The Potts model and Lorentzian polynomials on cones

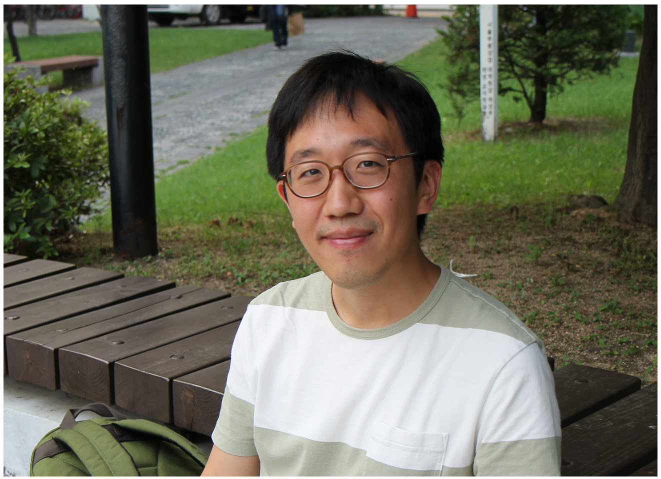
Petter Brändén

KTH Royal Institute of Technology

Algebraic Combinatorics and Mathematical Physics

University of Rome "Tor Vergata"
Jan 13-14, 2022

Based on joint work with



June Huh (Princeton)

Jonathan Leake (TU Berlin)

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  where $k(A)$ is the number of connected components of the subgraph (V, A).
- $Z_G(q, \mathbf{t})$ defines a probability distribution on subsets of E: Let X be a random subset of E. Then

$$\mathbb{P}[X = A] = \frac{1}{Z_G(q, \mathbf{t})} \cdot q^{k(A)} \prod_{e \in \cancel{E}\, A} t_e, \qquad t_e, q \geq 0.$$

- $\mathbb{P}$ is called the Fortuin-Kasteleyn random cluster model.

Positive and negative dependence

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- Known for the random spanning tree measure, which is a limit of Potts when $q \to 0$.
- Unknown for the random forest measure, which is another limit when $q \to 0$.

Negative dependence for the Potts model

- Ultra log-concavity: Let $r_k = \mathbb{P}[|X| = k]$ and $n = |E|$.

$$\frac{r_k^2}{\binom{n}{k}^2} \geq \frac{r_{k-1}}{\binom{n}{k-1}} \cdot \frac{r_{k+1}}{\binom{n}{k+1}}, \qquad 0 < k < n.$$

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- The proofs use Lorentzian polynomials.

$$\boxed{\partial_i \partial_j f \cdot f \leq 2\frac{d-1}{d} \cdot \partial_i f \cdot \partial_j f}$$

Fortuin-Kasteleyn representation

- Theorem (Fortuin-Kasteleyn, 1969). For positive integers q,

$$Z_G(q, \mathbf{t}) = \sum_{\sigma : V \to \{1,2,\ldots,q\}} \prod_{e=ij \in E} \big(1 + t_e \delta(\sigma(i), \sigma(j))\big)$$

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- Proved by June Huh in 2012 using Hodge theory.

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- Example. If $G = (V, E)$ is a graph, then

  $$r(S) = |V| - k(S)$$

  defines a graphic matroid.

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- Example. Let M be the matroid defined by $\mathbf{v}_1, \ldots, \mathbf{v}_m \in K^n$, and let $\mathcal{U}$ be the collection of all subspaces of K^n spanned by subsets of $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

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- The Möbius function, $\mu : \mathcal{L}(\mathrm{M}) \times \mathcal{L}(\mathrm{M}) \to \mathbb{Z}$, is defined by
  - $\mu(F, G) = 0$ unless $F \leq G$,
  - $\mu(F, F) = 1$, and
  - If $F < G$, then

$$\sum_{F \leq H \leq G} \mu(F, H) = 0.$$

Heron-Rota-Welsh conjecture

- The characteristic polynomial is

$$\chi_{\mathrm{M}}(t) = \sum_{F \in \mathcal{L}(\mathrm{M})} \mu(\varnothing, F) t^{r(E)-r(F)} = \sum_{k=0}^{r(E)} (-1)^{r(E)-k} w_k t^k.$$

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- Proved by Adiprasito, Huh and Katz (2018) by developing a Hodge theory for matroids.
- We will sketch a short and self-contained "polynomial" proof using Lorentzian polynomials on cones.

Lorentzian polynomials on cones

- For $\mathbf{w} \in \mathbb{R}^n$, let

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- Let $\mathcal{C}$ be an open convex cone in $\mathbb{R}^n$.
- f is called $\mathcal{C}$-Lorentzian if for all $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathcal{C}$,
  (P) $D_{\mathbf{v}_1} \cdots D_{\mathbf{v}_d} f > 0$, and
  (AF) For all $\mathbf{x} \in \mathbb{R}^n$,

$$(D_{\mathbf{x}} D_{\mathbf{v}_2} \cdots D_{\mathbf{v}_d} f)^2 \geq (D_{\mathbf{x}} D_{\mathbf{x}} \cdots D_{\mathbf{v}_d} f) \cdot (D_{\mathbf{v}_2} D_{\mathbf{v}_2} \cdots D_{\mathbf{v}_d} f).$$

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Lorentzian signature
$(+, -, -, -, \ldots)$

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- $\mathbb{R}^n_{>0}$-Lorentzian polynomials are called Lorentzian.

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- Suppose $\mathbf{u}, \mathbf{v} \in \overline{\mathcal{C}}$, and f is $\mathcal{C}$-Lorentzian. Write

$$f(s\mathbf{u} + t\mathbf{v}) = \sum_{k=0}^{d} a_k \binom{d}{k} s^k t^{d-k}.$$

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- Volume polynomials of convex bodies and projective varieties are Lorentzian.
- Various polynomials associated to matroids are Lorentzian.

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- For example,

$$\mathbf{y} = \Big(|S \setminus K| \cdot |L \setminus S|\Big)_{K\subset S\subset L} \in \mathcal{S}_K^L.$$

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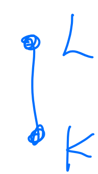

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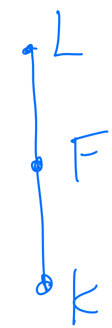

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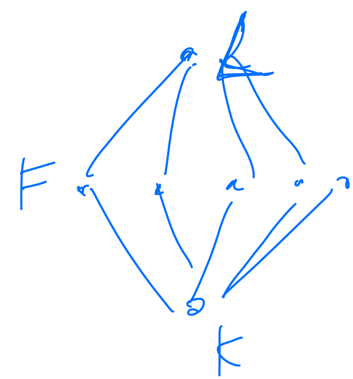

Polynomials associated to lattices of flats

- If $d(K,L) = 2$, then

$$2 \cdot \mathrm{pol}_K^L(\mathbf{t}) = \sum_{K \prec F \prec G \prec L} \left(2 \cdot t_F t_G - t_F^2 \cdot \frac{|L \setminus G|}{|L \setminus F|} - t_G^2 \cdot \frac{|F \setminus K|}{|G \setminus K|} \right)$$

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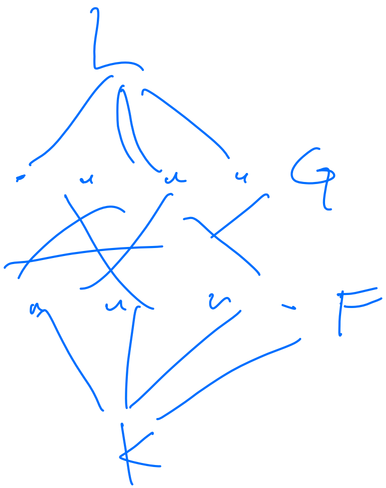

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- From this follows that if $d(K, L) = 2$, then pol_K^L is $\mathcal{S}_K^L$-Lorentzian.

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Heron-Rota-Welsh conjecture

- Consider the elements in the closure of $\mathcal{S}_{\varnothing}^{E}$:

$$\alpha = \left(\frac{|S|}{|E|}\right)_{\varnothing \subset S \subset E} \quad \text{and} \quad \beta = \left(\frac{|E \setminus S|}{|E|}\right)_{\varnothing \subset S \subset E}.$$

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- Theorem (Folklore). $\mathcal{S}_K^L$ is effective, with lineality space the set of modular functions.

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