Invariant manifolds for a Solar sail

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1. Background
2. Station Keeping around Equilibria
3. Dynamics near an asteroid
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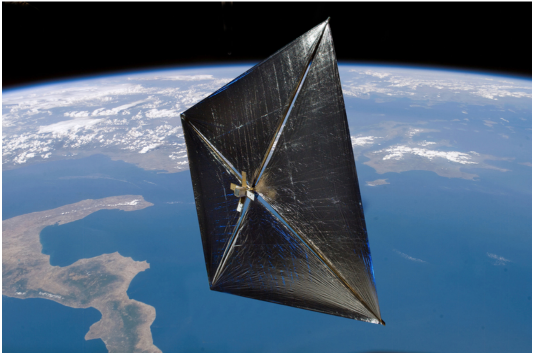
What is a Solar Sail ?

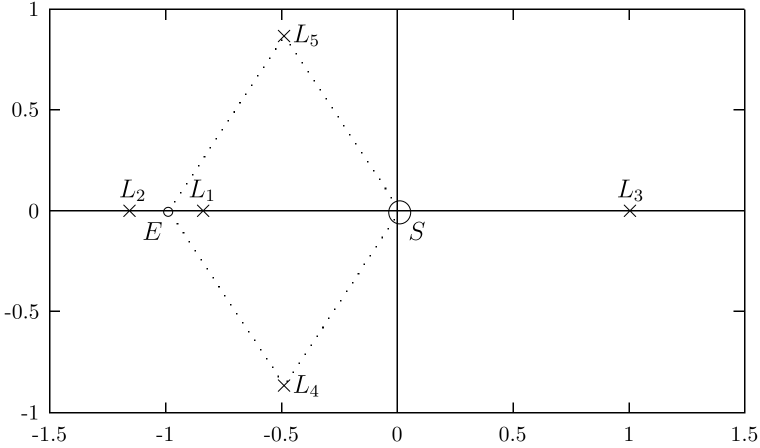
- It is a new concept of spacecraft propulsion that takes advantage of the Solar radiation pressure to propel a satellite. The impact of the photons emitted by the Sun on the surface of the sail and its further reflection produce momentum on it.
- Solar Sails open a new range of applications not accessible by a traditional spacecraft.



Several solar sails have already been in space:

- IKAROS: (Interplanetary Kite-craft Accelerated by Radiation Of the Sun). It is a Japan Aerospace Exploration Agency experimental spacecraft with a 14×14 m^2. sail. The spacecraft was launched on May 21st 2010, together with Akatsuki (Venus Climate Orbiter). On December 8th 2010, IKAROS passed by Venus at about 80,800 km distance.
- NanoSail-D2: On January 2011 NASA deployed a small solar sail (10 m^2, 4kg.) in a low Earth orbit. It reentered the atmosphere on September 17th 2011.
- LightSail-A: This is a small test spacecraft (32 m^2) of the Planetary Society. It has been launched on May 20th 2015 and it deployed its solar sail on June 7th 2015. It has reentered the atmosphere on June 14th 2015.

The Restricted Three-Body Problem



The Restricted Three-Body Problem

Defining momenta as $P_X = \dot{X} - Y$, $P_Y = \dot{Y} + X$ and $P_Z = \dot{Z}$, the equations of motion can be written in Hamiltonian form. The corresponding Hamiltonian function is

$$H = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

being $r_1^2 = (X-\mu)^2 + Y^2 + Z^2$ and $r_2^2 = (X-\mu+1)^2 + Y^2 + Z^2$.

The Solar Sail

As a first model, we consider a flat and perfectly reflecting Solar Sail: the force due to the solar radiation pressure is normal to the surface of the sail ($\vec{n}$), and it is defined by the *sail orientation* and the *sail lightness number*.

- The *sail orientation* is given by the normal vector to the surface of the sail, $\vec{n}$. It is parametrised by two angles, α and δ.
- The *sail lightness number* is given in terms of the dimensionless parameter β. It measures the effectiveness of the sail.

The acceleration of the sail due to the radiation pressure is given by:

$$\vec{a}_{sail} = \beta \frac{m_s}{r_{ps}^2} \langle \vec{r}_s, \vec{n} \rangle^2 \vec{n}.$$

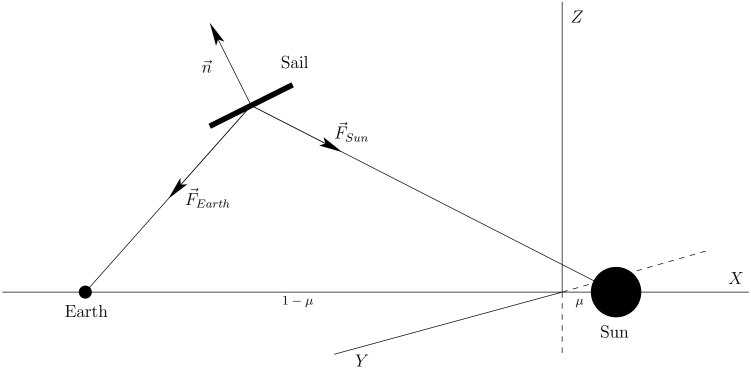
The Sail Effectiveness

The parameter β is defined as the ratio of the solar radiation pressure in terms of the solar gravitational attraction.

With nowadays technology, it is considered reasonable to take $\beta \approx 0.05$. This means that a spacecraft of 100 kg has a sail of 58×58 m^2.

A Dynamical Model

We use the Restricted Three Body Problem (RTBP) taking the Sun and Earth as primaries and including the solar radiation pressure.



Equations of Motion

The equations of motion are:

$$\begin{aligned}
\ddot{x} &= 2\dot{y} + x - (1-\mu)\frac{x-\mu}{r_{ps}^3} - \mu\frac{x+1-\mu}{r_{pe}^3} + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_x, \\
\ddot{y} &= -2\dot{x} + y - \left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) y + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_y, \\
\ddot{z} &= -\left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) z + \beta\frac{1-\mu}{r_{ps}^2}\langle\vec{r}_s,\vec{n}\rangle^2 n_z,
\end{aligned}$$

where $\vec{n} = (n_x, n_y, n_z)$ is the normal to the surface of the sail with

$$\begin{aligned}
n_x &= \cos(\phi(x,y)+\alpha)\cos(\psi(x,y,z)+\delta), \\
n_y &= \sin(\phi(x,y,z)+\alpha)\cos(\psi(x,y,z)+\delta), \\
n_z &= \sin(\psi(x,y,z)+\delta),
\end{aligned}$$

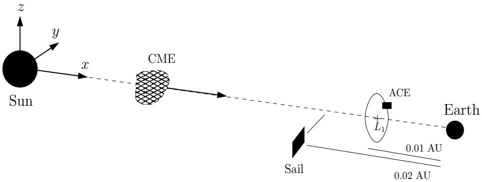
and $\vec{r}_s = (x-\mu, y, z)/r_{ps}$ is the Sun - sail direction.

Equilibrium Points

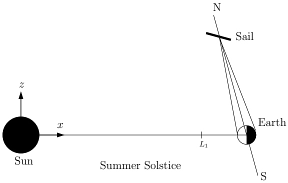
- The RTBP has 5 equilibrium points (L_i, $i = 1, \ldots, 5$). For small β, these 5 points are replaced by 5 continuous families of equilibria, parametrised by α and δ.
- For a small value of β, we have 5 disconnected families of equilibria near the classical L_i.
- For a fixed and larger β, these families merge into each other. We end up having two disconnected surfaces, S_1 and S_2, where S_1 is like a sphere and S_2 is like a torus around the Sun.
- All these families can be computed numerically by means of a continuation method.

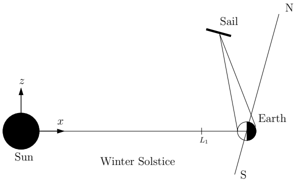
Interesting Missions Applications

Observations of the Sun provide information of the geomagnetic storms, as in the Geostorm Warning Mission.



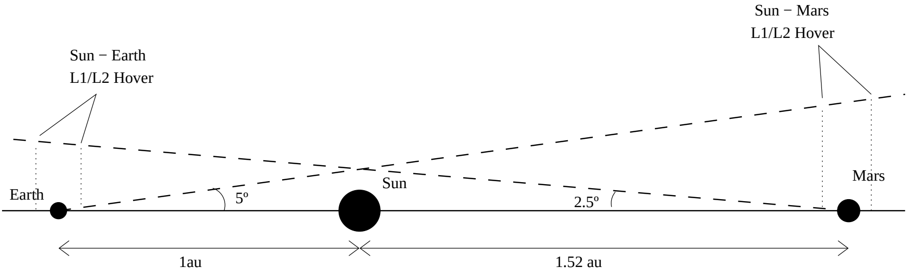
Observations of the Earth's poles, as in the Polar Observer.





Interesting Missions Applications

To ensure reliable radio communication between Mars and Earth even when the planets are lined up at opposite sides of the Sun.



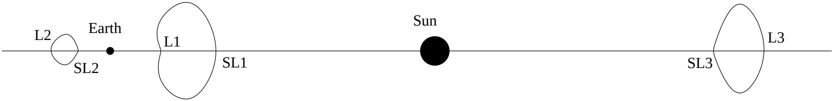
Periodic Motion Around Equilibria

We must add a constrain on the sail orientation to find bounded motion. One can see that when $\alpha = 0$ and $\delta \in [-\pi/2, \pi/2]$ (i.e. only move the sail vertically w.r.t. the Sun - sail line):

- The system is time reversible $\forall \delta$ by $R : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t)$ and Hamiltonian only for $\delta = 0, \pm\pi/2$.
- There are 5 disconnected families of equilibrium points parametrised by δ, we call them $FL_{1,\dots,5}$ (each one related to one of the Lagrangian points $L_{1,\dots,5}$).
- Three of these families ($FL_{1,2,3}$) lie on the $Y = 0$ plane, and the linear behaviour around them is of the type saddle$\times$centre$\times$centre.
- The other two families ($FL_{4,5}$) are close to $L_{4,5}$, and the linear behaviour around them is of the type sink$\times$sink$\times$source or sink$\times$source$\times$source.

We focus on ...

- We focus on the motion around the equilibrium on the FL_1 family close to SL_1 (they correspond to $\alpha = 0$ and $\delta \approx 0$).
- We fix $\beta = 0.051689$.
- We consider the sail orientation to be fixed along time.

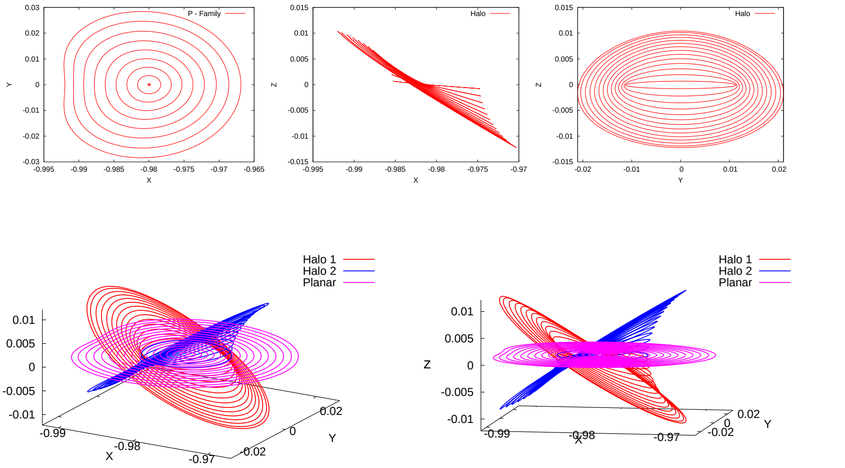


(Schematic representation of the equilibrium points on $Y = 0$)

Let us see the periodic motion around these points for a fixed sail orientation and show how it varies when we change, slightly, the sail orientation.

$\mathcal{P}$-Family of Periodic Orbits

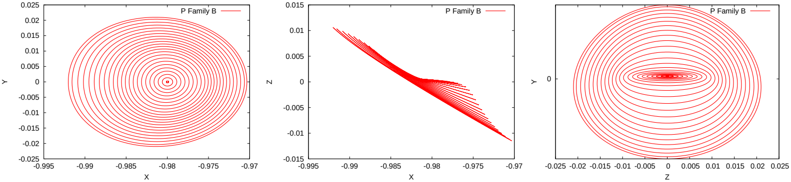
Periodic Orbits for $\delta = 0$.



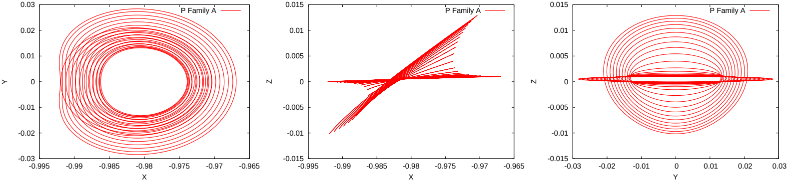
$\mathcal{P}$-Family of Periodic Orbits

Periodic Orbits for $\delta = 0.01$.

Main family of periodic orbits for $\delta = 0.01$

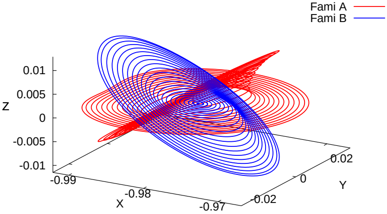


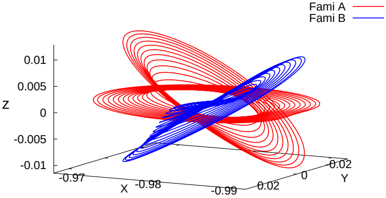
Secondary family of periodic orbits for $\delta = 0.01$

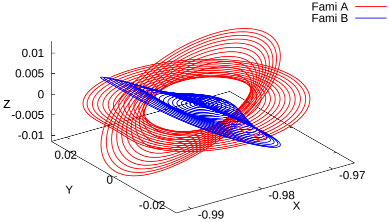


$\mathcal{P}$-Family of Periodic Orbits

Periodic Orbits for $\delta = 0.01$.

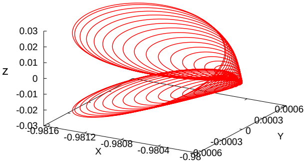




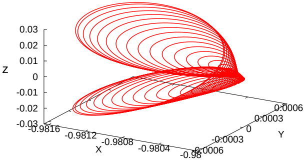


$\mathcal{V}$ - Family of Periodic Orbits

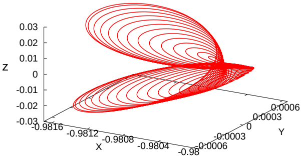
δ = 0



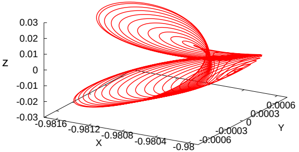
δ = 0.001



δ = 0.005



δ = 0.01



Station Keeping around Equilibria

Station Keeping around equilibria

Goal:

- Design station keeping strategy to maintain the trajectory of a solar sail close to an unstable equilibrium point.
- We want to use *Dynamical Systems Tools* to find a station keeping algorithm for a Solar Sail.

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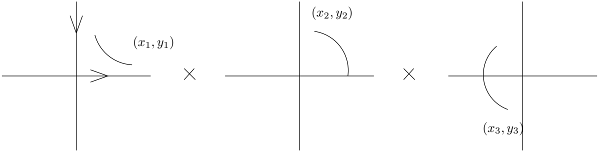
Idea:

- We focus on the linear dynamics around an equilibrium point and study how this one varies when the sail orientation changes.
- We want to change the sail orientation (i.e. the phase space) to make the natural dynamics act in our favour: keep the trajectory close to a given equilibrium point.

We focus on the previous missions, where the equilibrium points are unstable with two real eigenvalues, $\lambda_1 > 0, \lambda_2 < 0$, and two pair of complex eigenvalues, $\nu_{1,2} \pm \mathrm{i}\,\omega_{1,2}$, with $|\nu_{1,2}| << |\lambda_{1,2}|$.

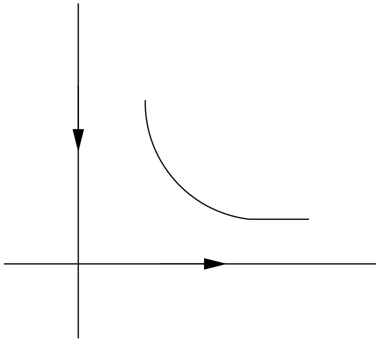
- The linear dynamics at the equilibrium point is of the type saddle $\times$ centre $\times$ centre.
- We describe the trajectory of the sail in three reference planes defined by the eigendirections.



- For small variations of the sail orientation, the equilibrium point, eigenvalues and eigendirections have a small variation. We will describe the effects of the changes on the sail orientation on each of these three reference planes.

Schematic Idea of the Station Keeping Strategy (I)

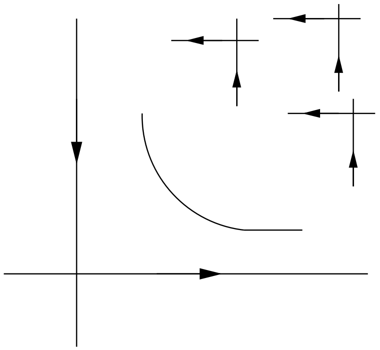
In the saddle projection of the trajectory:



- When we are close to the equilibrium point, p_0, the trajectory escapes along the unstable direction.

Schematic Idea of the Station Keeping Strategy (I)

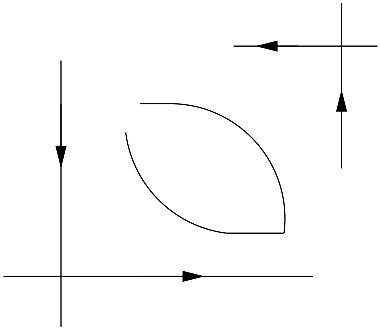
In the saddle projection of the trajectory:



- When we are close to the equilibrium point, p_0, the trajectory escapes along the unstable direction.
- If we change the sail orientation the equilibrium point is shifted. Now the trajectory will escape along the new unstable direction.

Schematic Idea of the Station Keeping Strategy (I)

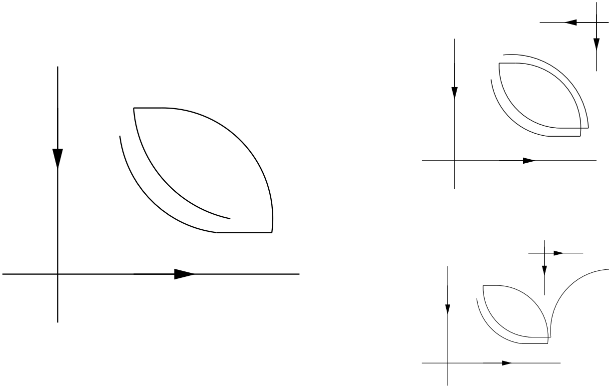
In the saddle projection of the trajectory:



- When we are close to the equilibrium point, p_0, the trajectory escapes along the unstable direction.
- If we change the sail orientation the equilibrium point is shifted. Now the trajectory will escape along the new unstable direction.
- We want to find a new sail orientation (α, δ) so that the trajectory will come close to the stable direction of p_0.

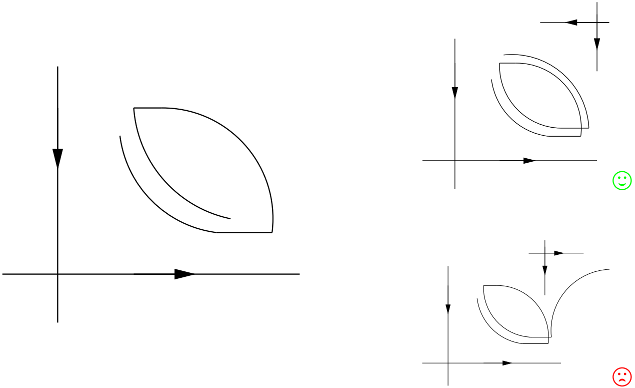
Schematic Idea of the Station Keeping Strategy (II)

In the saddle projection of the trajectory:



Schematic Idea of the Station Keeping Strategy (II)

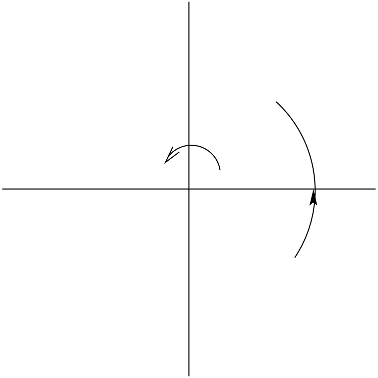
In the saddle projection of the trajectory:



- With these ideas we can control the instability due to the saddle.
- We need to take into account the centre projection of the trajectory, as it might grow.

Schematic Idea of the Station Keeping Strategy (III)

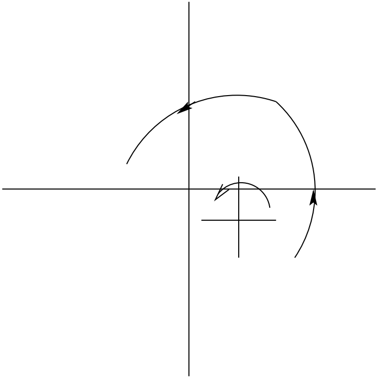
In the centre projection of the trajectory:



- When we are close to the equilibrium point the trajectory is a rotation.

Schematic Idea of the Station Keeping Strategy (III)

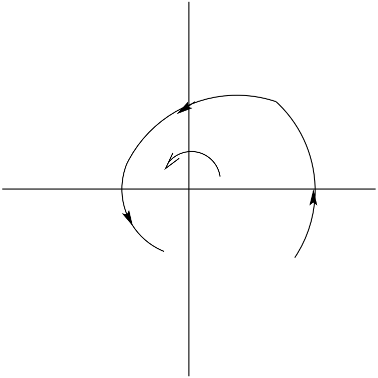
In the centre projection of the trajectory:



- When we are close to the equilibrium point the trajectory is a rotation.
- If we change the sail orientation the equilibrium point is shifted. Now the trajectory will rotate around the new equilibrium point.

Schematic Idea of the Station Keeping Strategy (III)

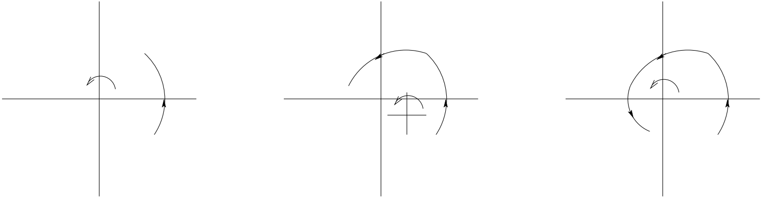
In the centre projection of the trajectory:



- When we are close to the equilibrium point the trajectory is a rotation.
- If we change the sail orientation the equilibrium point is shifted. Now the trajectory will rotate around the new equilibrium point.
- A sequence of changes on the sail orientation results in a sequence of rotations around the different equilibrium points.

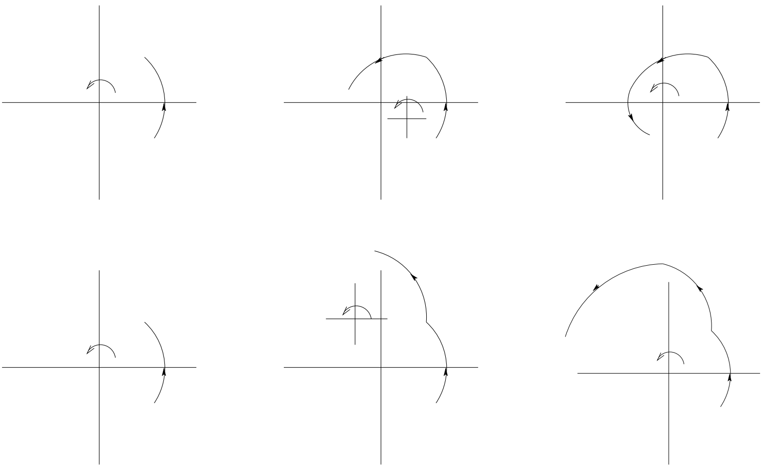
Schematic Idea of the Station Keeping Strategy (IV)

In the centre projection of the trajectory:



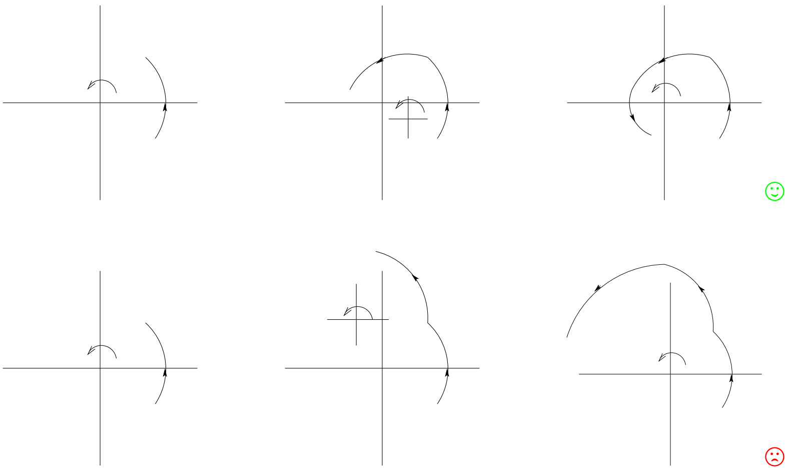
Schematic Idea of the Station Keeping Strategy (IV)

In the centre projection of the trajectory:



Schematic Idea of the Station Keeping Strategy (IV)

In the centre projection of the trajectory:



Schematic Idea of the Station Keeping Strategy (V)

In the centre projection of the trajectory:

- A sequence of changes on the sail orientation implies a sequence of rotations around different equilibrium points on the centre projection.
- As we have seen a sequence of rotations around different equilibrium points can result unbounded.
- How can we choose the position of the new equilibrium point on the centre projection to keep this projection bounded ?

Schematic Idea of the Station Keeping Strategy (V)

In the centre projection of the trajectory:

- A sequence of changes on the sail orientation implies a sequence of rotations around different equilibrium points on the centre projection.
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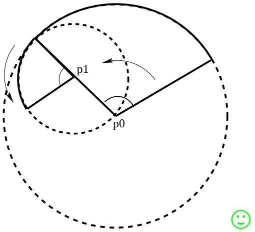
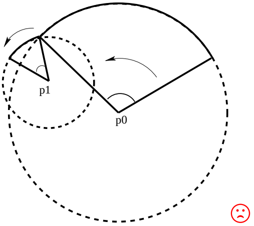
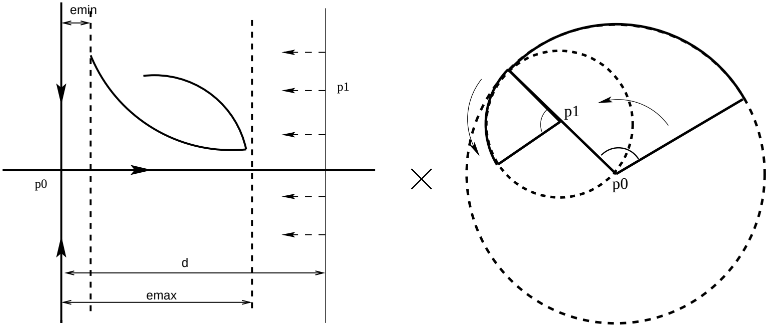




Schematic Idea of the Station Keeping Strategy (VI)

To control the saddle and centre projection we want the new equilibrium point to satisfy:



The constants ε_{min}, ε_{max} and d will depend on the mission requirements and the dynamics around the equilibrium point.

Results

We have applied this station keeping strategy to two different mission applications, the *Geostorm Warning Mission* and the *Polar Observer*.

For each mission:

- We have done a Monte Carlo simulation taking a 1000 random initial conditions.
- For each simulation we have applied the station keeping strategy for 30 years.
- We have tested the robustness of our strategy including random errors on the position and velocity determination, as well as on the orientation of the sail at each manoeuvre.

Results for the Geostorm

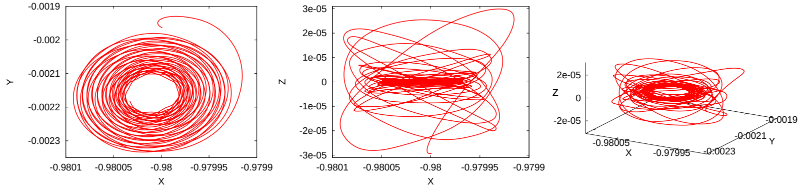
We take $\beta = 0.051689$ (i.e. a satellite of $130kg$ mass with a $67m \times 67m$ square sail).

	Success	Max. Time	Min. Time	Ang. Vari.
No Error	100 %	45.87 days	24.13 days	1.43°
Error Pos.	100 %	45.85 days	24.13 days	1.43°
Error Pos. & Ori. *	100 %	53.90 days	21.59 days	1.42°
Error Pos. & Ori. †	97 %	216.47 days	15.54 days	1.67°

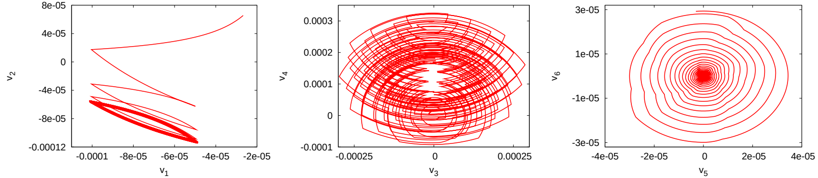
Statistics for the Geostorm mission taking 1000 simulations. Considering errors on the sail orientation of order 0.5° *(***) and* 2.2° *(*†*).*

Results for the Geostorm (No Errors in Manoeuvres)

XY and XZ and XYZ Projections

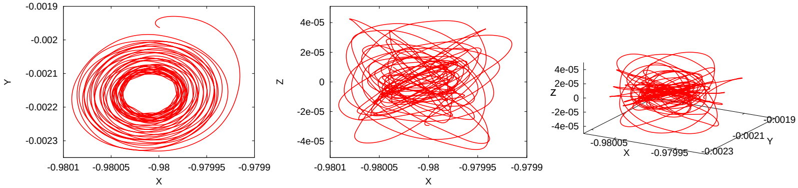


Saddle × Centre × Centre Projections

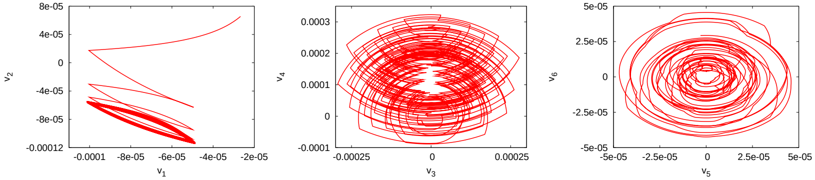


Results for the Geostorm (Errors in Manoeuvres)

XY and XZ and XYZ Projections

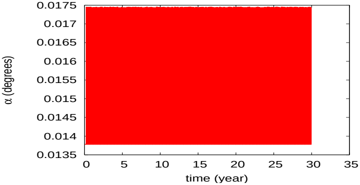


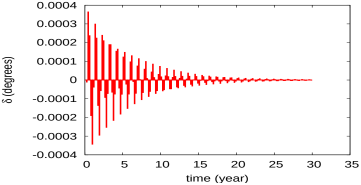
Saddle × Centre × Centre Projections



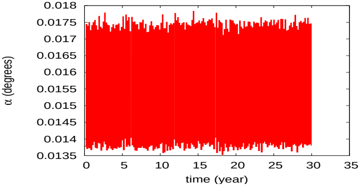
Results for the Geostorm

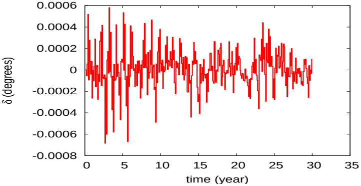
Variation of the sail orientation (No Errors in Manoeuvres)





Variation of the sail orientation (Errors in Manoeuvres)





Results

- We manage to maintain the trajectory close to the equilibrium point for 30 years.
- The most significant errors are the ones due to the sail orientation.
- This station keeping strategy does not require previous planning as the decisions taken by the sail only depend on its position in the phase space.
- The same ideas can be used to design strategies to move along the family of equilibrium points.

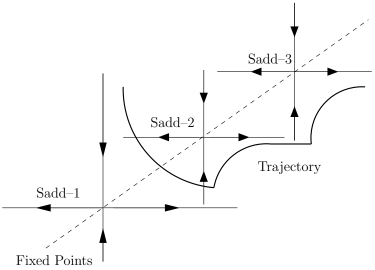
Navigation along families of equilibrium points

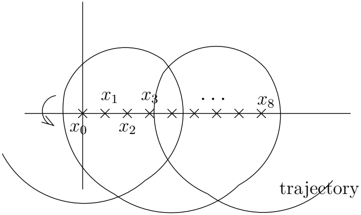
If we can control the sail so that it stays near a fixed point, we can use the same strategy to navigate along a family of equilibria.

The idea is to move the orientation of the sail so that the equilibrium point is displaced, and then to start the control algorithm for the new point.

Surfing along the family of equilibria

Scheme on the idea to surf along the family of equilibria.





Dynamics near an asteroid

The Hill's Problem

G.W. Hill introduced a simplified version of the Restricted Three-Body Problem to study the motion of the Moon.

In this model,

- Moon has zero mass,
- Earth is fixed at the origin,
- Sun is so far away that its gravitational attraction is constant.

This model is Hamiltonian.

The Hill's Problem

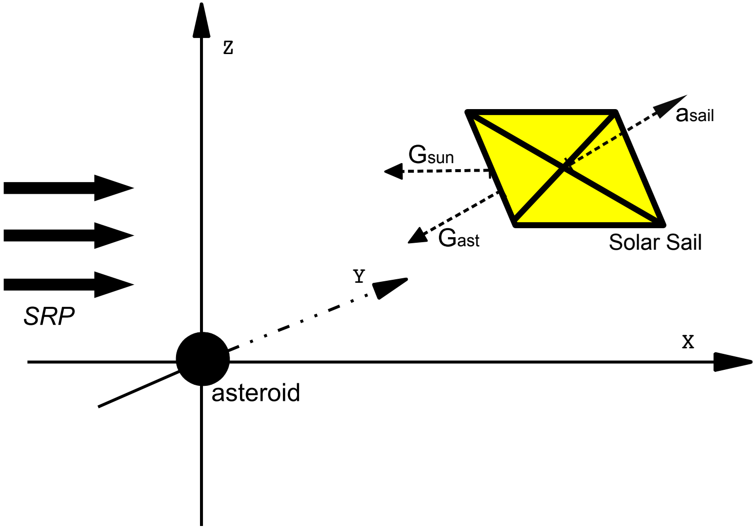
The equations of motion are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= 3x - \frac{x}{r^3}, \\ \ddot{y} + 2\dot{x} &= \frac{y}{r^3}, \\ \ddot{z} &= -z - \frac{z}{r^3}, \end{aligned}$$

where $r^2 = x^2 + y^2 + z^2$.

This model can be adequate as a first approximation to study the dynamics of a spacecraft near an asteroid.

The Augmented Hill Problem



Equations of motion

$$\begin{aligned}\ddot{X} - 2\dot{Y} &= -\frac{X}{r^3} + 3X + a_x,\\ \ddot{Y} + 2\dot{X} &= -\frac{Y}{r^3} + a_y,\\ \ddot{Z} &= -\frac{Z}{r^3} - Z + a_z,\end{aligned}$$

- (X, Y, Z) denotes the position of the solar sail in a rotating frame.
- $r = \sqrt{X^2 + Y^2 + Z^2}$.
- $\mathbf{a} = (a_x, a_y, a_z)$ is the acceleration given by the solar sail.

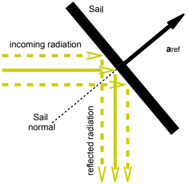
The normalised units of distance and time are $L = (\mu_{sb}/\mu_{sun})^{1/3} R$ and $T = 1/\omega$, where μ_{sb}, μ_{sun} are gravitational parameters for the small body and the Sun, R is Sun - asteroid mean distance, and $\omega = \sqrt{\mu_{sun}/R^3}$ is its frequency.

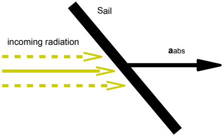
Solar Sail model

We consider the simplified model for a solar sail†: flat and non-perfectly reflecting

Reflectivity $\mathbf{F_r} = 2PA\,\langle \mathbf{r_s}, \mathbf{n}\rangle^2\ \mathbf{n}$

Absorption $\mathbf{F_a} = PA\,\langle \mathbf{r_s}, \mathbf{n}\rangle\ \mathbf{r_s}$





$$\mathbf{a_{sail}} = \frac{2PA}{m}\langle \mathbf{r_s}, \mathbf{n}\rangle \left(\rho\langle \mathbf{r_s}, \mathbf{n}\rangle\mathbf{n} + \frac{1}{2}(1-\rho)\mathbf{r_s} \right)$$

Where, if ρ is the reflectivity coefficient and a is the absorption coefficient, then $\rho + a = 1$ (*Notice that $\rho = 0$ corresponds to a solar-panel and $\rho = 1$ to a perfectly reflecting solar sail*).

Solar Sail model

The acceleration given by the solar sail $\mathbf{a} = (a_x, a_y, a_z)$ is:

$$\begin{aligned} a_x &= \beta(\rho \cos^3 \alpha \cos^3 \delta + 0.5(1-\rho) \cos \alpha \cos \delta), \\ a_y &= \beta(\rho \cos^2 \alpha \cos^3 \delta \sin \alpha), \\ a_z &= \beta(\rho \cos^2 \alpha \cos^2 \delta \sin \delta), \end{aligned}$$

Remarks:

- α and δ are the angles that define the orientation of the sail.
- In the normalised units $\beta = K_1(A/m)\mu_{sb}^{-1/3}$, where $K_1 \approx 7.8502$ if A is given in m^2 and m in kg.
- $\rho \approx 0$ corresponds to the performance of a solar panel.
- $\rho \approx 1$ corresponds to a high performance solar sail.

Hamiltonian function

Defining momenta as $P_X = \dot{X} - Y$, $P_Y = \dot{Y} + X$, $P_Z = \dot{Z}$, this system is described by the Hamiltonian function

$$\begin{aligned} H &= \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{1}{2}(2X^2 - Y^2 - Z^2) - \frac{1}{r} \\ &\quad -a_x X - a_y Y - a_z Z. \end{aligned}$$

Equilibrium points

It is well-known that, if we neglect the effect of the solar sail ($\beta = 0$) the system has two equilibrium points, $L_{1,2}$, symmetrically located around the asteroid, with coordinates $(\pm 3^{-1/3}, 0, 0)$.

If the sail is perpendicular to the Sun direction ($\alpha = \delta = 0$), the position of $L_{1,2}$ move towards the Sun as β increases.

Periodic orbits

- The equilibrium points are unstable (centre$\times$centre$\times$saddle).
- Each centre gives rise to a family of (unstable) periodic orbits.
- The two centres give rise to a Cantor family of (unstable) 2D tori.

To visualise the dynamics, we will perform the so-called reduction to the centre manifold.

It is based on performing a sequence of normalising transformations on the Hamiltonian function, with the only purpose of decoupling the centre directions from the hyperbolic ones.

To describe this process, let us assume that the equilibrium point has already been translated to the origin.

Second order normal form

Now, the Hamiltonian takes the form

$$H(q,p) = H_2(q,p) + \sum_{n\geq 3} H_n(q,p),$$

where $H_2 = \lambda_1 q_1 p_1 + \sqrt{-1}\omega_1 q_2 p_2 + \sqrt{-1}\omega_2 q_3 p_3$ and H_n denotes an homogeneous polynomial of degree n.

The Lie series method

The changes of variables are implemented by means of the Lie series method: if $G(q,p)$ is a Hamiltonian system, then the function $\hat{H}$ defined by

$$\hat{H} \equiv H + \{H, G\} + \frac{1}{2!}\{\{H, G\}, G\} + \frac{1}{3!}\{\{\{H, G\}, G\}, G\} + \cdots ,$$

is the result of applying a canonical change to H. This change is the time one flow corresponding to the Hamiltonian G. G is usually called the generating function of the transformation.

It is easy to check that, if P and Q are two homogeneous polynomials of degree r and s respectively, then $\{P, Q\}$ is a homogeneous polynomial of degree $r + s - 2$.

This property is very useful to implement in a computer a transformation given by a generating transformation G.

For instance, let us assume that we want to eliminate the monomials of degree 3, as it is usually done in a normal form scheme.

Let us select as a generating function a homogeneous polynomial of degree 3, G_3. Then, it is immediate to check that the terms of $\hat{H}$ satisfy

- degree 2: $\hat{H}_2 = H_2$,
- degree 3: $\hat{H}_3 = H_3 + \{H_2, G_3\}$,
- degree 4: $\hat{H}_4 = H_4 + \{H_3, G_3\} + \frac{1}{2!}\{\{H_2, G_3\}, G_3\}$,
- $\vdots$

Hence, to kill the monomials of degree 3 one has to look for a G_3 such that $\{H_2, G_3\} = -H_3$.

Let us denote

$$H_3(q,p) = \sum_{|k_q|+|k_p|=3} h_{k_q,k_p} q^{k_q} p^{k_p},$$
$$G_3(q,p) = \sum_{|k_q|+|k_p|=3} g_{k_q,k_p} q^{k_q} p^{k_p},$$

where $\eta_1 = \lambda_1$, $\eta_2 = \sqrt{-1}\omega_1$ and $\eta_3 = \sqrt{-1}\omega_2$. As

$$\{H_2, G_3\} = \sum_{|k_q|+|k_p|=3} \langle k_p - k_q, \eta\rangle \, g_{k_q,k_p} q^{k_q} p^{k_p}, \quad \eta = (\eta_1, \eta_2, \eta_3),$$

it is immediate to obtain

$$G_3(q,p) = \sum_{|k_q|+|k_p|=3} \frac{-h_{k_q,k_p}}{\langle k_p - k_q, \eta\rangle} q^{k_q} p^{k_p}.$$

Observe that $|k_q| + |k_p| = 3$ implies $\langle k_p - k_q, \eta\rangle \neq 0$. Note that G_3 is so easily obtained because of the "diagonal" form of H_2.

We are not interested in a complete normal form, but only in uncoupling the central directions from the hyperbolic one.

Hence, it is not necessary to cancel all the monomials in H_3 but only some of them. Moreover, as we want the radius of convergence of the transformed Hamiltonian to be as big as possible, we will try to choose the change of variables as close to the identity as possible. This means that we will kill the least possible number of monomials in the Hamiltonian.

To produce an approximate first integral having the center manifold as a level surface (see below), it is enough to kill the monomials $q^{k_q} p^{k_p}$ such that the first component of k_q is different from the first component of k_p

This implies that the generating function G_3 is

$$G_3(q,p) = \sum_{(k_q,k_p)\in\mathcal{S}_3} \frac{-h_{k_q,k_p}}{\langle k_p - k_q, \eta\rangle} q^{k_q} p^{k_p},$$

where $\mathcal{S}_n$, $n \geq 3$, is the set of indices (k_q, k_p) such that $|k_q| + |k_p| = n$ and the first component of k_q is different from the first component of k_p.

Then, the transformed Hamiltonian $\hat{H}$ takes the form

$$\hat{H}(q,p) = H_2(q,p) + \hat{H}_3(q,p) + \hat{H}_4(q,p) + \cdots,$$

where $\hat{H}_3(q,p) \equiv \hat{H}_3(q_1p_1, q_2, p_2, q_3, p_3)$ (note that $\hat{H}_3$ depends on the product q_1p_1, not on each variable separately).

This process can be carried out up to a finite order N, to obtain a Hamiltonian of the form

$$\bar{H}(q,p) = \bar{H}_N(q,p) + R_N(q,p),$$

where $H_N(q,p) \equiv H_N(q_1 p_1, q_2, p_2, q_3, p_3)$ is a polynomial of degree N and R_N is a remainder of order $N+1$ (note that H_N depends on the product $q_1 p_1$ while the remainder depends on the two variables q_1 and p_1 separately).

Neglecting the remainder and applying the canonical change given by $I_1 = q_1 p_1$, we obtain the Hamiltonian $\bar{H}_N(I_1, q_2, p_2, q_3, p_3)$ that has I_1 as a first integral.

Setting $I_1 = 0$ we obtain a 2DOF Hamiltonian, $\bar{H}_N(0, \bar{q}, \bar{p})$, $\bar{q} = (q_2, q_3)$, $\bar{p} = (p_2, p_3)$, that represents (up to some finite order N) the dynamics inside the center manifold.

Note the absence of small divisors during this process.

The denominators that appear in the generating functions, $\langle k_p - k_q, \eta \rangle$, can be bounded from below when $(k_q, k_p) \in \mathcal{S}_N$: using that η_1 is real and that $\eta_{2,3}$ are purely imaginary, we have

$$|\langle k_p - k_q, \eta \rangle| \geq |\lambda_1|, \qquad \text{for all } (k_q, k_p) \in \mathcal{S}_N, \quad N \geq 3.$$

For this reason, the divergence of this process is very mild.

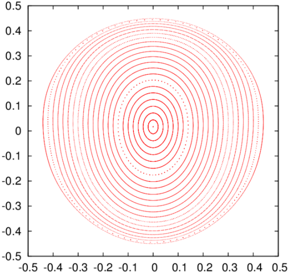
Displaying the dynamics

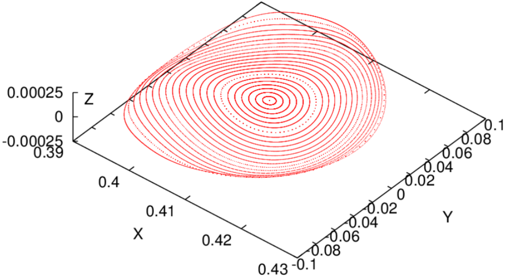
To display the dynamics, let us call (q_h, p_h) the variables in the normalised coordinates related to the horizontal oscillations, and (q_v, p_v) the variables related to the vertical oscillations.

We consider the Poincaré section $q_v = 0$ (in other words, we are "slicing" the vertical motions).

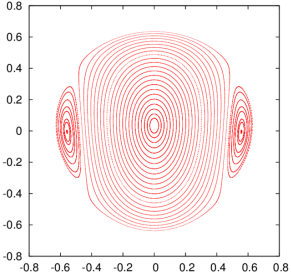
Let us consider the case $\alpha = \delta = 0$ and select the energy level $H_{cm} = 0.4$ (corresponding to $H = -4.519072$ in synodical coordinates).

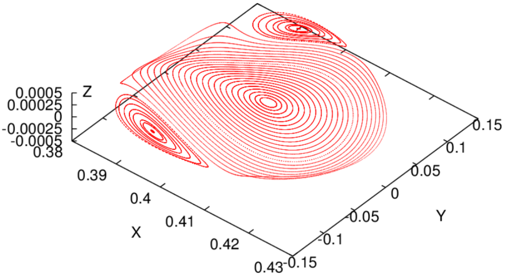
$H_{cm} = 0.4,\ \alpha = \delta = 0$



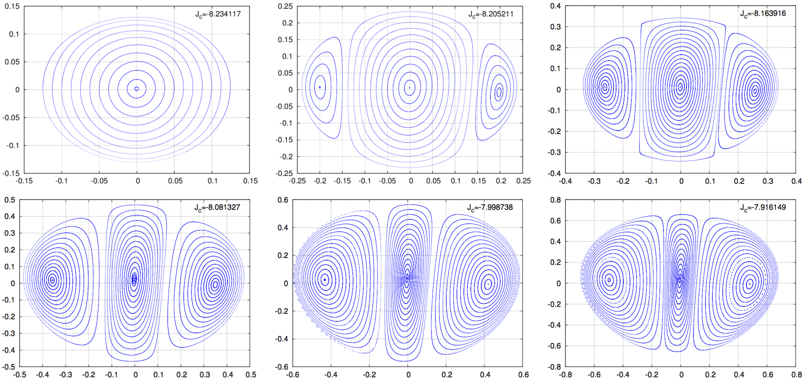


$H_{cm} = 0.8$, $\alpha = \delta = 0$

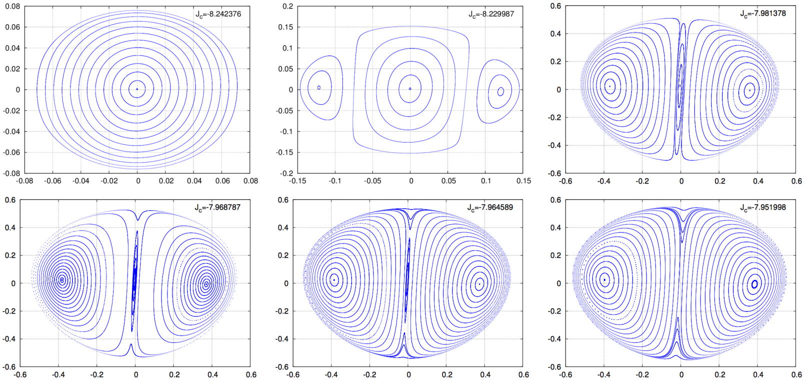




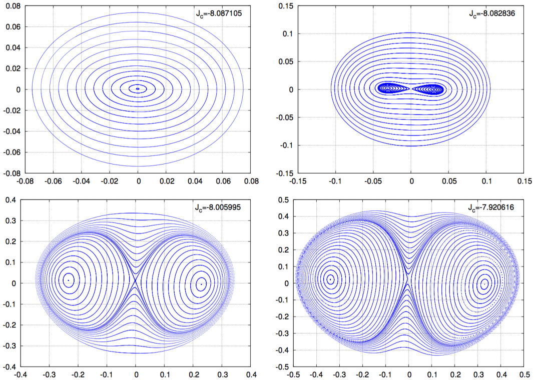
Dynamics on the Centre Manifold ($\alpha = 0.48, \delta = 0$)



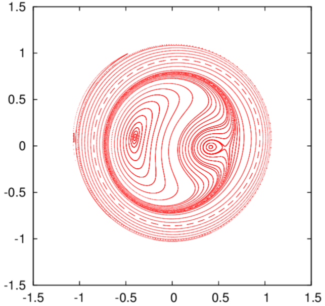
Dynamics on Centre Manifold ($\alpha = 0.50, \delta = 0$)

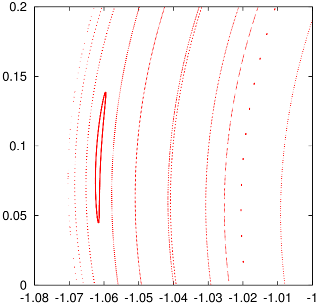


Dynamics on Centre Manifold ($\alpha = 0.52, \delta = 0$)



$H_{cm} = 2.3$, $\alpha = 0$, $\delta = 0.1$

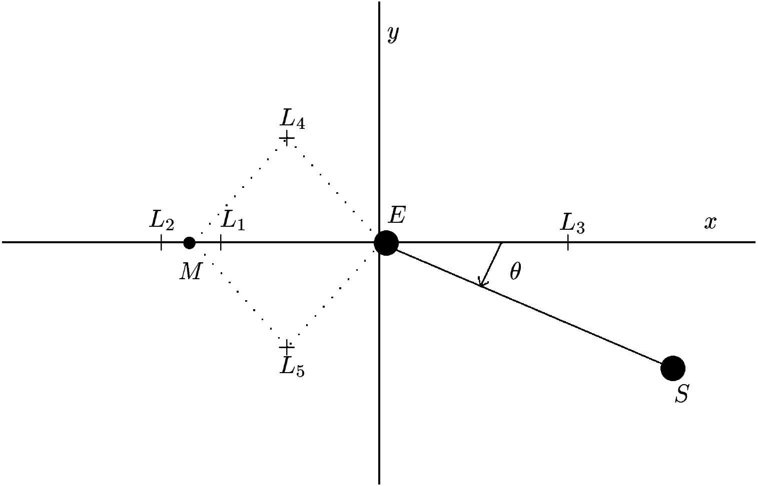




Periodic time-dependent effects

The Bicircular problem

It is a model for the study of the dynamics of a small particle in the Earth-Moon-Sun system.



The Bicircular problem

The BCP can be described by the Hamiltonian system,

$$
\begin{aligned}
H_{BCP} &= \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + y p_x - x p_y - \frac{1-\mu}{r_{PE}} - \frac{\mu}{r_{PM}} - \frac{m_S}{r_{PS}} \\
&\quad - \frac{m_S}{a_S^2}\left(y \sin\theta - x\cos\theta\right),
\end{aligned}
$$

where

$$
\begin{aligned}
r_{PE}^2 &= (x-\mu)^2 + y^2 + z^2, \\
r_{PM}^2 &= (x-\mu+1)^2 + y^2 + z^2, \\
r_{PS}^2 &= (x-x_S)^2 + (y-y_S)^2 + z^2,
\end{aligned}
$$

being $x_S = a_S \cos\theta$, $y_S = -a_S \sin\theta$ and $\theta = \omega_S t$.

The Augmented Bicircular model

The Hamiltonian function of the augmented model reads as:

$$H = H_{BCP} - \frac{\beta m_S}{a_S^2}\langle \vec{ss}, \vec{e}\rangle.$$

Here, $\vec{e} = (x, y, z)^T$ and the vector $\vec{ss} = (ss^x, ss^y, ss^z)$ is the orientation of the sail.

The system depends on two parameters β, the effectivity and of two angles (δ, α) that define the orientation.

On the computation of normal forms

We have implemented a software to cope with normal forms (and centre manifold reduction) around periodic orbits based on Lie transforms.

- By choosing suitable generating functions we are able to remove the monomials of a selected set. That is, we cast the original Hamiltonian to:

$$H_2 + \sum_{2<|k|\leq r, k\in M} N_k + \mathcal{R}^{[>r]}$$

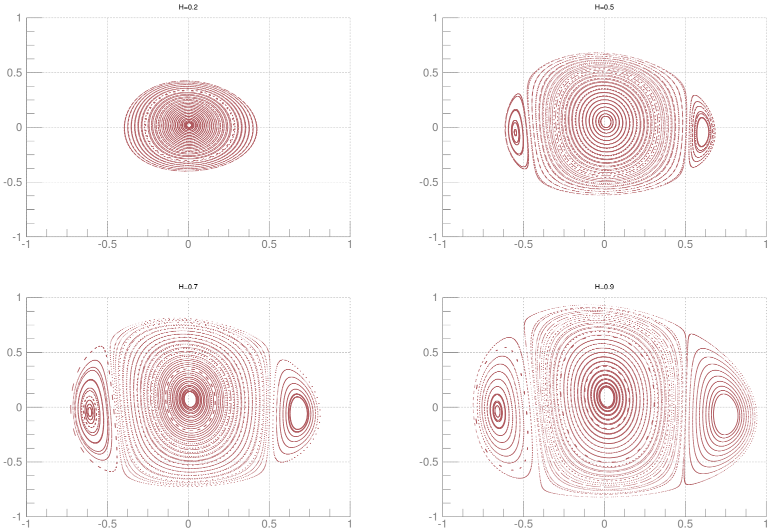
- The H_2 is arranged by using the Floquet theorem. Second order is reduced to constant coefficients. We can also remove the time dependence of the Hamiltonian. This introduces small divisors (even in the case of CMR).

On the computation of normal forms

- The main point of the implementation is the use of the public domain package FFTW3 (Frigo & Johnson, 2005). With this, operations with truncated Fourier series are of order, at most, $N \log N$ complex operations.
- A Taylor-Fourier arithmetic with this feature is very fast. For instance, a reduction to the centre manifold near a saddle$\times$ centre $\times$ centre p.o. ($N = 64$) up to order 10 takes about 17 seconds. Order 16 takes less than 3 minutes and order 20 about 20 minutes.

Next slide shows an horizontal section for the Centre Manifold of L_1 (BCP). The expansion used for the Hamiltonian is of order 12. The planar plots are obtained fixing the energy h at 0.2, 0.5, 0.7 and 0.9. Horizontal axis: q_1. Vertical axis: q_3.

Test example: The centre manifold of L_1 in the BCP



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