Denjoy dynamics and horseshoes on surfaces

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Horseshoes (HS) (1)

Definition

Let $f : M \to M$ be a diffeomorphism of a manifold.

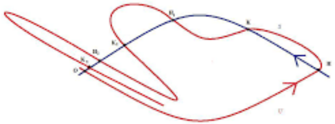
- A horseshoe for f is a f-invariant subset $H \subset M$ such that the Dynamics $f_{|H}$ is C^0 conjugate to the one of a non-trivial transitive subshift with finite type.
- A horseshoe for f is a σ_2-horseshoe when the Dynamics $f_{|H}$ is C^0 conjugate to the shift with two symbols.

Horseshoes (HS) (2)

- the shift σ is defined on $\Sigma_p = \{1, \dots, p\}^{\mathbb{Z}}$ by $\sigma((u_k)_{k\in\mathbb{Z}}) = (u_{k+1})_{k\in\mathbb{Z}}$;
- let $A = (a_{i,j})_{1\leq i,j\leq p}$ be a matrix with only 0s and 1s; we can define the subshift with finite type that is the restriction σ_A of σ to
$$\Sigma_A = \{(u_k)_{k\in\mathbb{Z}} \in \Sigma_p; \forall k \in \mathbb{Z}; a_{u_k,u_{k+1}} = 1\};$$
- we say that the subshift is transitive and non-trivial if all the entries of one power A^n are non zero; then σ_A is transitive and mixing.

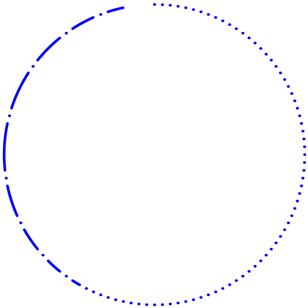
Horseshoes (HS) (3)

Examples. The first HS was introduced by S. Smale 1965 close to a transversal homoclinic intersection of a hyperbolic periodic point. This HS is hyperbolic. This was extended to the case of topologically transversal homoclinic intersection by Burns and Weiss 1995. Le Calvez and Tal 2018 use purely topological HSs for 2-dimensional homeomorphisms.



Denjoy example

- Let $\alpha \notin \mathbb{Q}/\mathbb{Z}$. The rotation $R_\alpha : \mathbb{T} \to \mathbb{T}, \theta \mapsto \theta + \alpha$ is minimal homeomorphism of the circle;
- "replace" each point $k\alpha$ of the orbit $(j\alpha)_{j\in\mathbb{Z}}$ by a small segment I_k. Your new homeomorphism $g : \mathbb{T} \to \mathbb{T}$ maps I_k onto I_{k+1}. The interior $\mathrm{Int}(I_k)$of I_k is then a wandering domain ;
- and $K = \mathbb{T} \backslash \bigcup_{k\in\mathbb{Z}} \mathrm{Int}(I_k)$ is a Cantor set such that the dynamic restricted to K is minimal.



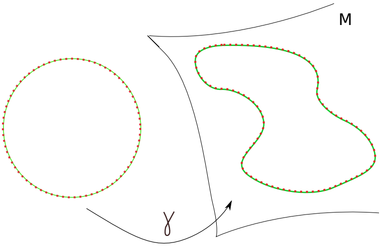
We can do the same thing for a at most countable number of orbits.

Denjoy sub-systems (DS) (1)

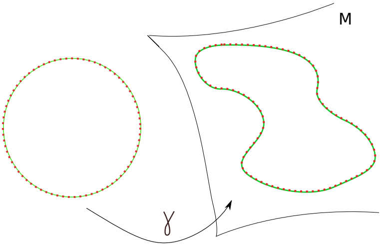
Definition

Let $f : M \to M$ be a C^k diffeomorphism of a manifold M. A C^k (resp. Lipschitz) Denjoy sub-system for f is a triplet (K, γ, h) where

- $\gamma : \mathbb{T} \to M$ is a C^k (resp. biLipschitz) embedding;
- $h : \mathbb{T} \to \mathbb{T}$ is a Denjoy example with $\Omega(h) = K \subset \mathbb{T}$;
- $f(\gamma(K)) = \gamma(K)$ and $\gamma \circ h_{|K} = f \circ \gamma_{|K}$.



Denjoy sub-systems (2)



Remarks.

- In this definition, $\gamma(\mathbb{T})$ is not necessarily invariant, but it is useful to define a circular order on the Cantor set $\gamma(K)$.
- We proved with P. Le Calvez 2017 that there is no C^2 DS.

Examples. Denjoy examples; Aubry-Mather sets.

Weak Denjoy sub-systems (WDS) (1)

Definition

Let $f : M \to M$ be a homeomorphism of a manifold M. A weak Denjoy sub-system for f is a triplet (K, j, h) where

- $h : \mathbb{T} \to \mathbb{T}$ is a Denjoy example with $\Omega(h) = K \subset \mathbb{T}$;
- $j : K \to M$ is a homeomorphism onto its image;
- $f(j(K)) = j(K)$ and $j \circ h_{|K} = f \circ j$.

Weak Denjoy sub-systems (2)

Remark. If (K, γ, h) is a DS, then $(K, \gamma_{|K}, h)$ is a WDS.

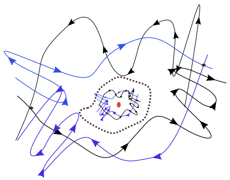
Theorem

Let (K, j, h) be a WDS of a surface homeomorphism. Then there exists a continuous DS (K, γ, h) such that $\gamma_{|K} = j$.

Idea of proof

- We use a result on planar Cantor sets: if two planar Cantor sets are homeomorphic, we can extend their homeomorphism into a homeomorphism of the plane.
- Observe that this result is not true in higher dimension.

Denjoy sub-systems versus Horseshoes



- These two kinds of Cantor Dynamics appear close to the generic fixed points of area preserving diffeomorphisms (Zehnder 73 and Aubry-Mather-Le Daeron 1982-3);
- a Denjoy Dynamics have zero entropy and no periodic points;
- horseshoes are the evidence of positive topological entropy for $C^{1+\alpha}$ diffeomorphisms (Katok 1980) and they have a dense set of periodic points.

Questions. Do horseshoes contain Denjoy sub-systems? Are Denjoy sub-systems contained in some horseshoes?

Rotation number (1)

Definition

Two WDS (K_1, j_1, h_1) and (K_2, j_2, h_2) for a same homeomorphism f are equivalent if $j_1(K_1) = j_2(K_2)$.

Theorem

Let (K_1, γ_1, h_1) and (K_2, γ_2, h_2) be two equivalent WDS. Then

- *there exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ h_1 = h_2 \circ h$;*
- *if we denote by $\rho(h_i)$ the rotation number of h_i, we have $\rho(h_1) = \pm\rho(h_2)$;*
- *if we denote by $\prec_{K_i}$ the circular order (for triplets of points) on $j_i(K_i)$ that is deduced from the one of $K_i \subset \mathbb{T}$ via the map j_i, then $\prec_{K_1} = \prec_{K_2}$ or $\prec_{K_1} = - \prec_{K_2}$, hence the two orders have the same intervals.*

Rotation number (2)

Idea of proof

- We associate to any continuous dynamical system $F : X \to X$ the equivalence relation $\mathcal{R}_F$

$$x \mathcal{R}_F y \Leftrightarrow \lim_{k \to +\infty} d(F^k x, F^k y) = 0.$$

- when (K, j, h) is a WDS for a homeomorphism f, then $K/\mathcal{R}_h \approx j(K)/\mathcal{R}_f \approx \mathbb{T}$.

Rotation number (3)

Notation. We can define the rotation number $\rho(K, j, h) \in \mathbb{T}/x \sim -x$, that satisfies that two equivalent Denjoy sub-systems have the same rotation number.

Topology on the set of WDS for f. The two weak Denjoy sub-systems (K_1, j_1, h_1) and (K_2, j_2, h_2) are close to each other if

- the two compact subsets $j_1(K_1)$ and $j_2(K_2)$ are close to each other for the Hausdorff distance on compact subsets of M;
- the two graphs of the circular orders $\prec_{K_1}$ and $\prec_{K_2}$ or $- \prec_{K_2}$ are close to each other for the Hausdorff distance on compact subsets of M^3.

Proposition

ρ *is continuous.*

Rotation number (4)

Idea of proof

- If two WDS are close from each other, we can encode their Dynamics by two sequences of $\{0, 1\}$-symbols that correspond to some itineraries that are close from each other;
- these sequences are the Sturmian sequences that are associated to the corresponding rotation numbers;
- the Sturmian sequences determine the terms of the continued fraction of the two rotations numbers.

Horseshoes always contain a family of weak Denjoy sub-systems (1)

Theorem

Let $f : M \to M$ be a C^k diffeomorphism and let $\mathcal{H}$ be a HS for f. Then there is a $N \geq 1$ and a continuous map $D : r \in (\mathbb{T}\backslash\mathbb{Q})/x \sim -x \mapsto (K_r, j_r, h_r)$ such that

- *$D(r) = (K_r, j_r, h_r)$ is a continuous WDS with rotation number r for f^N;*
- *$j_r(K_r) \subset \mathcal{H}$.*

Moreover, if $\mathcal{H}$ is a σ_2-HS, we have $N = 1$.

Horseshoes always contain a family of weak Denjoy sub-systems (2)

- To prove the theorem, we use Denjoy examples with one gap.
- It is possible to embed a family of WDS with a finite number p of gaps in a $\sigma_{\sup\{2,p\}}$-HS.
- But we cannot embed a WDS with a infinite countable number of gaps in a horseshoe, because the Dynamics on a HS is expansive.

Horseshoes always contain a family of weak Denjoy sub-systems (3)

Corollary

Let $f : M^{(2)} \to M^{(2)}$ *be a* C^k *diffeomorphism of a surface and let* $\mathcal{H}$ *be a HS for* f*. Then there is a* $N \geq 1$ *and a continuous map* $D : r \in (\mathbb{T} \backslash \mathbb{Q})/x \sim -x \mapsto (K_r, \gamma_r, h_r)$ *such that*

- $D(r) = (K_r, \gamma_r, h_r)$ *is a continuous DS with rotation number* r *for* f^N*;*
- $\gamma_r(K_r) \subset \mathcal{H}$*.*

Moreover, if $\mathcal{H}$ *is a* σ_2*-HS, we have* $N = 1$*.*

Horseshoes always contain a family of weak Denjoy sub-systems (4)

Idea of proof

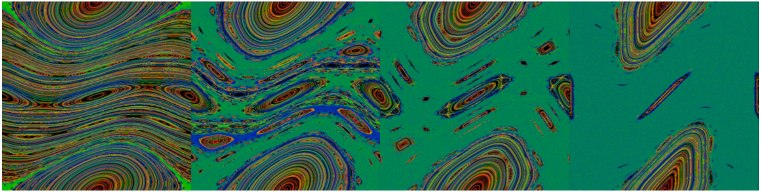
- We also use symbolic dynamics to obtain some Sturmian sequences;
- two rotation numbers that are close to each other give Sturmian sequences whose shift-orbits are close to each other (as sets);
- hence the corresponding compact subsets in the horseshoe are close to each other;
- also the not too small intervals that they determine are close to each other and then the order relations also.

The case of the conservative twist diffeomorphisms (1)

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of an exact symplectic twist diffeomorphism $f : \mathbb{A} \to \mathbb{A}$.

An Aubry-Mather set is a totally ordered compact set that contains only minimizing orbits. Then every Aubry-Mather set $\mathcal{A}$ has a rotation number $\rho(\mathcal{A}) \in \mathbb{R}$ and

- every Aubry-Mather set is a partial Lipschitz graph;
- if $(\mathcal{A}_n)$ is a sequence of Aubry-Mather sets such that the sequence of rotation numbers $(\rho(\mathcal{A}_n))$ converges to some $r \in \mathbb{R}$, then $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is relatively compact and any limit point of $(\mathcal{A}_n)$ is an Aubry-Mather set with rotation number r.



The case of the conservative twist diffeomorphisms (2)

- For every $r \in \mathbb{R} \backslash \mathbb{Q}$, there exists a unique maximal Aubry-Mather set $\mathcal{A}_r$ with rotation number r that contains every Aubry-Mather set with the same rotation number;
- Aubry-Mather sets $\mathcal{A}_r$ that have an irrational rotation number and that are not full graphs always contain a Lipschitz Denjoy sub-system;
- for every $r = \frac{p}{q} \in \mathbb{Q}$, there exists two Aubry-Mather set $\mathcal{A}_r^{\pm}$ with rotation number r that are maximal (for $\subset$) among the Aubry-Mather sets with the same rotation number and that are such that: $\forall x \in \tilde{\mathcal{A}}_r^+, \pi_1 \circ F^q(x) \geq \pi_1(x) + p$ (resp. $\forall x \in \tilde{\mathcal{A}}_r^-, \pi_1 \circ F^q(x) \leq \pi_1(x) + p$).

The case of the conservative twist diffeomorphisms (3)

Theorem

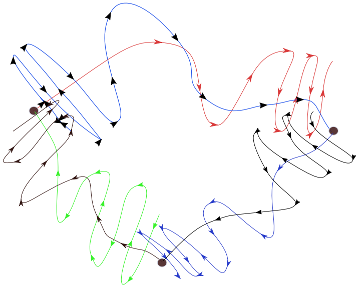
Let $f : \mathbb{A} \to \mathbb{A}$ be an exact symplectic twist diffeomorphism. Assume that $\mathcal{A}_r^+$ (resp. $\mathcal{A}_r^-$) is uniformly hyperbolic for some rational number $r \in \mathbb{Q}$. Let $\mathcal{V}_r$ be a neighbourhood of $\mathcal{A}_r^+$ (resp. $\mathcal{A}_r^-$). Then there exists a horseshoe $\mathcal{H}_r^+$ (resp. $\mathcal{H}_r^-$) for some f^N and $\varepsilon > 0$ such that

- $\mathcal{A}_r^+ \subset \mathcal{H}_r^+ \subset \mathcal{V}_r$ *(resp.* $\mathcal{A}_r^- \subset \mathcal{H}_r^- \subset \mathcal{V}_r$*);*
- *every Aubry-Mather set with rotation number in $(r, r+\varepsilon)$ (resp. $(r-\varepsilon, r)$) is contained in $\mathcal{H}_r^+$ (resp. $\mathcal{H}_r^-$);*
- *every point in $\mathcal{H}_r^+$ (resp. $\mathcal{H}_r^-$) has no conjugate points, i.e. has its orbit that is locally minimizing.*

The case of the conservative twist diffeomorphisms (4)

Corollary

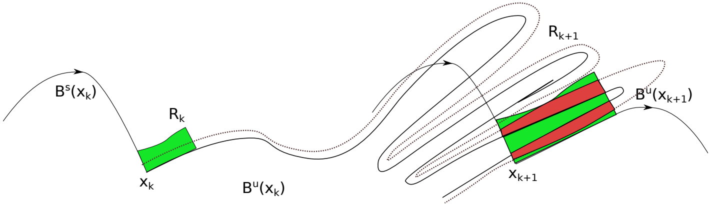
There exists a dense G_δ subset $\mathcal{G}$ of the set of C^k symplectic twist diffeomorphisms (for $k \geq 1$) such that for every $f \in \mathcal{G}$, there exist an open and dense subset $U(f)$ of $\mathbb{R}$ and a sequence $(r_n)_{n\in\mathbb{N}}$ in $U(f) \cap \mathbb{Q}$ such that every minimizing Aubry-Mather set with rotation number in $U(f)$ is hyperbolic and contained in a horseshoe associated to a minimizing hyperbolic Aubry-Mather set whose rotation number is r_n.



The case of the conservative twist diffeomorphisms (5)

Idea of proof

We carefully choose a Markov partition close to some heteroclinic cycle.



We also use a result of Le Calvez that implies that for a generic conservative twist diffeomorphisms, all the Aubry-Mather sets with a rational rotation number are hyperbolic.