Positive Geometries and Scattering Amplitudes

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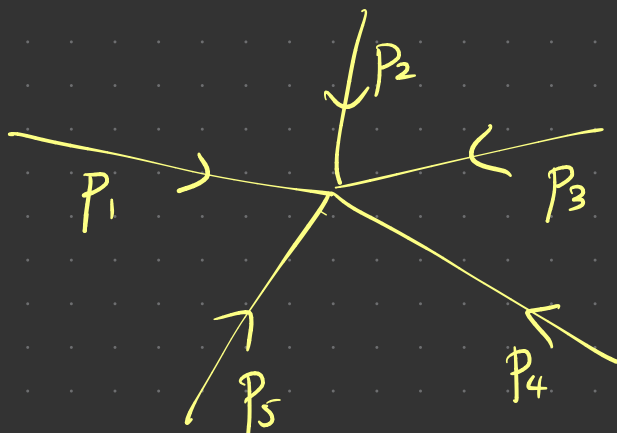
Joint work with N. Arkani-Hamed, Y. Bai

Scattering amplitudes are functions $A(p_1, p_2, \ldots)$ of the space-time momenta of n particles, and other data, that compute probabilities of various outcomes in elementary particle interactions



$p_i \in \mathbb{R}^4$

$$\mathcal{A}(p) = \sum_{\substack{\text{Feynman} \\ \text{diagrams } D}} \int_{\mathbb{R}^{4L}} \frac{P(x)}{Q(x)} \, d^{4L}x$$

$L = b_1(D)$
$= \text{"\# of loops"}$

- possible diagrams and integrand $\frac{P}{Q}$ depends on which quantum field theory is studied
- perturbative expansion $L = 0, 1, 2, \ldots$
- convergence issues
- wildly successful comparison with experiments

Parke-Taylor Formula (1986)

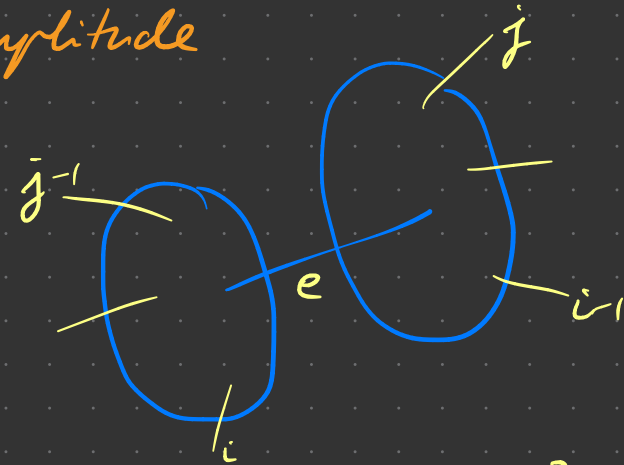
$$\mathcal{A}(1^+, 2^-, 3^-, \ldots, (n-1)^-, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

gluons (massless) in Yang-Mills at tree-level

n	4	5	6	7	8	9
# Feynman diagrams	4	25	220	2485	34300	559405

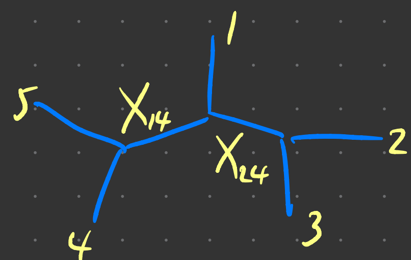
Planar, massless, tree-level ϕ^3-amplitude

$$\mathcal{A}_n = \sum_{\substack{\text{cubic cyclically}\\ \text{ordered}\\ \text{trees}}} \prod_e \frac{1}{X_e}$$



$$X_{i,j} = X_e = (p_i + \dots + p_{j-1})^2$$

$$\mathcal{A}_5 = \frac{1}{X_{13}X_{14}} + \frac{1}{X_{35}X_{13}} + \frac{1}{X_{14}X_{24}} + \frac{1}{X_{25}X_{35}} + \frac{1}{X_{24}X_{25}}$$

$\dots +$  $+ \dots$

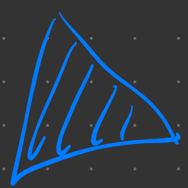
Definition

$(X, X_{\geq 0})$ **positive geometry**

X : projective, normal, d-dim variety

$X_{\geq 0}$: closed semialgebraic subset of $X(\mathbb{R})$

$X_{>0} = \mathrm{Int}(X_{\geq 0})$ open oriented d-dim manifold.



$\subset \mathbb{P}^2 = X$

$\partial X_{\geq 0} = X_{\geq 0} \setminus X_{>0}$

$\partial X := \overline{\partial X_{\geq 0}} \subset X$ Zariski closure

$= C_1 \cup C_2 \cup \dots \cup C_r \; (\cup \dots)$

$d-1$ dim components

$C_{i,\geq 0} = C_i \cap \partial X_{\geq 0}$.

$(C_i, C_{i,\geq 0})$ **boundary components**.

A <u>d-dim positive geometry</u> is

- $d=0$ $\quad X_{\geq 0} = X = pt \qquad \Omega(X, X_{\geq 0}) = \pm 1$
- $d>0$ (P1) Every boundary component $(C, C_{\geq 0})$ is a $(d-1)$-dim pos. geom.

(P2) There exists a unique nonzero rational (meromorphic) d-form $\Omega(X, X_{\geq 0})$ ← canonical form

s.t. $\quad \mathrm{Res}_C\, \Omega(X, X_{\geq 0}) = \Omega(C, C_{\geq 0}) \neq 0 \quad \forall (C, C_{\geq 0})$

and no other singularities.

$$\Omega(X, X_{\geq 0}) = \frac{df}{f} \wedge \Omega(C, C_{\geq 0}) + \dots$$

Examples

$d=1$ $\quad X=$ genus g curve

$\rightsquigarrow$ $g=0$ $\quad X=\mathbb{P}^1$

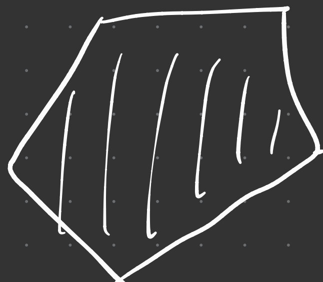
$X_{\geq 0}=$ union of closed intervals

$$\Omega(\mathbb{P}^1,[a,b]) = \frac{dx}{x-a} - \frac{dx}{x-b} = \frac{(b-a)}{(b-x)(x-a)}dx$$

$$\Omega(\mathbb{P}^1, \coprod_i [a_i,b_i]) = \sum_i \Omega(\mathbb{P}^1,[a_i,b_i])$$

$d = 2$ $\quad X = \mathbb{P}^2$ $\quad X_{\geq 0} \subset \mathbb{P}^2(\mathbb{R})$

YES



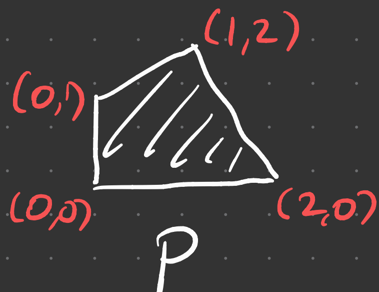
$$\Omega = C \frac{(1+2y)\,dx\,dy}{(1-x^2-y^2)(\sqrt{3}y+x)(\sqrt{3}y-x)}$$

NO

$\Omega = 0$

boundary is elliptic curve

Theorem $P \subset \mathbb{P}^d$ $\quad \dim P = d$ $\quad$ projective polytope is a positive geometry



$$\Omega = \frac{C\,(y-4x-4)}{xy\,(y-x-1)(2x+y-4)}\,dx\,dy$$

Idea of proof. $\quad P = \coprod_i T_i \quad$ triangulation

$$\Omega_P = \sum_i \Omega_{T_i}$$

Theorem $P \subset \mathbb{R}^d \subset \mathbb{P}^d$

$$\Omega_P(x) = \mathrm{Vol}\left(\left(P-x\right)^\vee\right) dx \qquad \text{for} \quad x \in \mathrm{Int}(P)$$

$$= \frac{\mathrm{adj}_{P^\vee}(x)}{\prod_{\text{facets } F} L_F}\, dx$$

adjoint of $P^\vee$

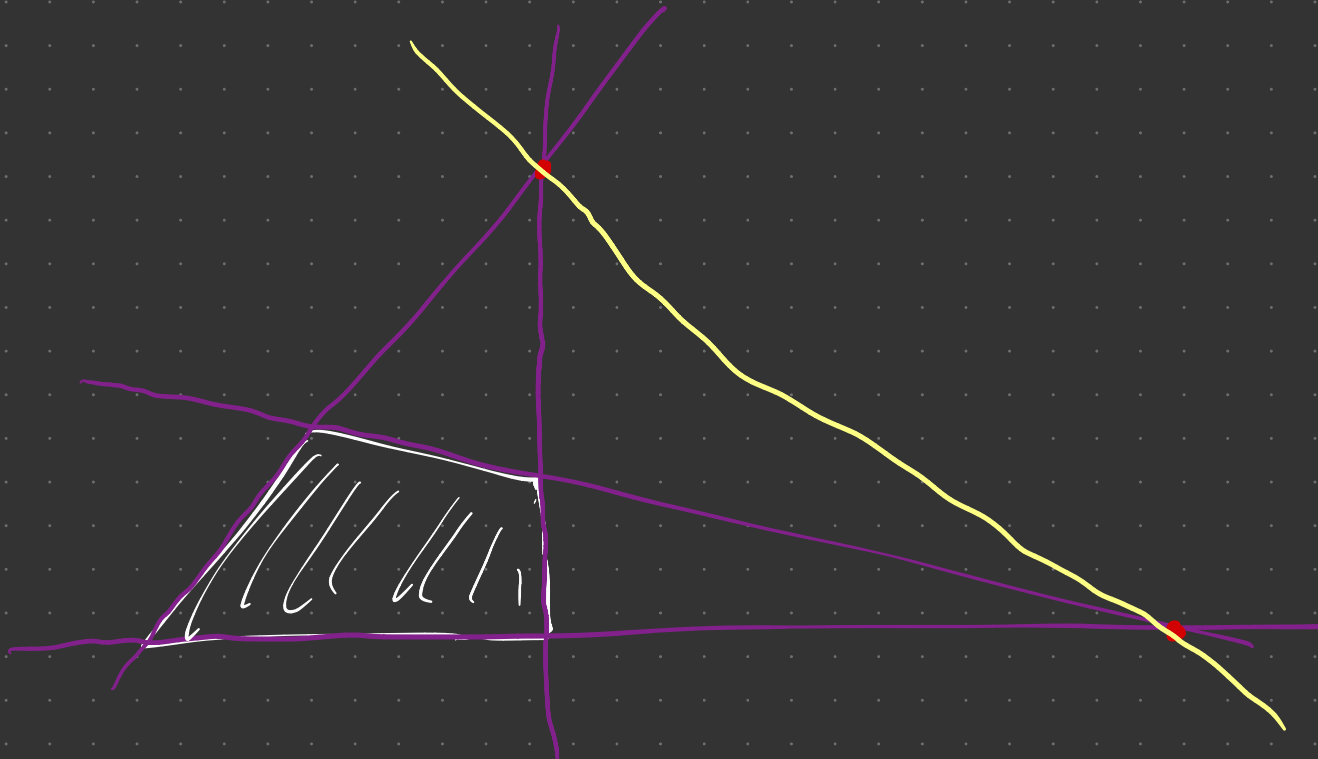
vanishes on F

Adjoint of $P^\vee$ = polynomial that vanishes on residual arrangement of P

[Kohn–Ranestad, Warren]

||

union of intersections of facet hyperplanes that do not contain any face of P.

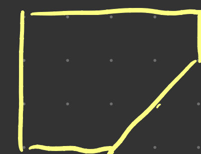


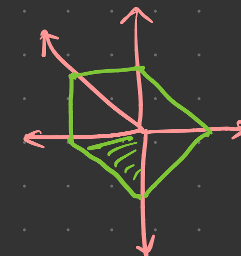
$adj_{\cup pv} =$

Arkani-Hamed, Bai, He, Yan

There exists a kinematic associahedron $P(p)$ such that

$$\Omega(P(p)) = \mathcal{A}_n^{\phi^3}\, dx$$





Feynman diagram formula:

$$\mathrm{Vol}((P(p)-x)^\vee) = \sum_v (P(p)-x)^\vee \cap C_v$$

vertices of P $\xleftrightarrow{1:1}$ cubic planar trees

Examples

- $(X_P, (X_P)_{\geq 0})$ projective normal toric variety

$$\Omega = \frac{dx_1}{x_1} \cdots \frac{dx_d}{x_d}$$

- $(Gr(k,n), Gr(k,n)_{\geq 0})$ totally nonnegative Grassmannian
- $(\overline{\mathcal{M}_{0,n}}, (\mathcal{M}_{0,n})_{>0})$ moduli space of n-pointed rational curves
- Conjecture

$$Z : \mathbb{R}^n \longrightarrow \mathbb{R}^{k+m}$$

$(Gr(k,k+m), Z(Gr(k,n)_{\geq 0}))$ Grassmann polytope

$(Gr(k,k+m), \mathcal{A}_{n,k,m})$ Amplituhedron

Integral functions

- $\mathcal{M}(z,\eta) = \int \Omega(\mathcal{A}_{n,k,4}(Z))\, \delta^{4k}(Y; Y_0)\ d^{4N}\phi$

  N=4 SYM planar tree amplitude

- $I(S) = \int_{(\mathcal{M}_{0,n})_{>0}} \Omega((\mathcal{M}_{0,n})_{>0}) \begin{bmatrix}\text{rational}\\ \text{factor}\end{bmatrix}^S$

  string amplitude

  $\int_0^1 u^s(1-u)^t \frac{du}{u(1-u)}$ Beta function

- $\Psi(q) = \int_{Gr(k,n)_{>0}} \Omega(Gr(k,n)_{>0})\ e^{\text{superpotential}}$

  Whittaker function

  $\int_0^\infty e^{-(x+\frac{q}{x})} \frac{dx}{x}$

  Bessel function

  $Gr(1,2)_{>0}$

General problem:

$$\boxed{\text{Find formulae for } \Omega(X, X_{\geq 0})}$$

- Topology? Contractible, balls, ?
- Face poset?
- Can every positive geometry be triangulated into simplex-like positive geometries?
- Convexity? $\Omega_P(x) > 0$ for $x \in \mathrm{Int}(P)$ polytope
  conjectured for $\mathcal{A}_{n,k,m}$ m even