y^2 + 2xy^2$

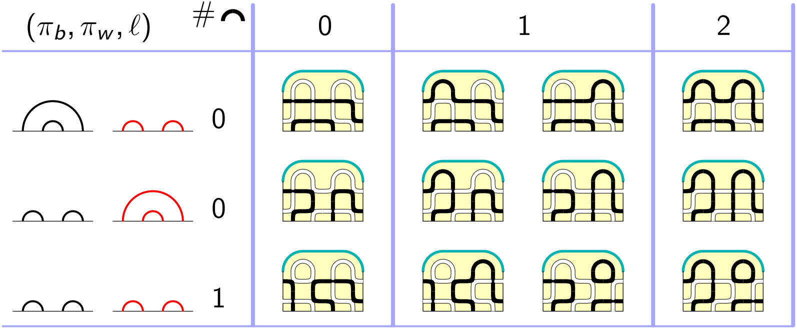
Part III

Smash together two failures
and see what happens...

Looking at UASM more closely

We shall "smash together the two failures above: ❶ we haven't proven any flavour of the Vertical Razumov–Stroganov conjectures; ❷ we never devised any flavour of Razumov–Stroganov conjectures, not even dihedral, involving the triple enumeration $\Psi_n(\pi_b, \pi_w, \ell)$

We will look more closely at the full list of FPL's in the simplest instance of Vertical RS, that is U-turn ASM's (UASM).



The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$

Let us call $\Psi_n(\pi_b, \pi_w, \tau, y)$ the generating function of UASM's at size n, with black/white link patterns π_b and π_w, and weight $\tau^\ell y^{\#\cap}$

Known: $Z_n(y) = \sum_{\pi_b,\pi_w} \Psi_n(\pi_b, \pi_w, 1, y)$ has an overall factor $(1+y)^n$

G. Kuperberg, *Symmetry classes of alternating-sign matrices under one roof*, Ann. of Math. **156** (2002)

Luigi Cantini and myself conjectured, also long ago, that this factorisation holds for the RS components

$$\Psi_n(\pi_b, y) = \sum_{\pi_w} \Psi_n(\pi_b, \pi_w, 1, y) = (1+y)^n \, \tilde{\Psi}_n(\pi_b)$$

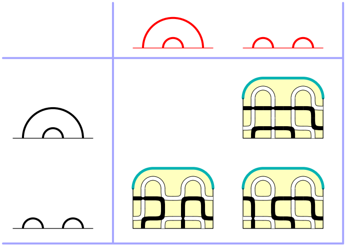
The new numerical investigation leads to the first of our "new conjectures":

Conjecture 1

$$\Psi_n(\pi_b, \pi_w, \tau, y) = (1+y)^n \, \Psi_{\pi_b,\pi_w}(\tau) \qquad \forall \; n, \tau, \pi_b, \pi_w$$

(only proven: $(1+y)^2$ divides $\Psi_n(\pi_b, \pi_w, \tau, y)$ for $n \geq 2$)

The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$



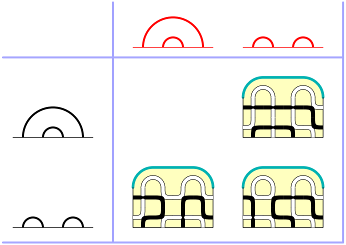
These sets of polynomials are better visualised as tables π_b vs. π_w...

$\longrightarrow$

	0	1
	1	τ

	0	0	0	0	1
	0	0	1	1	2τ
	0	1	τ	τ	τ^2+1
	0	1	τ	τ	τ^2+1
	1	2τ	τ^2+1	τ^2+1	$\tau^3+3\tau$

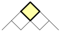
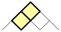
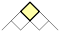
The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$



. . . or, equivalently, as tables λ vs. ρ with $\lambda = \lambda(\pi_b)$ and $\rho = \lambda(\pi_w)$

$\longrightarrow$

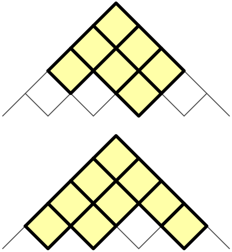
		
	0	1
	1	τ

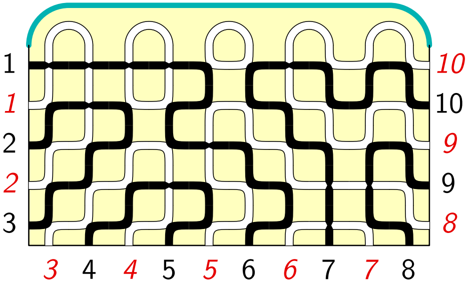
					
	0	0	0	0	1
	0	0	1	1	2τ
	0	1	τ	τ	τ^2+1
	0	1	τ	τ	τ^2+1
	1	2τ	τ^2+1	τ^2+1	$\tau^3+3\tau$

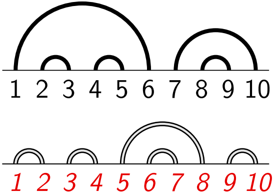
	0	1	2	2	3	3	3	4	4	4	5	5	5	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	1	1	1	3τ
2	0	0	0	0	0	0	1	1	1	2τ	2τ	2τ	$2+3\tau^2$
2	0	0	0	0	0	0	0	1	1	1	2τ	2τ	2τ	$2+3\tau^2$
3	0	0	0	0	0	0	1	τ	τ	τ	$1+\tau^2$	$1+\tau^2$	$1+\tau^2$	$\tau(3+\tau^2)$
3	0	0	0	0	0	0	1	τ	τ	τ	$1+\tau^2$	$1+\tau^2$	$1+\tau^2$	$\tau(3+\tau^2)$
3	0	0	0	0	1	1	2	4τ	3τ	4τ	$3+4\tau^2$	$2+4\tau^2$	$3+4\tau^2$	$\tau(10+5\tau^2)$
4	0	0	1	1	τ	τ	4τ	$2+3\tau^2$	$1+2\tau^2$	$2+3\tau^2$	$\tau(5+2\tau^2)$	$\tau(4+2\tau^2)$	$\tau(5+2\tau^2)$	$4+9\tau^2+2\tau^4$
4	0	0	1	1	τ	τ	3τ	$1+2\tau^2$	τ^2	$1+2\tau^2$	$\tau(2+\tau^2)$	$\tau(2+\tau^2)$	$\tau(2+\tau^2)$	$2+4\tau^2+\tau^4$
4	0	0	1	1	τ	τ	4τ	$2+3\tau^2$	$1+2\tau^2$	$2+3\tau^2$	$\tau(5+2\tau^2)$	$\tau(4+2\tau^2)$	$\tau(5+2\tau^2)$	$4+9\tau^2+2\tau^4$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$3+4\tau^2$	$\tau(5+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(5+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$2+4\tau^2$	$\tau(4+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(4+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+4\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$3+4\tau^2$	$\tau(5+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(5+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
6	1	3τ	$2+3\tau^2$	$2+3\tau^2$	$\tau(3+\tau^2)$	$\tau(3+\tau^2)$	$\tau(10+5\tau^2)$	$4+9\tau^2+2\tau^4$	$2+4\tau^2+\tau^4$	$4+9\tau^2+2\tau^4$	$\tau(10+7\tau^2+\tau^4)$	$\tau(10+7\tau^2+\tau^4)$	$\tau(10+7\tau^2+\tau^4)$	$8+24\tau^2+9\tau^4+\tau^6$

A large example:

$\Psi_{\ ,\ }(\tau) = \cdots + \tau^2 + \cdots$







$\#\{\bigcirc\} + \#\{\circledcirc\} = 2$

The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$

In the following, $\Psi_{\lambda,\rho}(\tau) \equiv \Psi_{\pi_b,\pi_w}(\tau)$ with abuse of notation

Conjecture 2

$$\deg\left(\Psi_{\lambda,\rho}(\tau)\right) = |\lambda| + |\rho| - |\delta_n|$$

In particular, $\Psi_{\lambda,\rho}(\tau) = 0$ if $|\lambda| + |\rho| < \binom{n}{2}$.

Conjecture 3

The $\Psi_{\lambda,\rho}(\tau)$'s are polynomials of defined parity.

Conjecture 4

The table has the $\mathbb{Z}_2^3$ symmetry: ❶ $\Psi_{\lambda,\rho}(\tau) = \Psi_{\rho,\lambda}(\tau)$;
❷ $\Psi_{\lambda,\rho}(\tau) = \Psi_{\rho',\lambda'}(\tau)$; ❸ $\Psi_{\lambda,\rho}(\tau) = \Psi_{\lambda,\rho'}(\tau)$.

❶: easily proven (Wieland + swap b/w);

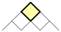
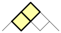
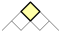
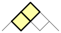
❷: easily corollary of Conjecture 1 (vertical reflection + swap b/w);

❸: rather mysterious.

Conjecture 5

The entries s.t. $|\lambda| + |\rho| = |\delta_n|$ are the Littlewood–Richardson coefficients $\Psi_{\lambda,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$.

		
	0	1
	1	τ

					
	0	0	0	0	1
	0	0	1	1	2τ
	0	1	τ	τ	τ^2+1
	0	1	τ	τ	τ^2+1
	1	2τ	τ^2+1	τ^2+1	$\tau^3+3\tau$

	0	1	2	2	3	3	3	4	4	4	5	5	5	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	1	1	1	3τ
2	0	0	0	0	0	0	1	1	1	2τ	2τ	2τ	$2+3\tau^2$
2	0	0	0	0	0	0	0	1	1	1	2τ	2τ	2τ	$2+3\tau^2$
3	0	0	0	0	0	0	1	τ	τ	τ	$1+\tau^2$	$1+\tau^2$	$1+\tau^2$	$\tau(3+\tau^2)$
3	0	0	0	0	0	0	1	τ	τ	τ	$1+\tau^2$	$1+\tau^2$	$1+\tau^2$	$\tau(3+\tau^2)$
3	0	0	0	0	1	1	2	4τ	3τ	4τ	$3+4\tau^2$	$2+4\tau^2$	$3+4\tau^2$	$\tau(10+5\tau^2)$
4	0	0	1	1	τ	τ	4τ	$2+3\tau^2$	$1+2\tau^2$	$2+3\tau^2$	$\tau(5+2\tau^2)$	$\tau(4+2\tau^2)$	$\tau(5+2\tau^2)$	$4+9\tau^2+2\tau^4$
4	0	0	1	1	τ	τ	3τ	$1+2\tau^2$	τ^2	$1+2\tau^2$	$\tau(2+\tau^2)$	$\tau(2+\tau^2)$	$\tau(2+\tau^2)$	$2+4\tau^2+\tau^4$
4	0	0	1	1	τ	τ	4τ	$2+3\tau^2$	$1+2\tau^2$	$2+3\tau^2$	$\tau(5+2\tau^2)$	$\tau(4+2\tau^2)$	$\tau(5+2\tau^2)$	$4+9\tau^2+2\tau^4$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$3+4\tau^2$	$\tau(5+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(5+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$2+4\tau^2$	$\tau(4+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(4+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+4\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$3+4\tau^2$	$\tau(5+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(5+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
6	1	3τ	$2+3\tau^2$	$2+3\tau^2$	$\tau(3+\tau^2)$	$\tau(3+\tau^2)$	$\tau(10+5\tau^2)$	$4+9\tau^2+2\tau^4$	$2+4\tau^2+\tau^4$	$4+9\tau^2+2\tau^4$	$\tau(10+7\tau^2+\tau^4)$	$\tau(10+7\tau^2+\tau^4)$	$\tau(10+7\tau^2+\tau^4)$	$8+24\tau^2+9\tau^4+\tau^6$

Conjecture 4

❶ $\Psi_{\lambda,\rho} = \Psi_{\rho,\lambda}$; ❷ $\Psi_{\lambda,\rho} = \Psi_{\rho',\lambda'}$; ❸ $\Psi_{\lambda,\rho} = \Psi_{\lambda,\rho'}$.

Conjecture 5

When $|\lambda| + |\rho| = |\delta_n|$ we have $\Psi_{\lambda,\rho} = c_{\lambda\rho}^{\delta_n}$ (Littlewood–Richardson)

Are these two conjectures even compatible?
Indeed, ❶ and ❷ are simple symmetries of LR coeffs
(with ❷ using the fact $\delta_n = (\delta_n)'$),
but why on Earth should we have $c_{\mu\nu}^{\lambda} = c_{\mu\nu'}^{\lambda}$?

Call $\mathcal{T} = \{\delta_n\}_{n\geq 1}$ and $\mathcal{M} = \{\lambda \,|\, c_{\mu\nu}^{\lambda} = c_{\mu\nu'}^{\lambda} \ \forall \mu, \nu\}$

Lemma

$$\mathcal{T} = \mathcal{M}$$

A property of the Littlewood–Richardson coefficients

Lemma

$\mathcal{T} = \{\delta_n\}_{n\geq 1}$ and $\mathcal{M} = \{\lambda \,|\, c^{\lambda}_{\mu\nu} = c^{\lambda}_{\mu\nu'}\ \forall \mu,\nu\}$ coincide.

<u>*Proof.*</u> The implication $\lambda \notin \mathcal{T} \Rightarrow \lambda \notin \mathcal{M}$ is easy
(recognise that $\lambda \notin \mathcal{T} \Leftrightarrow \lambda = [\alpha \circ \circ \bullet \beta]$ or $\lambda = [\alpha \circ \bullet \bullet \beta]$,
call $\mu = [\alpha \bullet \circ \circ \beta]$ or $\mu = [\alpha \bullet \bullet \circ \beta]$, and evaluate $c^{\lambda}_{\mu\,(2)}$, $c^{\lambda}_{\mu\,(1,1)}$)

The implication $\lambda \in \mathcal{T} \Rightarrow \lambda \in \mathcal{M}$ is interesting.
The crucial observation is that $T(x)|\delta_n\rangle = \bar{T}(x)|\delta_n\rangle$

that, using the commutation of T's and $\bar{T}$'s, implies on supersymmetric skew Schur functions $s_{\delta_n/\mu}(\vec{x}|\vec{y}) = s_{\delta_n/\mu}(\vec{y}|\vec{x})$

by the coproduct definition of LR's:
$\sum_\nu c^{\delta_n}_{\mu\nu}\, s_\nu(\vec{x}|\vec{y}) = s_{\delta_n/\mu}(\vec{x}|\vec{y}) = s_{\delta_n/\mu}(\vec{y}|\vec{x}) = \sum_\nu c^{\delta_n}_{\mu\nu}\, s_\nu(\vec{y}|\vec{x}) = \sum_\nu c^{\delta_n}_{\mu\nu}\, s_{\nu'}(\vec{x}|\vec{y}) = \sum_\nu c^{\delta_n}_{\mu\nu'}\, s_\nu(\vec{x}|\vec{y})$. By the linear independence of Schur functions $c^{\delta_n}_{\mu\nu} = c^{\delta_n}_{\mu\nu'}$ $\square$

A mystery plot

We have mentioned that there exists several deformations of Schur functions (Grothendiek, Hall–Littlewood, ...), many of them allow for a representation as an integrable Vertex Model, and even some representation *à la* Zinn-Justin of the corresponding structure constants (i.e., with the trick "sl_2 embeds into sl_3 in three ways").

✍ M. Wheeler and P. Zinn-Justin, *Littlewood–Richardson coefficients for Grothendieck polynomials from integrability*, J. für die Reine und Angewandte Math. **757** (2017); — *Hall polynomials, inverse Kostka polynomials and puzzles*, JCT-A **159** (2018).

Maybe there exists a basis/dual-basis of symmetric functions $\{f_\lambda\}$, $\{g_\lambda\}$, which are a τ-deformation of Schur fns., such that $\Psi_{\lambda,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$ or $\Psi_{\lambda,\rho}(\tau) = d^{\lambda\rho}_{\delta_n}$, for all pairs $\lambda, \rho \preceq \delta_n$?

Maybe we will have a result of the form $\Psi_{\lambda\rho}(\tau) = \sum_{P \in \mathcal{P}_{\lambda,\rho,\delta_n}} \tau^{x(P)}$ with $\mathcal{P}_{\lambda,\rho,\delta_n}$ some variant of Knutson–Tao puzzles, and $x(P)$ the number of tiles of some kind?

A mystery plot: collecting the hints

We shall suppose that these new functions exist, are still described by an integrable Vertex Model, and are given by a 'minimal' deformation of $T(x)$ and $\bar{T}(y)$ operators.

Which properties shall we reproduce?

1. The degree condition (and its corollary on which $\Psi_{\lambda,\rho}$ do vanish)
2. Polynomials of defined parity
3. The mysterious extra symmetry $\Psi_{\lambda,\rho} = \Psi_{\lambda,\rho'}$
4. The new T and $\bar{T}$ must still constitute a commuting family
5. $\langle \mu | T(x) | \lambda \rangle$ well-defined on infinite strings $\cdots \bullet \bullet \bullet [\cdots] \circ \circ \circ \cdots$

Which generalisations we do not want?

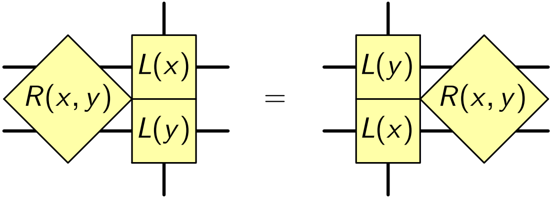
1. We do not "change δ_n" (e.g., try $\Psi_{\lambda,\rho}(\tau) = \sum_{\theta \succeq \delta_n} c^{\theta}_{\lambda\rho} \tau^{|\theta/\delta_n|}$)
2. We only investigate Vertex Models with "spin $\frac{1}{2}$" horizontal and vertical spaces

The reason is that

we want our proof of $c^{\delta_n}_{\lambda\rho} = c^{\delta_n}_{\lambda\rho'}$ to extend to $\Psi_{\lambda,\rho}(\tau)$ almost verbatim

5VM and 6VM $RLL = LLR$ relations

The standard technique from Integrable Systems is to construct a $RLL = LLR$ relation (a version of Yang–Baxter when the spaces are not all equal), that is, for L the tile-weights appearing in the transfer matrices T and $\bar{T}$, devise a matrix R such that



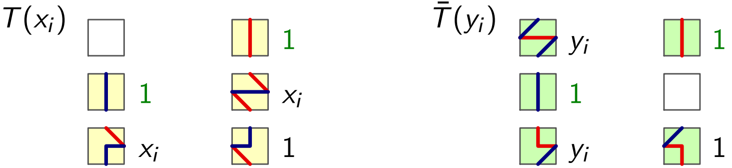
5VM and 6VM $RLL = LLR$ relations

For the (non–free-fermionic) 5-Vertex Model, this can be done unambiguously, once we take into account:

❶ weight well-defined on infinite strings;

❷ gauge invariance;

❸ covariance under reparametrisation;



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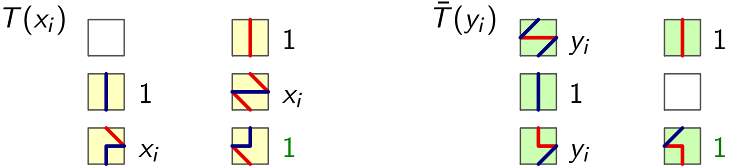

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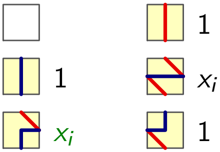
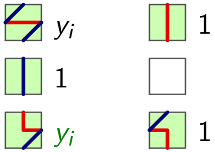
$T(x_i)$



$\bar{T}(y_i)$



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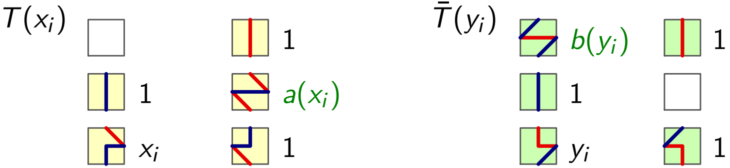
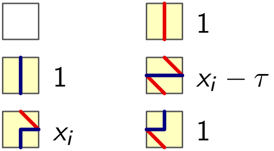


$$a(x_1) - x_1 = a(x_2) - x_2 = b(y_1) - y_1 = b(y_2) - y_2$$

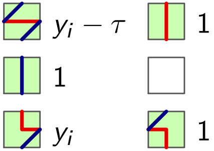
5VM and 6VM $RLL = LLR$ relations

$T(x)$ and $\bar{T}(y)$ for the (non–free-fermionic) 5-Vertex Model extension

$T^{5v}(x_i)$

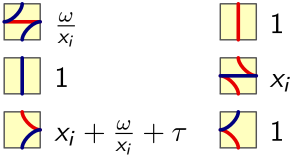


$\bar{T}^{5v}(y_i)$



A similar procedure for the (non–free-fermionic) 6-Vertex Model gives, again unambiguously

$T^{6v}(x_i)$



$$\bar{T}^{6v}(y_i) = T^{6v}(\omega/y_i)$$

Note that now the $f_{\lambda/\mu}(\vec{x})$'s are Laurent polynomials in x_i's

Non-FF 5VM and dual Canonical Grothendieck polynomials

The FF 5VM operators T and $\bar{T}$ act on integer partitions as

$$\langle \mu | T(x) | \lambda \rangle = \begin{cases} x^{|\lambda/\mu|} & \mu \preceq \lambda \,;\, \lambda/\mu \text{ hor. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \mu | \bar{T}(y) | \lambda \rangle = \begin{cases} y^{|\lambda/\mu|} & \mu \preceq \lambda \,;\, \lambda/\mu \text{ vert. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$s_{\lambda/\mu}(x_1, \dots, x_n | y_1, \dots, y_m) = \langle \mu | T(x_1) \cdots T(x_n) \bar{T}(y_1) \cdots \bar{T}(y_m) | \lambda \rangle$$

1	1	3	4	4	4
2	3				
4	6				
6					

$x_1^2 \, x_2 \, x_3^2 \, x_4^4 \, x_6^2$

Non-FF 5VM and dual Canonical Grothendieck polynomials

The non-FF 5VM operators T and $\bar{T}$ act on integer partitions as

$$\langle \mu | T^{5v}(x) | \lambda \rangle = \begin{cases} x^{K(\lambda/\mu)} (x-\tau)^{|\lambda/\mu| - K(\lambda/\mu)} & \mu \preceq \lambda \,;\, \lambda/\mu \text{ hor. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \mu | \bar{T}^{5v}(y) | \lambda \rangle = \begin{cases} y^{K(\lambda/\mu)} (y-\tau)^{|\lambda/\mu| - K(\lambda/\mu)} & \mu \preceq \lambda \,;\, \lambda/\mu \text{ vert. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\lambda/\mu}(x_1, \dots, x_n | y_1, \dots, y_m) = \langle \mu | T^{5v}(x_1) \cdots T^{5v}(x_n) \bar{T}^{5v}(y_1) \cdots \bar{T}^{5v}(y_m) | \lambda \rangle$$

1	1	3	4	4	4
2	3				
4	6				
6					

$$x_1 \, (x_1 - \tau) \, x_2 \, x_3^2 \, x_4^2 \, (x_4 - \tau)^2 \, x_6^2$$

Towards an expansion of f_λ's over Schur functions

Remark: $f_{\lambda/\mu}(\vec{x}|\vec{y})$ are homogeneous of degree $|\lambda/\mu|$ in x_i's, y_j's and τ (so that in fact only the cases $\tau = 0$ (Schur) and $\tau = 1$ do matter)

As a result, we cannot hope that the structure constants of the f_λ's are *tout court* our $\Psi_{\lambda,\rho}(\tau)$. Our best hope is that they reproduce the leading coefficient of the polynomials, i.e. the coeff. of degree $|\lambda| + |\rho| - \binom{n}{2}$ in τ.

It is easily seen that $f_\lambda = \sum_{\mu \preceq \lambda} B_\lambda{}^\mu \tau^{|\lambda/\mu|} s_\mu$, where $\preceq$ is the inclusion order, and $B_\lambda{}^\mu \in \mathbb{Z}$. Some more work shows that (call $\ell = \ell(\lambda)$)

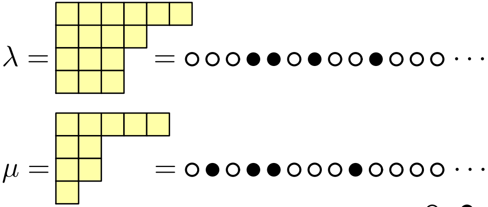
1. $B_\lambda{}^\mu = 0$ if $\ell(\lambda) \neq \ell(\mu)$
2. $\prod_{i=1}^{\ell} x_i$ divides $f_\lambda(x_1, \dots, x_\ell)$
3. If $\lambda_\ell \geq 2$, then $\prod_{i=1}^{\ell}(x_i - \tau)$ divides $f_\lambda(x_1, \dots, x_\ell)$
4. If $\lambda_\ell = 1$, then $f_\lambda(x_1, \dots, x_\ell) = x_\ell f_\rho(x_1, \dots, x_\ell) + \mathcal{O}(x_\ell^2)$, with $\mu = (\lambda_1, \dots, \lambda_{\ell-1})$

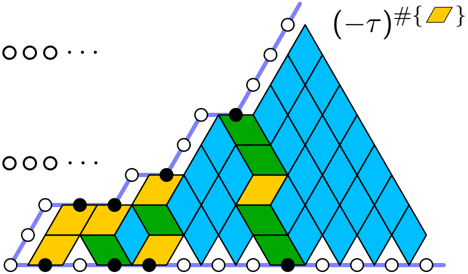
Expansion of f_λ's over Schur functions

Expansion $f_\lambda = \sum_{\mu \preceq \lambda} B_\lambda{}^\mu \tau^{|\lambda/\mu|} s_\mu$

$$B_\lambda{}^\mu = (-1)^{|\lambda/\mu|} \det \left[\binom{\lambda_i - 1}{\mu_j - j + i - 1} \right]_{i,j=1\dots,\ell}$$

$$(B^{-1})^\lambda{}_\mu = \det \left[\binom{\lambda_i - i + j - 1}{\mu_j - 1} \right]_{i,j=1\dots,\ell}$$





Determinantal formula for the f_λ's

Determinantal formula for f_λ

$$f_\lambda(x_1,\dots,x_\ell) = \frac{1}{\Delta(\vec{x})} \det \left[(x_j - \tau)^{\lambda_i - 1} x_j^{\ell-i+1} \right]_{i,j=1\dots,\ell}$$

All these results allow to identify the f_λ's with functions that have already arised in various places in the literature

A. Borodin, *On a family of symmetric rational functions*, Adv. in Math. **306** (2014) [Sect. 8.4, identified by the determinant formula]

K. Motegi and T. Scrimshaw, *Refined Dual Grothendieck Polynomials, Integrability, and the Schur Measure*, Sém. Lothar. Combin. **85** (2021) [pag. 5, with $t_i \to \tau$, identified by the $B_{\lambda\mu}$ formula]

A. Gunna and P. Zinn-Justin, *Vertex models for Canonical Grothendieck polynomials and their duals*, `arXiv:2009.13172` (Sept. 2020) [Sect. 3.4.3, identified from the branching rule]

Note that in these papers the f_λ's arise from a bosonic Vertex Model!

A nice fact

A nice corollary of the determinantal formula for the f_λ's (on a minimal-size alphabet) is a simple expression for f_{δ_n}

$$f_{\delta_n}(x_1, \dots, x_{n-1}) = \left(\prod_{i=1}^{n-1} x_i \right) \frac{\Delta(\{(x_i - \tau) x_i\})}{\Delta(\{x_i\})}$$

$$= \left(\prod_{i=1}^{n-1} x_i \right) \left(\prod_{1 \leq i < j \leq n-1} (x_i + x_j - \tau) \right)$$

This may make the evaluation of structure constants $c_{\mu\nu}^{\delta_n}$, $d_{\delta_n}^{\mu\nu}$ considerably simpler than the one of generic $c_{\mu\nu}^{\lambda}$'s and $d_{\lambda}^{\mu\nu}$'s

Troubles ahead...

So, we had hopes that the structure constants of our new basis $\{f_\lambda\}$ may be related to our UASM enumeration vectors

In particular, we hoped to reproduce the leading coefficient of the enumeration polynomials, namely

Conjecture

$$f_\mu(\vec{x}) f_\nu(\vec{x}) = \sum_\lambda c^\lambda_{\mu\nu} f_\lambda(\vec{x}) \qquad [\tau^{|\lambda|+|\rho|-\binom{n}{2}}]\Psi_{\lambda,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$$

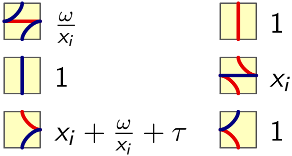
This conjecture holds up to $n = 4$, and "almost holds at $n = 5$ (holds after a 'tiny' combination of rows and columns, and still may be a bug of my program...)

Furthermore, our $\Psi_{\lambda,\rho} = \Psi_{\lambda,\rho'}$ works out of the box for the coproduct coefficients $d^{\lambda\rho}_{\delta_n}$, not for the product ones $c^{\delta_n}_{\lambda\rho}$! But the $d^{\lambda\rho}_{\delta_n}$ are non-zero 'on the wrong side' of the degree inequality, so they are not viable candidates!

The non-FF 6VM extension

Can we hope of going besides the leading coefficient?
Recall that we have proposed a higher level of generalisation, involving a 6-Vertex Model extension

$T^{6v}(x_i)$



$\bar{T}^{6v}(y_i) = T^{6v}(\omega/y_i)$

Now the $f_{\lambda/\mu}(\vec{x})$'s are Laurent polynomials in the x_i's, and are homogeneous in x_i's, τ and $\sqrt{\omega}$

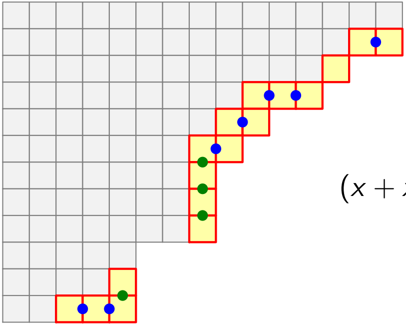
The 5VM case is obtained via the singular limits
$\lim_{\omega\to 0} T^{6v}(x) = T^{5v}(x-\tau)$ and $\lim_{\omega\to 0} T^{6v}(\omega/x) = \bar{T}^{5v}(x-\tau)$

Homogeneity allows to fix one parameter among τ and ω (say, $\omega = 1$), and have an algebra of functions where τ really matters

The 6VM operators T act on integer partitions as

$$\langle \mu | T^{6v}(x) | \lambda \rangle = \begin{cases} (x + x^{-1} + \tau)^{K(\lambda/\mu)} x^{\#\{\bullet\} - \#\{\bullet\}} & \mu \preceq \lambda\,;\, \lambda/\mu \text{ a `ribbon' (no } \boxplus\text{)} \\ 0 & \text{otherwise} \end{cases}$$

$$f^{6v}_{\lambda/\mu}(x_1, \dots, x_n) = \langle \mu | T^{6v}(x_1) \cdots T^{6v}(x_n) | \lambda \rangle$$



$$(x + x^{-1} + \tau)^4 x^{7-4}$$

The algebra of $f^{6v}_{\lambda/\mu}$'s

Now, as the f^{6v}_{λ}'s are Laurent polynomials in the x_i's, it is not even clear that they induce an algebra that coincides with $span_{\mathbb{K}}(\{f^{6v}_{\lambda}\})$ (instead of being strictly larger)

Nonetheless, we conjecture

Conjecture

$$f^{6v}_{\mu} f^{6v}_{\nu} = \sum_{\lambda} c^{\lambda}_{\mu\nu} f^{6v}_{\lambda} \qquad c^{\lambda}_{\mu\nu} \in \mathbb{N}(\tau)$$

$$f^{6v}_{\lambda/\mu} = \sum_{\nu} d^{\mu\nu}_{\lambda} f^{6v}_{\nu} \qquad d^{\mu\nu}_{\lambda} \in \mathbb{N}(\tau)$$

Interestingly, the coefficients $c^{\lambda}_{\mu\nu}$ have both the right degree and the parity property for a "$\Psi_{\lambda,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$" conjecture (although, unfortunately, this very conjecture is false)

A work in progress

This is clearly a work in progress, with many things going on...
I summarise my perspective through a few questions that I find interesting:

- How can we prove our conjectures on the $\Psi_{\lambda,\rho}(\tau)$ enumerations?
- Does the conjecture $[\tau^{|\lambda|+|\rho|-\binom{n}{2}}]\Psi_{\lambda,\rho}(\tau) = (c^{\delta_n}_{\lambda\rho})^{5v}$ stand still?
- There is any hope for a conjecture of the form $\Psi_{\lambda,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$, for some family of functions?
- There is a puzzle description of the $c^{\lambda}_{\mu\nu}$ and $d^{\mu\nu}_{\lambda}$ structure constants, for the 5VM and the 6VM generalisations of the T, $\bar{T}$ formalism?
  [this should be work in progress of A. Gunna and P. Zinn-Justin]

Thank you for listening!