On the restricted three-body problem with crossing singularities

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The restricted three-body problem

Three-body problem: Sun, Earth, asteroid.

Restricted problem: the asteroid does not influence the motion of the two larger bodies.

Equations of motion of the asteroid:

$$\ddot{\boldsymbol{y}} = -G\left[m_{\odot}\frac{(\boldsymbol{y}-\boldsymbol{y}_{\odot}(t))}{|\boldsymbol{y}-\boldsymbol{y}_{\odot}(t)|^3} + m_{\oplus}\frac{(\boldsymbol{y}-\boldsymbol{y}_{\oplus}(t))}{|\boldsymbol{y}-\boldsymbol{y}_{\oplus}(t)|^3}\right]$$

- $\boldsymbol{y}$ is the unknown position of the asteroid;
- $\boldsymbol{y}_{\odot}(t), \boldsymbol{y}_{\oplus}(t)$ are known functions of time, solutions of the two-body problem Sun-Earth.

The restricted three–body problem

In heliocentric coordinates

$$\ddot{\boldsymbol{x}} = -k^2 \left[\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} + \mu \left(\frac{(\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} - \frac{\boldsymbol{x}'}{|\boldsymbol{x}'|^3} \right) \right]$$

where

- $\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{y}_\odot, \boldsymbol{x}' = \boldsymbol{y}_\oplus - \boldsymbol{y}_\odot$;
- $k^2 = Gm_\odot$, $\mu = \frac{m_\oplus}{m_\odot}$ is a small parameter;
- $-k^2\mu\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$ is the direct perturbation of the planet on the asteroid;
- $k^2\mu\frac{\boldsymbol{x}'}{|\boldsymbol{x}'|^3}$ is the indirect perturbation, due to the interaction Sun-planet.

Hint! We can model the dynamics of an asteroid in the solar system by summing up the contribution of each planet to the perturbation.

Canonical formulation of the problem

Use Delaunay's variables $\mathcal{Y} = (L, G, Z, \ell, g, z)$ for the motion of the asteroid:

$$\begin{cases} L = k\sqrt{a} \\ G = L\sqrt{1-e^2} \\ Z = G\cos I \end{cases} \qquad \begin{cases} \ell = n(t - t_0) \\ g = \omega \\ z = \Omega \end{cases}$$

These are canonical variables, representing the osculating orbit, solution of the 2-body problem Sun-asteroid.

Denote by $\mathcal{Y}' = (L', G', Z', \ell', g', z')$ Delaunay's variables for the planet.

Canonical formulation of the problem

Hamilton's equations are

$$\dot{\mathcal{Y}} = \mathbb{J}\, \nabla_{\mathcal{Y}} \mathcal{H}\,,$$

where

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1, \qquad \epsilon = \mu k^2, \qquad \mathbb{J} = \begin{bmatrix} O_3 & -I_3 \\ I_3 & O_3 \end{bmatrix}.$$

$$\mathcal{H}_0 = -\frac{k^4}{2L^2} \qquad \text{(unperturbed part)},$$

$$\mathcal{H}_1 = -\left(\frac{1}{|\mathcal{X} - \mathcal{X}'|} - \frac{\mathcal{X} \cdot \mathcal{X}'}{|\mathcal{X}'|^3} \right) \qquad \text{(perturbing function)}.$$

Here $\mathcal{X}, \mathcal{X}'$ denote $\boldsymbol{x}, \boldsymbol{x}'$ as functions of $\mathcal{Y}$, $\mathcal{Y}'$.

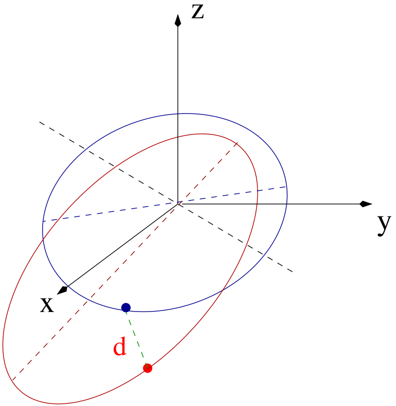
The Keplerian distance function

Let (E_j, v_j), $j = 1, 2$ be the orbital elements of two celestial bodies on Keplerian orbits with a common focus:
E_j represents the trajectory of a body,
v_j is a parameter along it. Set $V = (v_1, v_2)$.

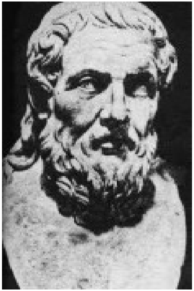
For a given two-orbit configuration $\mathcal{E} = (E_1, E_2)$, we introduce the Keplerian distance function

$$\mathbb{T}^2 \ni V \mapsto d(\mathcal{E}, V) = |\mathcal{X}_1 - \mathcal{X}_2|$$

We are interested in the local minimum points of d.



Is there still something that we do not know about distance of points on conic sections?



ἐθεώρουν σε σπεύδοντα μετασχεῖν τῶν πεπραγμένων ἡμῖν κωνικῶν (1)

(Apollonius of Perga, *Conics*, Book I)

(1) I observed you were quite eager to be kept informed of the work I was doing in conics.

Critical points of d^2

Gronchi SISC (2002), CMDA (2005)

- Apart from the case of two concentric coplanar circles, or two overlapping ellipses, d^2 has finitely many critical points.
- There exist configurations with 12 critical points, and 4 local minima of d^2.
  This is thought to be the maximum possible, but a proof is not known yet.[1]
- A simple computation shows that, for non-overlapping trajectories, the number of crossing points is at most two.

(1) Albouy, Cabral and Santos, 'Some problems on the classical n-body problem' CMDA **113/4**, 369-375 (2012)

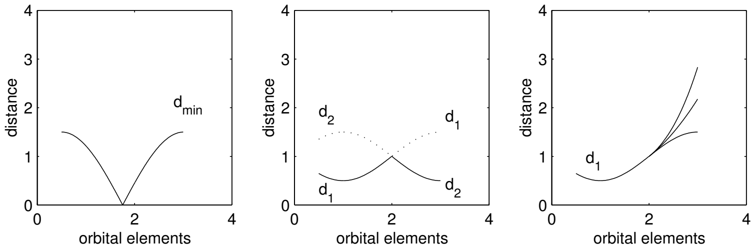
The orbit distance

Let $V_h = V_h(\mathcal{E})$ be a local minimum point of $V \mapsto d^2(\mathcal{E}, V)$. Consider the maps

$$
\begin{aligned}
\mathcal{E} &\mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h) , \\
\mathcal{E} &\mapsto d_{min}(\mathcal{E}) = \min_h d_h(\mathcal{E}) .
\end{aligned}
$$

The map $\mathcal{E} \mapsto d_{min}(\mathcal{E})$ gives the orbit distance.

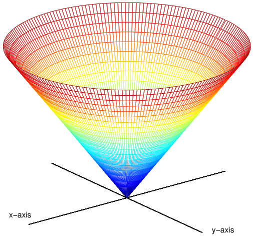
Singularities of d_h and d_{min}

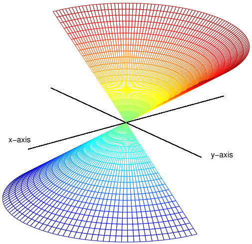


(i) d_h and d_{min} are not differentiable where they vanish;

(ii) two local minima can exchange their role as absolute minimum thus d_{min} loses its regularity without vanishing;

(iii) when a bifurcation occurs the definition of the maps d_h may become ambiguous after the bifurcation point.



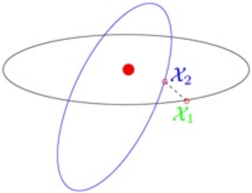


Toy problem:

$$f(x,y) = \sqrt{x^2 + y^2} \qquad \tilde{f}(x,y) = \begin{cases} -f(x,y) & \text{for } x > 0 \\ f(x,y) & \text{for } x < 0 \end{cases}$$

Can we smooth the maps $d_h(\mathcal{E})$, $d_{min}(\mathcal{E})$ through a change of sign?

Local smoothing of d_h at a crossing singularity



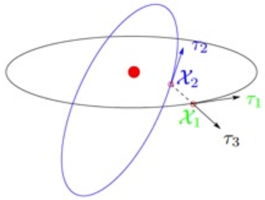
Smoothing d_h, the procedure for d_{min} is the same.

- Consider the points on the two orbits

$$\mathcal{X}_1^{(h)} = \mathcal{X}_1(E_1, v_1^{(h)})\,; \qquad \mathcal{X}_2^{(h)} = \mathcal{X}_2(E_2, v_2^{(h)})\,.$$

  corresponding to the local minimum point $V_h = (v_1^{(h)}, v_2^{(h)})$ of d^2;

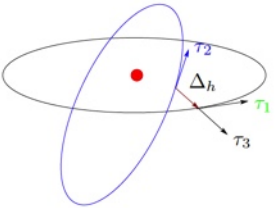
Local smoothing of d_h at a crossing singularity



- introduce the tangent vectors to the trajectories E_1, E_2 at these points:

$$\tau_1 = \frac{\partial \mathcal{X}_1}{\partial v_1}(E_1, v_1^{(h)}) , \qquad \tau_2 = \frac{\partial \mathcal{X}_2}{\partial v_2}(E_2, v_2^{(h)}) ,$$

and their cross product $\tau_3 = \tau_1 \times \tau_2$;



- define also

$$\Delta = \mathcal{X}_1 - \mathcal{X}_2 \,, \qquad \Delta_h = \mathcal{X}_1^{(h)} - \mathcal{X}_2^{(h)} \,.$$

The vector Δ_h joins the points attaining a local minimum of d^2 and $|\Delta_h| = d_h$.

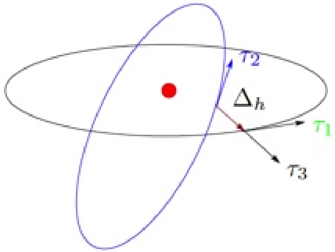
Note that $\Delta_h \times \tau_3 = 0$

Smoothing the crossing singularity

Gronchi and Tommei, DCDS-B (2007)

smoothing rule:

$$\tilde{d}_h = \mathrm{sign}(\tau_3 \cdot \Delta_h) d_h$$



$\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ is an analytic map in a neighborhood of most crossing configurations.

The averaging method

The averaging principle is used to study the qualitative behavior of solutions of ODEs in perturbation theory, see Arnold, Kozlov, Neishtadt (1997).

unperturbed
$$\begin{cases} \dot{\phi} = \omega(I) \\ \dot{I} = 0 \end{cases} \qquad \phi \in \mathbb{T}^n, I \in \mathbb{R}^m$$

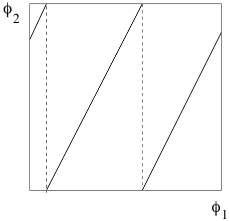
perturbed
$$\begin{cases} \dot{\phi} = \omega(I) + \epsilon f(\phi, I, \epsilon) \\ \dot{I} = \epsilon g(\phi, I, \epsilon) \end{cases}$$

averaged
$$\dot{J} = \epsilon G(J)\,, \quad G(J) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} g(\phi, J, 0)\, d\phi_1 \ldots d\phi_n$$

Averaging over 2 angular variables

Using the averaged equations corresponds to substituting the time average with the space average.

Case of 2 angles: a problem occurs if there are resonant relations of low order between the motions $\phi_1(t), \phi_2(t)$, i.e. if $h_1\dot{\phi}_1 + h_2\dot{\phi}_2 = 0$, with h_1, h_2 small integers.



Averaged equations

Gronchi and Milani, CMDA (1998)

Averaged Hamilton's equations:

$$\dot{\overline{Y}} = \epsilon \, \mathbb{J} \, \overline{\nabla_Y \mathcal{H}_1} \,, \tag{1}$$

with $Y = (G, Z, g, z)$.
If no orbit crossing occurs, (1) are equal to

$$\dot{\overline{Y}} = \epsilon \, \mathbb{J} \, \nabla_Y \overline{\mathcal{H}_1} \tag{2}$$

with

$$\overline{\mathcal{H}}_1 = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \mathcal{H}_1 \, d\ell \, d\ell' = -\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{1}{|\mathcal{X} - \mathcal{X}'|} \, d\ell \, d\ell'$$

The average of the indirect term of $\mathcal{H}_1$ is zero.

Crossing singularities

If there is an orbit crossing, then averaging on the fast angles ℓ, ℓ' produces a singularity in the averaged equations:

we take into account every possible position on the orbits, thus also the collision configurations.

$$\overline{\mathcal{H}}_1 = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{1}{|\mathcal{X}-\mathcal{X}'|}\,d\ell\,d\ell'$$

and

$$\left|\mathcal{X}(E_1, v_1^{(h)}) - \mathcal{X}'(E_2, v_2^{(h)})\right| = 0\ .$$

Near-Earth asteroids and crossing orbits

(433) Eros: the first near-Earth asteroid (NEA, with $q = a(1-e) \leq 1.3$ au), discovered in 1898; it crosses the trajectory of Mars.

from NEAR mission (NASA)

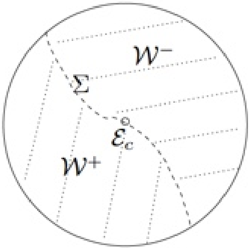
Today (March 19, 2019) we know about 19800 NEAs: several of them cross the orbit of the Earth during their evolution.

Derivative jumps

Let $\mathcal{E}_c$ be a non–degenerate crossing configuration for d_h, with only 1 crossing point.
Given a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$, we set

$$\mathcal{W}^+ = \mathcal{W} \cap \{\tilde{d}_h > 0\} \,,$$

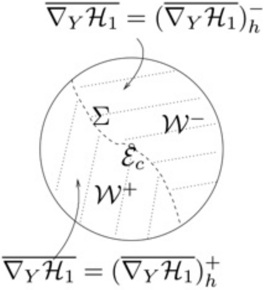
$$\mathcal{W}^- = \mathcal{W} \cap \{\tilde{d}_h < 0\} \,.$$



The averaged vector field $\overline{\nabla_Y \mathcal{H}_1}$ is not defined on $\Sigma = \{d_H = 0\}$.

Gronchi and Tardioli, DCDS-B (2013)

The averaged vector field $\overline{\nabla_Y \mathcal{H}_1}$ can be extended to two Lipschitz–continuous vector fields $(\overline{\nabla_Y \mathcal{H}_1})_h^{\pm}$ on a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$. The components of the extended fields, restricted to $\mathcal{W}^+$, $\mathcal{W}^-$ respectively, correspond to $\frac{\overline{\partial \mathcal{H}_1}}{\partial y_k}$.



Moreover the following relations hold:

$$\begin{aligned}\mathrm{Diff}_h\left(\frac{\partial\overline{\mathcal{H}_1}}{\partial y_k}\right) &\stackrel{def}{=} \left(\frac{\partial\overline{\mathcal{H}_1}}{\partial y_k}\right)_h^- - \left(\frac{\partial\overline{\mathcal{H}_1}}{\partial y_k}\right)_h^+ = \\ &= -\frac{1}{\pi}\left[\frac{\partial}{\partial y_k}\left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}}\right)\tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}}\frac{\partial\tilde{d}_h}{\partial y_k}\right],\end{aligned}$$

where y_k is a component of Delaunay's elements Y.

Generalized solutions

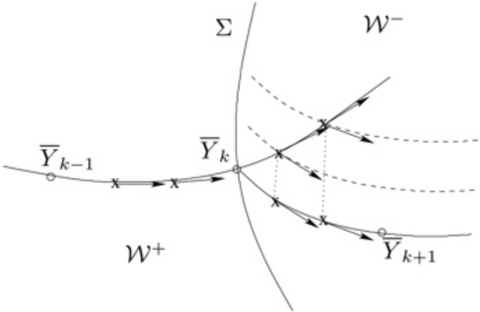
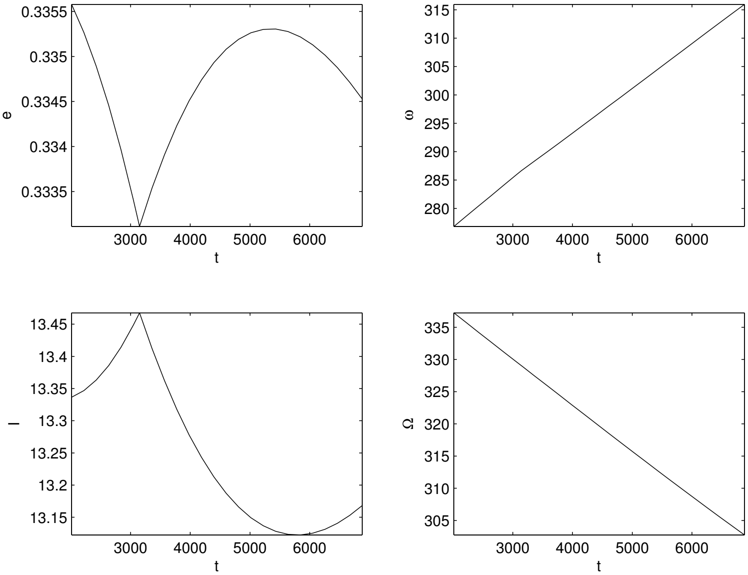


Figure: Runge-Kutta-Gauss method and continuation of the solutions of equations (1) beyond the singularity.

The averaged solutions are piecewise–smooth

Averaged evolution of (1620) Geographos



Proper elements for NEAs: (1620) Geographos

NEODyS-2

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- Additional notes

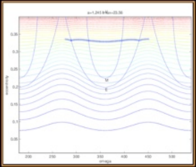
Proper elements and proper frequencies

Prop. ele.

e_{min}	0.3287
e_{max}	0.3359
i_{min}	13.301
i_{max}	13.935
ω_{min}	0
ω_{max}	0
D	0

Prop. freq.

g	4.403
s	-21.972
lf	0



(Click on the image to enlarge)

Encounter conditions

	Venus	Earth	Mars	Jupiter	Saturn
Crossings	0	4	4	0	0

u	θ	φ	λ	u_y
0.3888	86.6198	309.67	208.597	0.0229
0.3888	86.6198	50.3296	71.299	0.0229
0.3888	86.6198	129.67	41.2507	0.0229
0.3888	86.6198	230.33	278.582	0.0229

Note: "crossings" is the number of crossings per revolution or libration period of omega.

NEODyS ∴ Objects ∴ Geographos ∴ Proper elements

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Proper elements for NEAs: (2102) Tantalus

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(2102) Tantalus ▹ PROPER ELEMENTS

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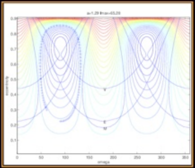
Proper elements and proper frequencies

Prop. ele.

e_{min}	0.2235
e_{max}	0.8553
i_{min}	36.179
i_{max}	64.59
ω_{min}	47.732
ω_{max}	132.277
D	84.545

Prop. freq.

g	0
s	-15.95
lf	-27.614



(Click on the image to enlarge)

Encounter conditions

	Venus	Earth	Mars	Jupiter	Saturn
Crossings	4	4	0	0	0

u	θ	φ	λ	u_y
1.129	117.708	124.004	235.645	-0.525
1.129	117.708	345.305	91.1116	-0.525
1.129	117.708	165.305	86.6798	-0.525
1.129	117.708	304.004	302.147	-0.525

Note: "crossings" is the number of crossings per revolution or libration period of omega.

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Secular evolution of the orbit distance

Define the secular evolution of the minimal distances

$$\overline{d}_h(t) = \tilde{d}_h(\overline{\mathcal{E}}(t)) \,, \qquad \overline{d}_{min}(t) = \tilde{d}_{min}(\overline{\mathcal{E}}(t))$$

in an open interval containing a crossing time t_c.

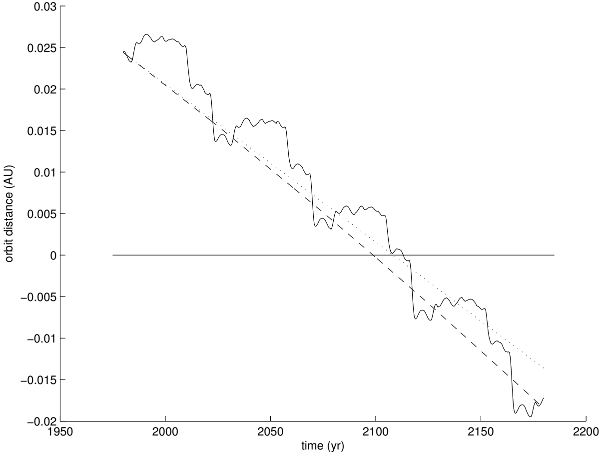
Assume t_c is a crossing time and $\mathcal{E}_c = \overline{\mathcal{E}}(t_c)$ is a non-degenerate crossing configuration with only one crossing point, i.e. $d_h(\mathcal{E}_c) = 0$. Then there exists an interval (t_a, t_b), $t_a < t_c < t_b$ such that $\overline{d}_h \in C^1((t_a, t_b); \mathbb{R})$.

idea of the proof:

$$\begin{aligned} \lim_{t \to t_c^-} \dot{\overline{d}}_h(t) - \lim_{t \to t_c^+} \dot{\overline{d}}_h(t) &= \mathrm{Diff}_h\big(\overline{\nabla_Y \mathcal{H}_1}\big) \cdot \epsilon\, \mathbb{J}_2 \nabla_Y \tilde{d}_h \Big|_{\mathcal{E}=\mathcal{E}_c} \\ &= -\frac{\epsilon}{\pi\sqrt{\det \mathcal{A}_h}} \,\{\tilde{d}_h, \tilde{d}_h\}_Y \Big|_{\mathcal{E}=\mathcal{E}_c} = 0\,, \end{aligned}$$

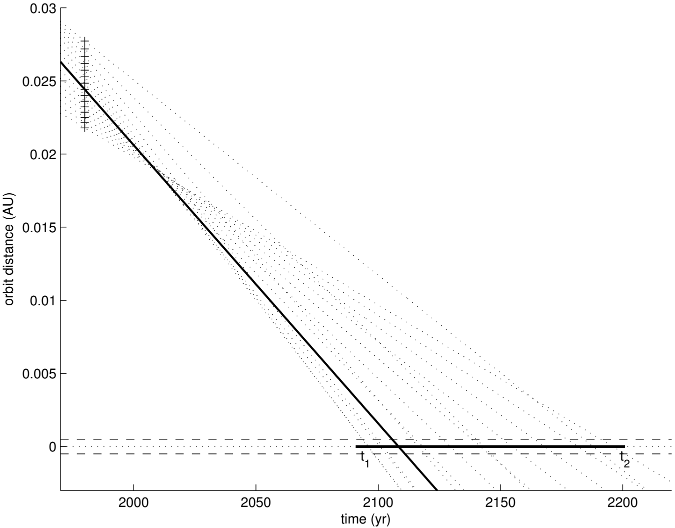
The secular evolution of $\tilde{d}_{min}$ is more regular than that of the orbital elements in a neighborhood of a planet crossing time.

Evolution of the orbit distance for 1979 XB



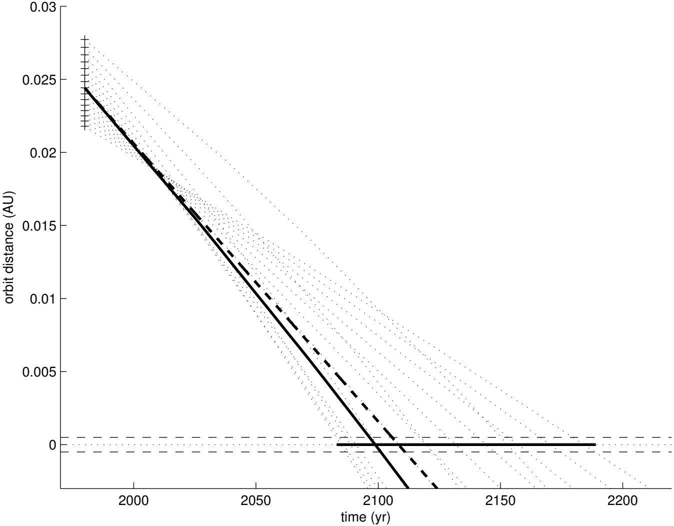
Transition through a planet crossing for 1979 XB

linearized secular evolution



Transition through a planet crossing for 1979 XB

nonlinear secular evolution



Mean motion resonances

Marò and Gronchi, SIADS (2018)

Resonance condition:

$$hn + h'n' = 0, \qquad h, h' \in \mathbb{Z}.$$

Extended Hamiltonian:

$$\tilde{\mathcal{H}} = \mathcal{H}_0 + n'L' + \epsilon\mathcal{H}_1$$

Resonant normal form to order N:

$$\mathscr{H}_N(V, L, L'; X) = \sum_{\mathsf{k}\in\mathcal{R}, |\mathsf{k}|\le N} \hat{\mathcal{H}}_{\mathsf{k}}(L, L'; X) e^{i\mathsf{k}\cdot V}.$$

Here $V = (\ell, \ell')$, X are the other (secular) variables,

$$\mathcal{R} = \{\mathsf{k} = (k, k') \in \mathbb{Z}^2 : \exists n \in \mathbb{Z} \text{ with } \mathsf{k} = n\mathsf{h}\},$$

$\mathsf{h} = (h, h')$, and

$$\hat{\mathcal{H}}_{\mathsf{k}}(L, L'; X) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \tilde{\mathcal{H}}(\mathcal{V}, L, L'; X) e^{-i\mathsf{k}\cdot\mathcal{V}} d\mathcal{V}.$$

$\mathcal{V} = (\ell, \ell')$ when the latter are integration variables.

Resonant normal form

Note that

$$\mathscr{H}_N(V,L,L';X) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} D_N(\mathsf{h}\cdot\mathcal{V} - \mathsf{h}\cdot V)\tilde{\mathcal{H}}(\mathcal{V},L,L';X)d\mathcal{V},$$

where

$$D_N(x) = \sum_{|n|\leq N} e^{inx} = \frac{\sin((N+1/2)x)}{\sin(x/2)}$$

is the Dirichlet kernel.

We introduce a canonical transformation Ψ through the relations

$$\begin{pmatrix}\sigma\\ \sigma'\end{pmatrix} = A\begin{pmatrix}\ell\\ \ell'\end{pmatrix}, \quad \begin{pmatrix}S\\ S'\end{pmatrix} = A^{-T}\begin{pmatrix}L\\ L'\end{pmatrix},$$

with

$$A = \begin{bmatrix} h & h' \\ 0 & 1/h \end{bmatrix}, \qquad A^{-T} = \begin{bmatrix} 1/h & 0 \\ -h' & h \end{bmatrix}.$$

$\sigma = h\ell + h'\ell'$ is the resonant angle.

Resonant normal form

Set $X = (G, Z, g, z)$ and let us define

$$\mathscr{K}_N(\sigma, S, T; X) = \mathscr{H}_N \circ \Psi^{-1}(\sigma, \tau, S, T; X).$$

Fix $N_{\max}$ and take the resonant normal form in the new variables

$$\mathscr{K}_{N_{\max}} = \mathcal{K}_0 + \epsilon(\overline{\mathcal{K}_1} + \mathcal{K}_{res}^{N_{\max}}), \tag{3}$$

with

$$\mathcal{K}_0(S; S') = \mathcal{H}_0\Big(hS, h'S + \frac{S'}{h}\Big) = -\frac{k^4}{2(hS)^2} + n'\Big(h'S + \frac{S'}{h}\Big),$$

$$\overline{\mathcal{K}_1}(S, X; S') = \overline{\mathcal{H}_1}(hS, h'S + \frac{S'}{h}, X) = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{1}{d(\ell, \ell')} d\ell d\ell',$$

$$\mathcal{K}_{res}^{N_{\max}}(S, \sigma, X; S') = -\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \big(D_{N_{\max}}(\mathsf{h} \cdot \mathcal{V} - \sigma) - 1\big)\mathcal{H}_1\Big(\mathcal{V}, hS, h'S + \frac{S'}{h}, X\Big) d\mathcal{V}.$$

Since $\mathscr{K}_{N_{\max}}$ does not depend on σ', the value of S' is constant.

Resonant normal form

Equations for the motion of the asteroid:

$$\dot{\mathcal{Y}} = \mathbb{J}\nabla_{\mathcal{Y}}\mathscr{K}_{N_{\max}},$$

where $\mathcal{Y} = (S, G, Z, \sigma, g, z)$, or, in components,

$$\dot{S} = -\frac{\partial \mathscr{K}_{N_{\max}}}{\partial \sigma} = -\epsilon\frac{\partial \mathcal{K}_{res}^{N_{\max}}}{\partial \sigma},$$

$$\dot{G} = -\frac{\partial \mathscr{K}_{N_{\max}}}{\partial g} = -\epsilon\Big(\frac{\partial \mathcal{K}_{res}^{N_{\max}}}{\partial g} + \frac{\partial \overline{\mathcal{K}_1}}{\partial g}\Big),$$

$$\dot{Z} = -\frac{\partial \mathscr{K}_{N_{\max}}}{\partial z} = -\epsilon\Big(\frac{\partial \mathcal{K}_{res}^{N_{\max}}}{\partial z} + \frac{\partial \overline{\mathcal{K}_1}}{\partial z}\Big),$$

$$\dot{\sigma} = \frac{\partial \mathscr{K}_{N_{\max}}}{\partial S} = \frac{hk^4}{(hS)^3} + n'h' + \epsilon\Big(\frac{\partial \mathcal{K}_{res}^{N_{\max}}}{\partial S} + \frac{\partial \overline{\mathcal{K}_1}}{\partial S}\Big),$$

$$\dot{g} = \frac{\partial \mathscr{K}_{N_{\max}}}{\partial G} = \epsilon\Big(\frac{\partial \mathcal{K}_{res}^{N_{\max}}}{\partial G} + \frac{\partial \overline{\mathcal{K}_1}}{\partial G}\Big),$$

$$\dot{z} = \frac{\partial \mathscr{K}_{N_{\max}}}{\partial Z} = \epsilon\Big(\frac{\partial \mathcal{K}_{res}^{N_{\max}}}{\partial Z} + \frac{\partial \overline{\mathcal{K}_1}}{\partial Z}\Big).$$

Mean motion resonances

If $\mathcal{E}_c$ corresponds to a crossing configuration with Jupiter, then the following relation holds in a neighborhood $\mathcal{W}$

$$\mathrm{Diff}_h\left(\frac{\partial \mathscr{K}_{N_{\max}}}{\partial y_i}\right) = \epsilon\left[\left(\frac{\partial \overline{\mathcal{K}}_1}{\partial y_i}\right)_h^- - \left(\frac{\partial \overline{\mathcal{K}}_1}{\partial y_i}\right)_h^+ + \left(\frac{\partial \mathcal{K}_{res}^{N_{\max}}}{\partial y_i}\right)_h^- - \left(\frac{\partial \mathcal{K}_{res}^{N_{\max}}}{\partial y_i}\right)_h^+\right]$$

$$= -\frac{\epsilon}{\pi} D_{N_{\max}}(\sigma - \mathsf{h}\cdot V_h)\left[\frac{\partial}{\partial y_i}\left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}}\right)\tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}}\frac{\partial \tilde{d}_h}{\partial y_i}\right].$$

Let $\sigma_c = \mathsf{h}\cdot V_h$. We observe that

$$\lim_{N_{\max}\to\infty} D_{N_{\max}}(\sigma - \sigma_c) = \delta_{\sigma_c},$$

that is, for $N_{\max} \to \infty$, the Dirichlet kernel converges in the sense of distributions to the Dirac delta centered in σ_c.

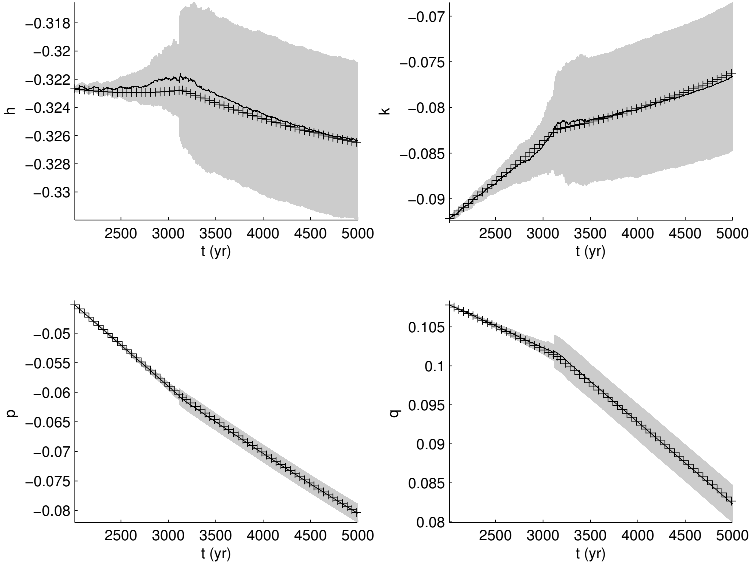
Recent work (in progress)

Joint work (in progress) with M. Fenucci

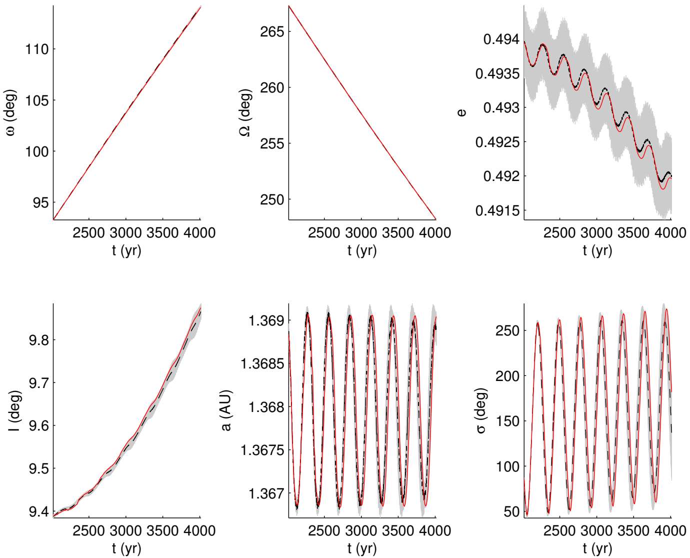
Open questions:

- Can we prove that the averaged solutions are a good approximation of the solutions of the full equations?
- What happens in case of close approaches with some planet?
- In case of mean motion resonances, can we prove that the solutions of Hamilton's equations for the resonant normal form are a good approximation of the solutions of the full equations?

Comparison for (1620) Geographos: 64 clones



Comparison in the resonant case: 64 clones, $N_{\max} = 15$



The arithmetic mean

Let $V = (\ell, \ell')$ and I be the other variables. Consider the arithmetic mean

$$\hat{I}_N(t) = \frac{1}{N} \sum_{j_1,j_2=1}^{N} I_{j_1,j_2}(t)$$

where

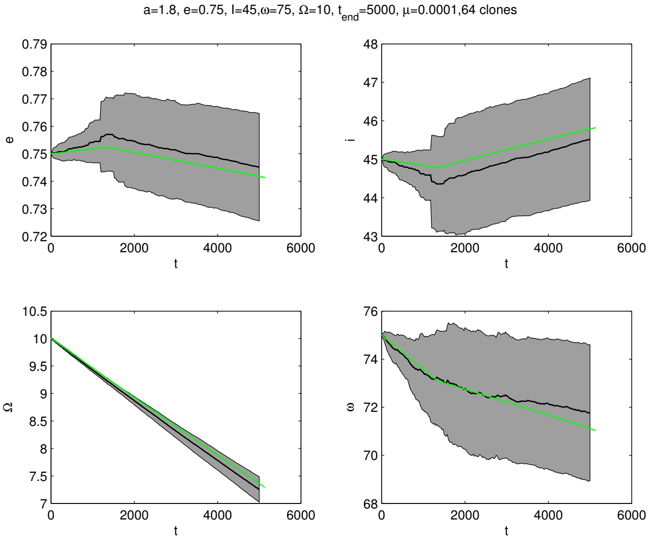
$$I_{j_1,j_2}(t) = I(t; I(0), V_{j_1,j_2}(0)), \quad V_{j_1,j_2}(0) = \frac{2\pi}{N}(j_1, j_2), \qquad j_1, j_2 = 1, \ldots, N.$$

The solutions $I_{j_1,j_2}(t)$ are computed through Kustaanheimo-Stiefel regularization of binary collisions.
Consider also the standard deviation of the solutions:

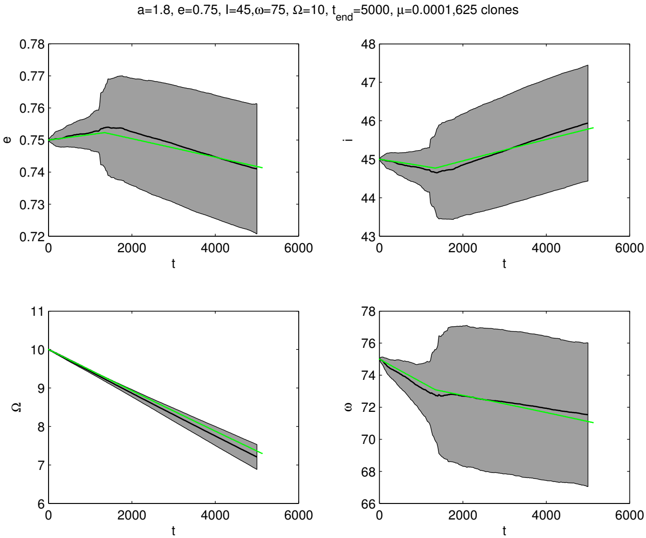
$$\mathrm{std}_I(t) = \left(\sum_{j_1,j_2=1}^{N} \frac{(I_{j_1,j_2}(t) - \hat{I}_N(t))^2}{N^2 - 1} \right)^{1/2}$$

Then compare $\hat{I}_N(t)$ with the solutions of Hamilton's equations for the normal form, for different values of N.

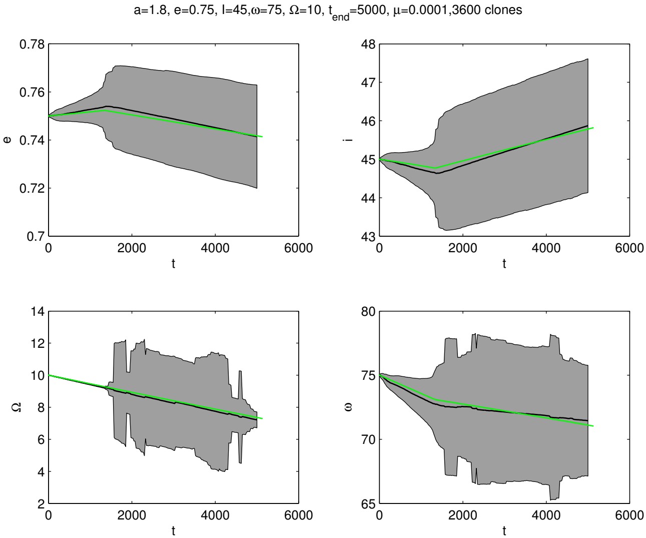
Crossing case: 64 clones



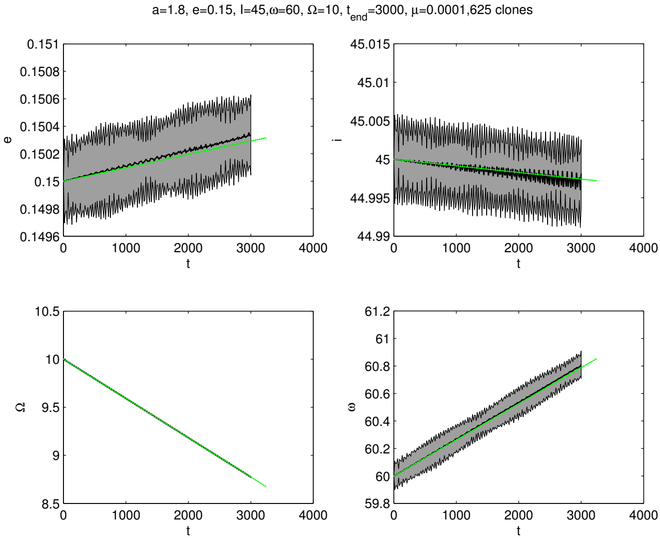
Crossing case: 625 clones

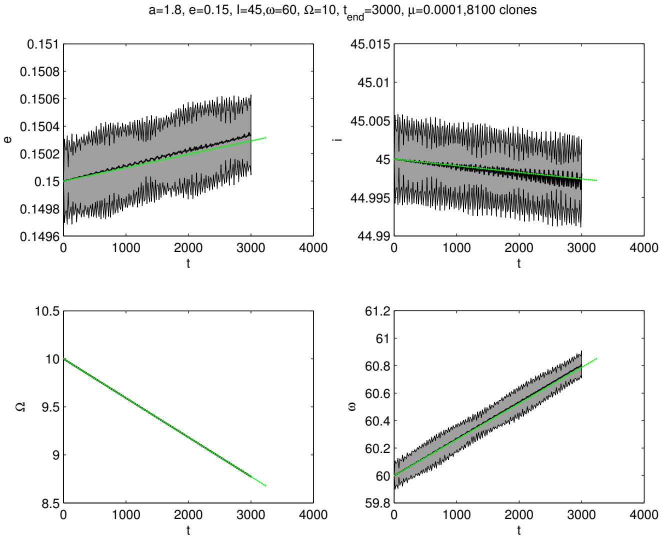


Crossing case: 3600 clones



non-crossing case: 625 clones




a=1.8, e=0.15, I=45,ω=60, Ω=10, t_end=3000, μ=0.0001,8100 clones
e
0.151
0.1508
0.1506
0.1504
0.1502
0.15
0.1498
0.1496
i
45.015
45.01
45.005
45
44.995
44.99
Ω
10.5
10
9.5
9
8.5
ω
61.2
61
60.8
60.6
60.4
60.2
60
59.8
0
1000
2000
3000
4000
t

Conclusions and future work

- We can compute the secular evolution of planet crossing asteroids, by averaging over the fast angles: the solutions are piecewise–smooth;
- the orbit distance along the averaged evolution is more regular than the orbital elements;
- We can compute the long term evolution of planet crossing asteroids also in case of mean motion resonances;
- the arithmetic mean of the solutions seems to be close to the solution of Hamilton's equations for the normal form of the Hamiltonian, both in the non-resonant and in the resonant case.

Thanks for your attention!

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