Slope problems in the theory of semigroups of holomorphic self-maps of the unit disc

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Dynamical Systems: from geometry to mechanics
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  3. Others: poles and fractional singularities of the vector field, rate of convergence, contact arcs, synchronization formulae for fixed points, ...

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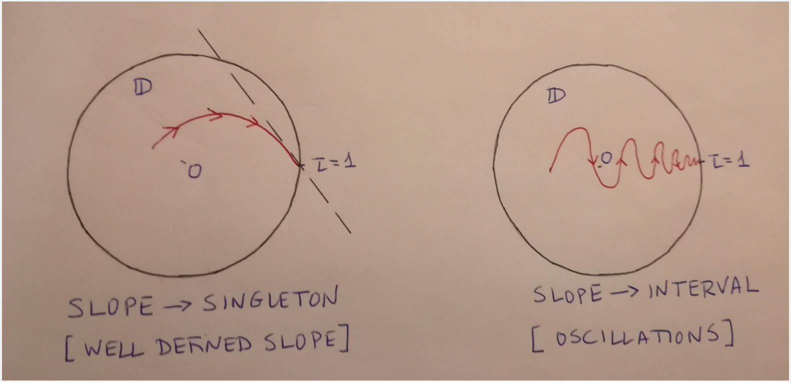

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1. $S_{\pi/\lambda}$ is the open strip $\{x + iy \in \mathbb{C} : x \in (0, \pi/\lambda), y \in \mathbb{R}\}$.

2. h is a univalent function from $\mathbb{D}$ into $S_{\pi/\lambda}$
(hence $h(\mathbb{D})$ is a simply connected domain).

3. $h \circ \varphi_t(z) = h(z) + it, \quad z \in \mathbb{D}, \; t \geq 0$
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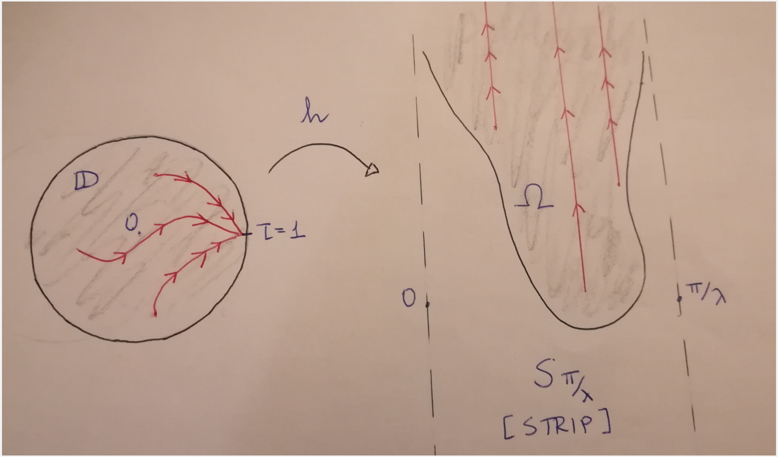

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Let (φ_t) be a hyperbolic semigroup in $\mathbb{D}$ with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$, spectral value $\lambda > 0$ and model $(S_{\pi/\lambda}, h, z \mapsto z + it)$.

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In particular, for all $z \in \mathbb{D}$,

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- In other words, the trajectories "are asymptotically tangential" to the boundary of $\mathbb{D}$.

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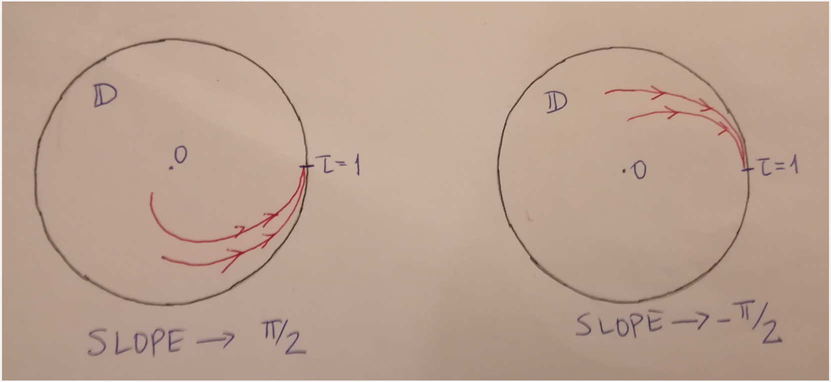

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- In the rest of the talk, we mainly treat the second question.

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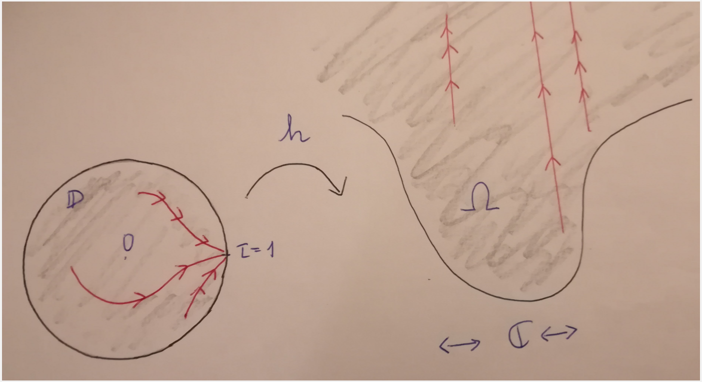

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- For t large enough and asymptotically, $\delta^{+}_{\Omega,p}(t)$ and $\delta^{-}_{\Omega,p}(t)$ measure (in a normalized way) the symmetrical aspect of Ω with respect to the trajectory $t \mapsto p + it$.

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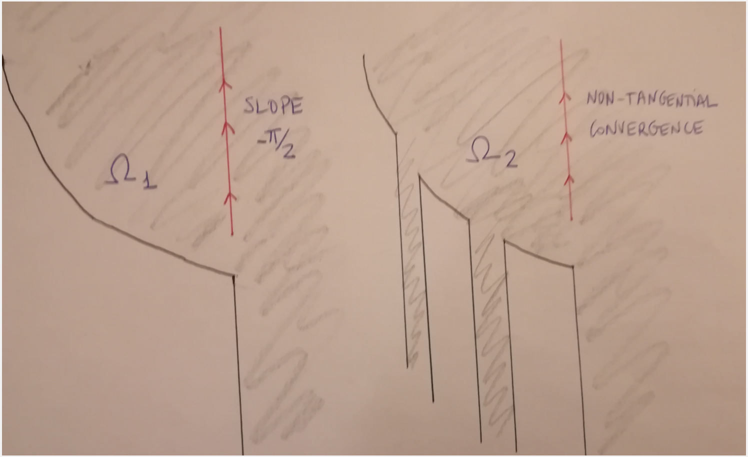
$$c\delta^+_{\Omega,p}(t) \leq \delta^-_{\Omega,p}(t) \leq C\delta^+_{\Omega,p}(t).$$

Parabolic-zero semigroups: models and slopes (III)



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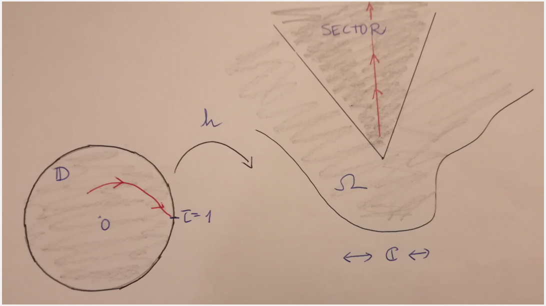

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- A similar result is true for general non-elliptic semigroups.