Parameter determination for energy balance models with memory

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Mathematical Models and Methods in Earth and Space Sciences, Rome, March 19-22, 2019

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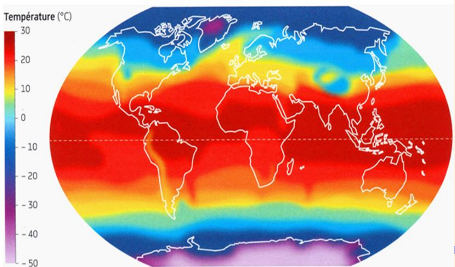
I. Climate models : general considerations

I. Climate models : purposes

- Better understanding of past (and future) climates,
- Better understanding the sensitivity to some relevant solar and terrestrial parameters,
- Involve a long time scaling ($\neq$ weather prediction models).

Hierarchy in the class of climate models :

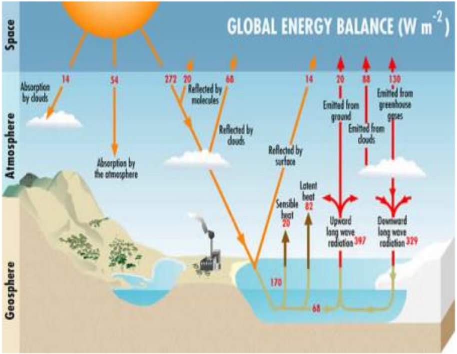
- $0-D$: $u(t)$, mean annual or seasonal Earth temperature average on the Earth,
- $2-D$: $u(t,m)$: mean annual or seasonal Earth temperature average ($m \in$ manifold $\mathcal{S}^2$),
- $3-D$: General Circulation Model ($u(t,m,h)$),
- GCM coupled with Glaceology, Celestial Mechanics, Geophysics...,
- $1-D$: $u(t,\varphi)$: mean annual or seasonal temperature average on the latitude circles around the Earth ; $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ parametrizes the latitude :



Energy Balance Models :

Introduced by Budyko (1969), Sellers (1969)

Global energy balance



Source: Bureau of Meteorology, *The greenhouse effect and climate change*, Bureau of Meteorology, 2003, p. 7.

The mean annual or seasonal temperature average on the Earth : u satisfies

variation of $u = +$absorbed energy $-$ reflected energy $+$ diffusion,

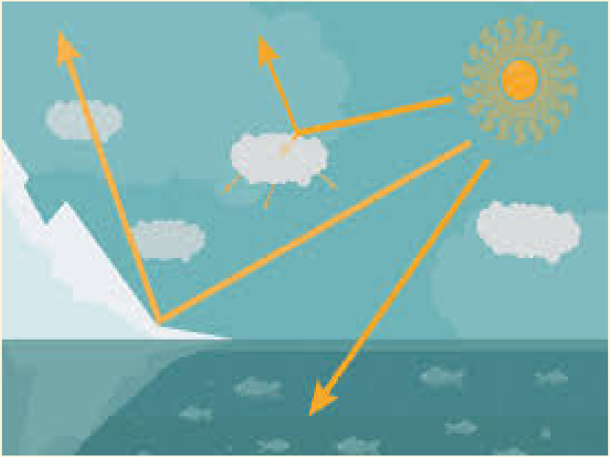
hence a **reaction-diffusion equation** of the form

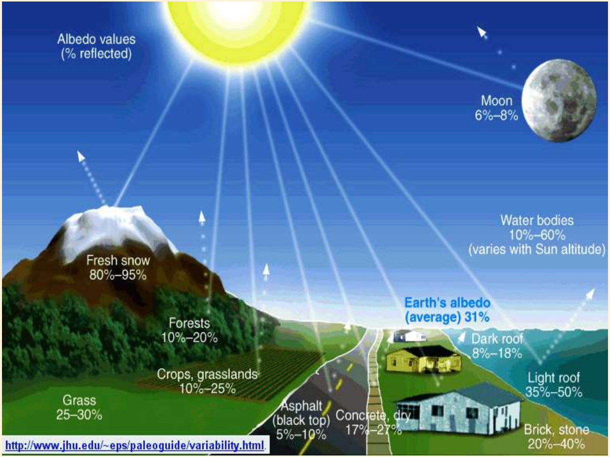
$$c(t,x)u_t - \text{ diffusion } = R_a - R_e,$$

where

- $c(t,x)$: heat capacity,
- diffusion $=$ div $(F_c + F_a)$ with
  - F_c the conduction heat flux,
  - F_a the advection heat flux,
- $R_a =$ absorbed solar radiation, $= QS(t,x)\beta(u)$:
  - Q : Solar constant,
  - $S(t,x)$: distribution of solar radiation,
  - $\beta(u)$: "planetary coalbedo" ($=$ the fraction absorbed according the average temperature),
- $R_e =$: emitted radiation (depends on the amount of greenhouse gases, clouds and water vapor in the atmosphere, increases with u).

EBMs : absorbed solar radiation




Albedo values
(% reflected)
Moon
6%–8%
Fresh snow
80%–95%
Water bodies
10%–60%
(varies with Sun altitude)
Forests
10%–20%
Earth's albedo
(average) 31%
Dark roof
8%–18%
Crops, grasslands
10%–25%
Light roof
35%–50%
Grass
25–30%
Asphalt
(black top)
5%–10%
Concrete, dry
17%–27%
Brick, stone
20%–40%
http://www.jhu.edu/~eps/paleoguide/variability.html.

Modelization for the absorbed solar radiation :

$$R_a = (1 - \alpha(...))\, Q(...) :$$

- Q : high-frequency solar radiation (depends at least on time and on and the space location) ;
- $1 - \alpha$: co-albedo (fraction of absorbed energy) ;
- α : albedo (fraction of reflected energy) ;
  - α nonincreasing, from α_+ to α_- (ice reflects more than non ice),
  - Sellers : $\alpha(u)$ smooth / Budyko : $\alpha(u)$ discontinuous,
  - Bhattacharya-Ghil-Vulis (1982) : $\alpha(u,$ memory effect$)$; (memory effect : interesting to take into account the long response times of the ice sheets to temperature changes).

EBMs : diffusion and emitted radiation

- **diffusion** $=$ div $(k(...)\nabla u)$:
  - $k = k_0$ positive constant, and averaging along the parallels, $x = \sin(\text{latitude})$: 1D model, degenerate parabolic equation (and possibly quasilinear) :

$$k_0((1-x^2)u_x)_x, \quad x \in (-1,1);$$

  - Sellers (1969), Ghil (1976) : 1D, $k(u)$,
  - Stone (1972) : $k(x, \nabla u) = k_1(x)|\nabla u|$ (manifold, rotating atmosphere),
  - Diaz (1993) : $k(x, \nabla u) = k_1(x)|\nabla u|^{p-2}$ (manifold) ;
- **emitted radiation** :

$$\textit{Sellers} : R_e = c\sigma u^4 \quad / \quad \textit{Budyko} : R_e = a + bu,$$

(where σ : Stefan-Boltzmann constant, $c = c(u)$: emissivity).

EBMs : mathematical problems and directions

Parabolic equation,

- $1 - D$: degenerate diffusion coefficient,
- $2 - D$: on a manifold,
- with nonlinear source terms, and possibly quasilinear,
- possibly with discontinuous coefficients (Budyko),
- possibly with non local terms (memory).

Has been studied :

- multiple steady states (S-shaped bifurcation, parameter : solar radiation) (Ghil (1976))
- internal/external stability of steady states (Ghil (1976))
- existence of solutions, uniqueness/non uniqueness (in prescribed classes) (Diaz (1993), Hetzer (1996, 2011))
- dynamics, long-time asymptotic behavior (Hetzer (1991))
- free boundary value problem : snow lines (Diaz (1993))
- coeffs : uniqueness, inverse problems (Ghil et al (2014))

II. 1D Sellers climate model with memory

II. Sellers climate model with memory

$$\begin{cases} u_t - (\rho_0(1-x^2)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3 u + \mathbf{f}(\mathbf{H}), \\ \rho_0(1-x^2)u_x = 0, \quad x = \pm 1, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \end{cases}$$

where

- 1-D parametrization $x = \sin(\varphi) \in (-1,1)$ with $\varphi =$ the latitude,
- absorbed energy,
- emmited energy ,
- **memory term** : (to take into account the long response times of the ice sheets to temperature changes (Ghil et al (1982, 2014)) :

$$\mathbf{H}(\mathbf{t},\mathbf{x},\mathbf{u}) = \int_{-\tau}^{0} k(s,x)u(t+s,x)\,ds.$$

Sellers model : Inverse problem question

- **Goal :** study an inverse problem that consists in recovering the insolation function $q(x)$ in the Sellers model with memory using partial measurements of the solution,
- **Difficulties :** degeneracy + nonlinearity + nonlocal,
- **Results :**
  - well-posedness,
  - uniqueness result under pointwise measurements,
  - Lipschitz stability under localized measurements.
- **Motivations :**
  - conference 'Mathematical approach to Climate Change Impact' INdAM workshop, Roma (Italy) March 13-17, 2017 (in particular K. Fraedrich),
  - many works : Ghil (1976, 2014), Hetzer (1996, 2011), Diaz (2002), Yamamoto (1996, 2006)...

Sellers model : precise assumptions

- $\rho(x) = \rho_0(1-x^2)$, $\rho_0 > 0$, $x \in (-1,1)$,
- $\beta \in \mathcal{C}^2(\mathbb{R})$, $\beta, \beta', \beta'' \in L^\infty(\mathbb{R})$, $\beta(\cdot) \geq \beta_1 > 0$,
- $q \in L^\infty(I)$,
- $r \in \mathcal{C}^1(\mathbb{R})$, $r, r' \in L^\infty(\mathbb{R})$, $r(\cdot) \geq r_1 > 0$,
- $\varepsilon \in \mathcal{C}^2$, $\varepsilon, \varepsilon', \varepsilon'' \in L^\infty(\mathbb{R})$, $\varepsilon(\cdot) \geq \varepsilon_1 > 0$,
- memory term :
  - kernel $k \in C^1([-\tau, 0] \times [-1,1], \mathbb{R})$,
  - nonlinearity $f \in \mathcal{C}^2(\mathbb{R})$, $f, f', f'' \in L^\infty(\mathbb{R})$.

1D Sellers model : functional setting

- natural space :

$$V := \{w \in L^2(I) : w \in AC_{loc}(I), \sqrt{\rho} w_x \in L^2(I)\} \quad \hookrightarrow L^p(I) \forall p \geq 1;$$

- operator $A : D(A) \subset L^2(I) \to L^2(I)$ in the following way :

$$\begin{cases} D(A) := \{u \in V : \rho u_x \in H^1(I)\} \\ Au := (\rho_0(1-x^2)u_x)_x, \quad u \in D(A) : \end{cases}$$

$(A, D(A))$ is a self-adjoint operator and it is the infinitesimal generator of an analytic and compact semigroup $\{e^{tA}\}_{t \geq 0}$ in $L^2(I)$ that satisfies

$$|||e^{tA}|||_{\mathcal{L}(L^2(I))} \leq 1.$$

(Campiti-Metafune-Pallara (1998))

1D Sellers model : definition of mild solution

$$\begin{cases} \dot{u}(t) = Au(t) + G(t,u) + \mathbf{F}(\mathbf{u^{(t)}}) & t \in [0, T] \\ u(s) = u_0(s) & s \in [-\tau, 0], \end{cases} \tag{1}$$

with

$$G(t,u) = \text{ local source terms}, \quad \mathbf{F}(\mathbf{u^{(t)}}) = \text{memory term}$$

Definition

Given $u_0 \in C([-\tau, 0]; V)$, a function

$$u \in H^1(0, T; L^2(I)) \cap L^2(0, T; D(A)) \cap C([-\tau, T]; V)$$

is called a mild solution of (1) on $[0, T]$ if $u(s) = u_0(s)$ for all $s \in [-\tau, 0]$, and if for all $t \in [0, T]$, we have

$$u(t) = e^{tA} u_0(0) + \int_0^t e^{(t-s)A} \big(G(s,u) + \mathbf{F}(\mathbf{u^{(s)}}) \big) \, ds.$$

Memory Sellers model : well-posedness result

Theorem

(Cannarsa-Malfitana-M (2018)) Consider u_0 such that

$$u_0 \in C([-\tau, 0]; V) \quad \textit{and} \quad u_0(0) \in D(A) \cap L^\infty(I).$$

Then, for all $T > 0$, the problem (1) has a unique mild solution u on $[0, T]$.

Proof :

- local existence (fixed point, contraction)
- uniqueness (Gronwall's lemma),
- global existence of the maximal solution.

(Memory Sellers model : global existence)

based on the following boundedness property :

Theorem

Consider $u_0 \in C([-\tau, 0]; V)$ *and* $u_0(0) \in D(A) \cap L^\infty(I)$, $T > 0$ *and* u *a mild solution of* (1) *defined on* $[0, T]$. *Let us denote*

$$M_1 := \left(\frac{||q||_{L^\infty(I)}||r||_{L^\infty(\mathbb{R})}||\beta||_{L^\infty(\mathbb{R})} + ||f||_{L^\infty(\mathbb{R})}}{\varepsilon_1} \right)^{\frac{1}{4}}$$

and $M := \max\{||u_0(0)||_{L^\infty(I)}, M_1\}$. *Then* u *satisfies*

$$||u||_{L^\infty((0,T)\times I)} \leq M.$$

III. Budyko climate model with memory

III. Budyko climate model with memory : the problem

$$\begin{cases} u_t - (\rho_0(1-x^2)u_x)_x = r(t)q(x)\beta(u) - (a+bu) + \mathbf{f}(\mathbf{H}), \\ \rho_0(1-x^2)u_x = 0, \quad x = \pm 1, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \end{cases}$$

where coalbedo :

$$\beta(u) = \begin{cases} a_i, & u < \overline{u}, \\ [a_i, a_f], & u = \overline{u}, \\ a_f, & u > \overline{u}, \end{cases}$$

where $a_i < a_f$ (and the threshold temperature $\bar{u} := -10^\circ$).
Well-posedness ? differential inclusion :

$$\begin{cases} u_t - (\rho(x)u_x)_x \in r(t)q(x)\beta(u) - (a+bu) + f(H(u)), \\ \rho(x)u_x = 0, \quad x = \pm 1, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], x \in I. \end{cases} \tag{2}$$

Memory Budyko model : the notion of solution

Definition

Given $u_0 \in C([-\tau, 0); V)$, a function

$$u \in H^1(0, T; L^2(I)) \cap L^2(0, T; D(A)) \cap C([-\tau, T]; V)$$

is called a *mild solution* of (2) on $[-\tau, T]$ iff

- $u(s) = u_0(s)$ for all $s \in [-\tau, 0]$;
- there exists $g \in L^2([0, T]; L^2(I))$ such that
  - u satisfies

    $$\forall t \in [0, T], \quad u(t) = e^{tA} u_0(0) + \int_0^t e^{(t-s)A} g(s)\, ds,$$

  - and g satisfies the inclusion

    $$g(t, x) \in r(t)q(x)\beta(u(t, x)) - (a + bu(t, x)) + f(H(t, x, u)) \quad \text{a.e.} \; .$$

Memory Budyko model : global existence

Theorem

(Cannarsa-Malfitana-M (2018)) Assume that

$$u_0 \in C([-\tau, 0], V) \quad \textit{and} \quad u_0(0) \in D(A) \cap L^\infty(I).$$

Then (2) has a mild solution u, which is global in time (i.e. defined in $[0, +\infty)$ and mild on $[0, T]$ for all $T > 0$).

Proof :

- regularization of the coalbedo : β_j smooth, $\beta_j \to \beta$,
- $\rightsquigarrow$ approximate problem $\mathcal{P}_j$,
- the approximate problem has a solution u_j,
- $u_{j'} \to u_\infty$ solution of the original problem :
  - subsequence $u_{j'} \to u_\infty$ (compactness arguments)
  - u_∞ solution of the original problem (differential inclusion).

Memory Budyko model : elements of proof

- approximation : $\beta_j : \mathbb{R} \to \mathbb{R}$ which is of class C^2, nondecreasing, and

$$\begin{cases} \beta_j(u) = a_i, & u \leq \bar{u} - \frac{1}{j}, \\ \beta_j(u) = a_f, & u \geq \bar{u} + \frac{1}{j}; \end{cases}$$

- the approximate problem has a unique mild solution

$$u_j \in H^1(0, T; L^2(I)) \cap L^2(0, T; D(A)) \cap C([-\tau, T]; V);$$

- compactness arguments for $(u_j)_j$:
  - uniform L^∞ bound on u_j and $V \hookrightarrow L^2(I)$ compact $\Longrightarrow$ the set of traces $\{u_j(t), j \geq 1\}$ is relatively compact in $L^2(I)$;
  - integral representation formula

$$u_j(t) = e^{tA}u_0(0) + \int_0^t e^{t-s)A}\gamma_j(s)\, ds$$

    $\rightsquigarrow$ $(u_j)_j$ is equicontinuous in $C([0, T]; L^2(I))$;
  - conclusion with the Ascoli-Arzela (Vrabie (1987)) : the family $(u_j)_j$ is relatively compact in $C([0, T]; L^2(I))$.

- compactness arguments for $(\gamma_j)_j$:

$$\gamma_j(t,x) := r(t)q(x)\beta_j(u_j(t,x)) - (a + bu_j(t,x)) + f(H_j(t,x)),$$

is bounded in $L^2(0,T;L^2(I))$ hence weakly relatively compact in $L^2(0,T;L^2(I))$.

- Then, we can extract from $(u_j,\gamma_j)_j$ a subsequence $(u_{j'},\gamma_{j'})_{j'}$ such that

$$u_{j'} \to u_\infty \text{ in } C([0,T];L^2(I)) \quad \text{and} \quad \gamma_{j'} \rightharpoonup \gamma_\infty \text{ in } L^2(0,T;L^2(I)).$$

- Conclusion :
  - $j' \to \infty \implies$ the functions u_∞ and γ_∞ satisfy

$$\forall t \in [0,T], \quad u_\infty(t) = e^{tA}u_0(0) + \int_0^t e^{(t-s)A}\gamma_\infty(s)\,ds;$$

  - this gives that

$$u_\infty \in H^1(0,T;L^2(I)) \cap L^2(0,T;D(A)) \cap C([-\tau,T];V);$$

  - and we have the good differential inclusion

$$\gamma_\infty(t,x) \in r(t)q(x)\beta(u_\infty(t,x)) - (a + bu_\infty(t,x)) + f(H_\infty(t,x)).$$

IV. Memory Sellers model : inverse problems results

IV. 1D Memory Sellers model : uniqueness/stability of the insolation function ?

$$\begin{cases} u_t - (\rho(x)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3u + f(H), & t > 0, x \in I, \\ \rho(x)u_x = 0, & x \in \partial I, \\ u(s,x) = u_0(s,x), & s \in [-\tau, 0], x \in I, \end{cases} \tag{3}$$

$$\begin{cases} \tilde{u}_t - (\rho(x)\tilde{u}_x)_x = r(t)\tilde{q}(x)\beta(\tilde{u}) - \varepsilon(\tilde{u})|\tilde{u}|^3\tilde{u} + f(\tilde{H}), & t > 0, x \in I, \\ \rho(x)\tilde{u}_x = 0, & x \in \partial I, \\ \tilde{u}(s,x) = \tilde{u}_0(s,x), & s \in [-\tau, 0], x \in I : \end{cases} \tag{4}$$

$u = \tilde{u}$ on a "small" set $\Longrightarrow$ $q = \tilde{q}$?

$u - \tilde{u}$ "small on a small set" $\Longrightarrow$ $q - \tilde{q}$ small ?

IV. Motivation for these inverse problems Ghil et al (2014)

- Goal of the Energy Balance Models (with Memory) : toy models to understand a part of the climate evolution
- With suitable tuning of the parameters : EBMs simulations give reasonable results for the observed present climate (North-Mengel-Short (1983).
- Once fitted, EBM(M) can be used to estimate the temporal response patterns to various scenarios (climate change).
- BUT in practice, the model parameters cannot be measured directly (intertwined effects of several physical processes ; hence measure the solution, and fit the parameters, with robust and efficient methods (Yamamoto-Zou (2001), Ghil et al (2014)).

IV. Uniqueness of the insolation function under pointwise measurements : assumptions

- the set of admissible initial conditions : we consider

$$\mathcal{U}^{(pt)} = C^{1,2}([-\tau, 0] \times [-1, 1]),$$

- the set of admissible coefficients : we consider

$$\mathcal{Q}^{(pt)} := \{q \text{ is Lipschitz-continuous and piecewise analytic on } I\},$$

- the memory kernel : support condition :

$$\exists \delta > 0 \text{ s.t. } k(s, \cdot) \equiv 0 \quad \forall s \in [-\delta, 0] \tag{5}$$

with $\delta < \tau$.

IV. Uniqueness of the insolation function under pointwise measurements : result

Theorem

(Cannarsa-Malfitana-M (2018)) Consider

- *two insolation functions* $q, \tilde{q} \in \mathcal{Q}^{(pt)}$
- *an initial condition* $u_0 = \tilde{u}_0 \in \mathcal{U}^{(pt)}$

and let u *be the solution of* (3) *and* $\tilde{u}$ *the solution of* (4).
Assume that

- *the memory kernel satisfies* (5),
- r *and* β *are positive,*
- *there exists* $x_0 \in I$ *and* $T > 0$ *such that*

$$\forall t \in (0, T), \quad u(t, x_0) = \tilde{u}(t, x_0), \ \textit{and} \ u_x(t, x_0) = \tilde{u}_x(t, x_0).$$

Then $q \equiv \tilde{q}$ *on* $(-1, 1)$.

(Extension of Roques-Checkroun-Cristofol-Soubeyrand-Ghil (2014))

IV. Uniqueness of the insolation function under pointwise measurements : measurement zone

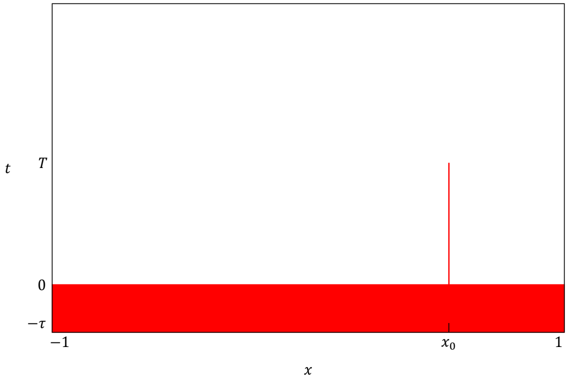


Figure – Space - time measurement region which can lead to unique coefficient determination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IV. 1D Memory Sellers model : Lipschitz stability of the insolation function ?

$$\begin{cases} u_t - (\rho(x)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3u + f(H), \quad t > 0, x \in I, \\ \rho(x)u_x = 0, \quad x \in \partial I, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], x \in I, \end{cases} \tag{6}$$

$$\begin{cases} \tilde{u}_t - (\rho(x)\tilde{u}_x)_x = r(t)\tilde{q}(x)\beta(\tilde{u}) - \varepsilon(\tilde{u})|\tilde{u}|^3\tilde{u} + f(\tilde{H}), \quad t > 0, x \in I, \\ \rho(x)\tilde{u}_x = 0, \quad x \in \partial I, \\ \tilde{u}(s,x) = \tilde{u}_0(s,x), \quad s \in [-\tau, 0], x \in I : \end{cases} \tag{7}$$

$$\|q - \tilde{q}\| \quad \leq \quad C \, \|u - \tilde{u}\|_{\dots} \quad ?$$

IV. Lipschitz stability of the insolation function under localized measurements : assumptions

- the set of admissible initial conditions : given $M > 0$,

$$\mathcal{U}_M^{(loc)} := \{u_0 \in C([-\tau, 0]; V \cap L^\infty(-1,1)), \\ u_0(0) \in D(A), Au_0(0) \in L^\infty(I), \\ \sup_{t \in [-\tau,0]} \left(\|u_0(t)\|_V + \|u_0(t)\|_{L^\infty}\right) + \|Au_0(0)\|_{L^\infty(I)} \leq M\},$$

- the set of admissible coefficients : given $M' > 0$,

$$\mathcal{Q}_{M'}^{(loc)} := \{q \in L^\infty(I) : \|q\|_{L^\infty(I)} \leq M'\},$$

- the memory kernel : the same support condition :

$$\exists \delta > 0 \text{ s.t. } k(s, \cdot) \equiv 0 \quad \forall s \in [-\delta, 0]$$

with $\delta < \tau$.

IV. Lipschitz stability of the insolation function under localized measurements : result

Theorem

(Cannarsa-Malfitana-M (2018)) Assume that r *and* β *are positive. Consider* $0 < T' < \delta$, $t_0 \in [0, T')$, $T > T'$, $M, M' > 0$. *Then there exists* $C(t_0, T', T, M, M') > 0$ *such that, for all* $u_0, \tilde{u}_0 \in \mathcal{U}_M^{(loc)}$, *for all* $q, \tilde{q} \in \mathcal{Q}_{M'}^{(loc)}$, *the solution* u *of* (6) *and the solution* $\tilde{u}$ *of* (7) *satisfy*

$$
\begin{aligned}
\|q - \tilde{q}\|^2_{L^2(I)} \leq C\Big(& \|u(T') - \tilde{u}(T')\|^2_{D(A)} \\
& + \|u_t - \tilde{u}_t\|^2_{L^2((t_0,T)\times(a,b))} + \|u_0 - \tilde{u}_0\|^2_{C([-\tau,0];V)}\Big). \qquad (8)
\end{aligned}
$$

Remark : extension of Tort-Vancostenoble (2012)

IV. Lipschitz stability of the insolation function under localized measurements : measurement zone

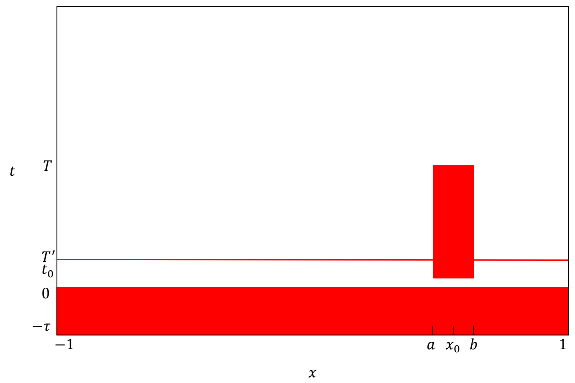

Figure – Space - time measurement region which can lead to Lipschitz stability


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2D Sellers model : recovering of the insolation function

- $\omega \subset \mathcal{S}^2$,
- set of insolation functions :

$$\mathcal{Q}_B := \{q \in L^\infty(\mathcal{M}) : \|q\|_{L^\infty(\mathcal{M})} \leq B\}.$$

- set of initial conditions :

$$\mathcal{U}_A := \{u^0 \in D(\Delta_{\mathcal{M}}) \cap L^\infty(\mathcal{M}) : \Delta_{\mathcal{M}} u^0 \in L^\infty(\mathcal{M}), \\ \|u_0\|_{L^\infty(\mathcal{M})} + \|\Delta_{\mathcal{M}} u_0\|_{L^\infty(\mathcal{M})} \leq A\},$$

$$\begin{cases} u_t - \Delta_{\mathcal{M}} u = r(t) q(m) \beta(u) - \varepsilon(u)|u|^3 u, & m \in \mathcal{S}^2, \\ u(0,m) = u_0(m), \end{cases}$$

$$\begin{cases} \tilde{u}_t - \Delta_{\mathcal{M}} \tilde{u} = r(t) \tilde{q}(m) \beta(\tilde{u}) - \varepsilon(\tilde{u})|\tilde{u}|^3 \tilde{u}, & m \in \mathcal{S}^2, \\ u(0,m) = u_0(m). \end{cases}$$

2D Sellers model : recovering of the insolation function : result

Theorem

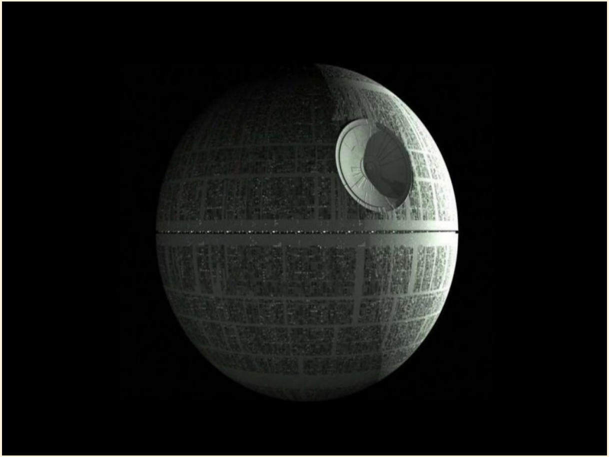
M-Tort-Vancostenoble (2017) $\forall T_0 \in \mathcal{U}$, $\forall D > 0$, $\exists C > 0$ *such that* $\forall q_1, q_2 \in \mathcal{Q}_D$,

$$\|q - \tilde{q}\|^2_{L^2(\mathcal{M})} \leq C\|(u - \tilde{u})(\tau', \cdot)\|^2_{D(A)} + C\|u_t - \tilde{u}_t\|^2_{L^2((t_0,\tau)\times\omega)}.$$

Tools :

- Carleman estimates on the manifold,
- maximum principles (nonlinear terms),
- stereographic projection (uniformisation theorem of Riemann in a general case).

2D Sellers model : Lipschitz stability : measurement zone



IV. Typical questions in the literature

Different problems of the same kind :

- $$\begin{cases} u_t - \Delta u = g(t,x) & (t,x) \in (0,T) \times \Omega \\ u(t,\cdot) = 0 & (t,x) \in [0,T) \times \partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega : \end{cases}$$

  goal : determine the source term g from partial measurements of u (uniqueness ? stability (logarithmic, Holder, Lipschitz) ? numerical reconstructions ?)

- $$\begin{cases} u_t - (a(x)u_x)_x + b(x)u + \int_0^t c(t-s)u(s)\,ds = 0 & \\ u(t,\cdot) = 0 & (t,x) \in [0,T) \times \\ u(0,x) = u_0(x) & x \in \Omega : \end{cases}$$

  goal : determine some (or all) the coefficients $a(x)$, $b(x)$, $c(t)$ from partial measurements of u (uniqueness ? stability (logarithmic, Holder, Lipschitz) ? numerical reconstructions ?)

Inverse source problem for the linear heat equation

Various approaches and results

- **"Simple" models** (constant coefficients/depending only on x or only on t...) : elegant and sharp techniques :
  - Fourier series (moment method, biorthogonal families),
  - Laplace transform,
  - Volterra integral equations...

  $\rightsquigarrow$ sharp results (explicit formula of the solution...)
  Cannon 1968, Lorenzi-Sinestrari (1988), Lorenzi (1989...), Bukhgeim (1993), Gentili (1991), Grasselli (1992), Yamamoto (1993), Janno-Wolfersdorf (1996), Choulli-Yamamoto (2006)...
- **nonlinear models** (or coefficients in x and t) : local/global Carleman estimates $\rightsquigarrow$ uniqueness, Holder/Lipschitz stability : Bukhgeim/Klibanov 1981, Klibanov (1992), Isakov (1990, 1998...), Imanuvilov/Yamamoto (1998)
- **Use of analyticity properties** $\rightsquigarrow$ uniqueness under measurements at one point (in $1-D$) (Roques-Cristofol (2010), Roques-Checkroun-Cristofol-Soubeyrand-Ghil (2014))

Inverse source problem for the linear heat equation

Literature on the subject

Founding papers using GCE :
Puel/Yamamoto 1996 + 1997 (linear wave equation)
Imanuvilov/Yamamoto 1998 (linear heat equation)

Other Lipschitz stability results for parabolic equations :
Yamamoto/Zou 2001 (simultaneous reconstruction of 2 quantities)
Cristofol/Gaitan/Ramoul 2006 (systems)
Benabdallah/Dermenjian/Le Rousseau 2007 + Benabdallah/Gaitan/Le Rousseau 2009 (discontinuous diffusion coefficient)
Cannarsa/Tort/Yamamoto 2010 (degenerate diffusion coefficient)
Ignat/Pazoto/Rosier 2012 (networks)

Lipschitz stability results for other equations :
Hyperbolic equations : Imanuvilov/Yamamoto 2001, Komornik/Yamamoto 2002, Bellassoued/Yamamoto 2006, Liu/Triggiani 2011
Schrodinger equation : Baudouin/Puel 2002, Mercado/Osses/Rosier 2008, Cardoulis/Gaitan 2010, Liu/Triggiani 20111 ...

Inverse problems : a basic remark for 1^{st} order ODE

Consider the following ODE :

$$\begin{cases} u'(t) + \lambda u(t) = g, & u(0) = u_0, \\ v'(t) + \lambda v(t) = h, & v(0) = u_0. \end{cases}$$

Then $w := u - v$ solves

$$w'(t) + \lambda w(t) = g - h, \quad w(0) = 0 :$$

hence

$$w(t) = \frac{g-h}{\lambda}(1 - e^{-\lambda t}),$$

$$w(s^*) = 0 \quad \Longrightarrow \quad g - h = 0 :$$

$$u(s^*) = v(s^*) \quad \Longrightarrow \quad g = h.$$

But false with $g(t)$, $h(t)$:

$$u(s^*) = v(s^*) \quad \text{does not imply} \quad g = h.$$

Inverse problems : a basic remark for the heat equation

Consider

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = g(x), \quad u(0,x) = u_0(x), \\ v_t(t,x) - \Delta v(t,x) = h(x), \quad v(0,x) = u_0(x). \end{cases}$$

Then decompose into Fourier series :

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(x), \quad v(t,x) = \sum_{n=1}^{\infty} v_n(t)\varphi_n(x),$$

hence

$$\begin{cases} u_n'(t) + \lambda_n u_n(t) = (g, \varphi_n), \quad u_n(0) = (u_0, \varphi_n), \\ v_n'(t) + \lambda_n v_n(t) = (h, \varphi_n), \quad v_n(0) = (u_0, \varphi_n), \end{cases}$$

hence

$$u(s^*) = v(s^*) \implies g = h.$$

Main tools of proof of the uniqueness/stability result

- For the Lipschitz stability result :
  - reduction to some (non standard) inverse source problem for a linear equation,
  - inverse source problems methods (in particular Imanuvilov-Yamamoto (1998),
  - adapted Global Carleman Estimates
    - 1D : for degenerate parabolic equations : Cannarsa-M-Vancostenoble (2008),
    - 2D : on a manifold M-Tort-Vancostenoble (2017),
  - maximum principles to deal with nonlinear terms.
- For the uniqueness result :
  - analyticity,
  - strong maximum principle, Hopf's lemma.

V. Perspectives

V. Perspectives

Open questions :

- (Recover the insolation function with fewer measurements ? (Li-Yamamoto-Zou (2009) ? To remove the kernel assumption ?)
- Budyko's model ? (maximal graph ; many mathematical difficulties : non uniqueness, snow lines... (Diaz (1993))
    - sometimes several solutions, sometimes uniqueness ("non degenerate functions") (Diaz (93)),
    - the solutions of the approximate problem depend strongly on the approximation,
    - but inverse problems results ? (influence of nonuniqueness ?)
    - bilinear control for approx. controllability ? (Floridia (2013)
- more elaborated Sellers models ? (quasilinear in u (Ghil (1976)), p-Laplacian (Diaz (1993)) ?)

Thank you for your attention !