Singularities of solutions of the Hamilton-Jacobi equation. A toy model: distance to a closed subset

Albert Fathi
joint work with Piermarco Cannarsa and Wei Cheng

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Distance to a subset

Rather than starting right away the general framework, we will first consider the case of the distance function to a closed subset in Euclidean space.

Consider $C \subset \mathbb{R}^k$ a closed subset of $\mathbb{R}^k$. The distance on $\mathbb{R}^k$ is the Euclidean distance.

The distance function $d_C : \mathbb{R}^k \to \mathbb{R}$ to the closed subset C is defined by

$$d_C(x) = \inf_{c \in C} \|x - c\|,$$

where $\|\cdot\|$ is the Euclidean norm. The function d_C is Lipschitz with Lipschitz constant 1. Therefore it is differentiable a.e. We denote by $\mathrm{Sing}^*(d_C)$ the set of points in $\mathbb{R}^k \setminus C$ where d_C is not differentiable.

Our goal is to give topological properties of $\mathrm{Sing}^*(d_C)$.

Theorem

The set $\mathrm{Sing}^*(d_C)$ is locally path-connected and even locally contractible.

When C is a C^2 sub-manifold, the theorem is known for the closure of $\mathrm{Sing}^*(d_C)$.
Note that in
Mantegazza, C. & Mennucci, A. C.,*Hamilton-Jacobi equations and distance functions on Riemannian manifolds.* Appl. Math. Optim. 47 (2003), no. 1, 1–25.
there is an example of a $C^{1,1}$ closed convex curve γ in the plane such that the closure of $\mathrm{Sing}^*(d_\gamma)$ is not locally connected and has > 0 Lebesgue measure.

This theorem for a general closed subset C is quite strong. In fact, the set $\mathrm{Sing}^*(d_C)$ is the singularity set of a Lipschitz function. But the only restriction on the singularity set of a Lipschitz function is that it should have measure 0. The "general such set" is definitely not locally connected. If you are a little bit more sophisticated, you might object to this argument.

It is known that the function d_C^2 is the sum of a C^∞ and a concave function. In fact, for a given $y \in \mathbb{R}^k$, the function $x \mapsto \|x-y\|^2 - \|x\|^2 = -2 < y, x > + \|y\|^2$ is an affine function of x. Hence $d_C^2(x) - \|x\|^2 = \inf_{c \in C} \|x-c\|^2 - \|x\|^2$ is concave. This implies that $d_C^2(x) = [d_C^2(x) - \|x\|^2] + \|x\|^2$ is indeed the sum of a concave function and a smooth function.

Therefore, we should rather expect the singularities of d_C^2 to be the singularities of a "general concave function". For a "general" concave function $\varphi : \mathbb{R} \to \mathbb{R}$, the singularities of φ are the jumps of the derivative φ'. These jumps are countable and dense in $\mathbb{R}$ in the "general" case. But, a locally connected countable subset of $\mathbb{R}$ has only isolated points, and cannot be dense in $\mathbb{R}$, or not even in any non trivial interval contained in $\mathbb{R}$.
There is no a priori reason why $\mathrm{Sing}^*(d_C)$ should be locally connected.

Theorem (Global Homotopy)

For every bounded connected component $U \subset \mathbb{R}^k \setminus C$, the inclusion $\mathrm{Sing}^*(d_C) \cap U \subset U$ is a homotopy equivalence.

In fact, this theorem was first proved in:
Lieutier, A., *Any open bounded subset of $\mathbb{R}^n$ has the same homotopy type as its medial axis.* Comput. Aided Des. 36, 1029–1046 (2004).
In Computer Science, if U is an open subset of $\mathbb{R}^n$, the set $\mathrm{Sing}^*(d_{\partial U}) \cap U$ is called the medial axis of U.
Our proof allows to give a version of the Global Homotopy theorem for non bounded connected components of $\mathbb{R}^k \setminus C$.

Singularities of the Hamilton-Jacobi Equation

The result on the distance function follows from a more general result on viscosity solutions of the (evolution) Hamilton-Jacobi equation

$$\partial_t U + H(x, \partial_x U) = 0, \tag{HJE}$$

that we will explain now.

We need to consider a connected manifold M (without boundary).

A point in the tangent (resp. cotangent) bundle TM (resp. T^*M) will be denoted by (x, v) (resp. (x, p)) with $x \in M$ and $v \in T_x M$ (resp. $p \in T_x^* M$).

H is a function $H : T^*M \to \mathbb{R}$ which is called Hamiltonian.

A classical solution of (HJE) is a differentiable function $U : [0, +\infty[\times M \to \mathbb{R}$ which satisfy the (evolution) Hamilton-Jacobi equation (HJE) at every point of $]0, +\infty[\times M$.

Singularities of the Hamilton-Jacobi Equation

This equation

$$\partial_t U + H(x, \partial_x U) = 0, \qquad \text{(HJE)}$$

usually does not admit global smooth solutions $U : [0, +\infty[\times M \to \mathbb{R}$. Therefore a weaker notion of solution is necessary.

Viscosity solutions have been the successful such theory–at least if the Hamiltonian $H(x, p)$ is convex in p, which is the case when H is Tonelli (explained below).

These viscosity solutions $U : [0, +\infty[\times M \to \mathbb{R}$ of (HJE) are not necessarily differentiable everywhere. We will describe the topology of their set of singularities $\mathrm{Sing}^*(U)$, ie. the set $\mathrm{Sing}^*(U)$ of points $(t, x) \in]0, +\infty[\times M$ where U is not differentiable.

Our results (proofs) need the Tonelli hypothesis.

Luckily, in the Tonelli case, the viscosity solutions can be described, via the so-called Lax-Oleinik semi-group without having to go through the whole theory of viscosity solution.

Tonelli Hamiltonian

We first explain what is a Tonelli Hamiltonian.
We will assume that M is endowed with a *complete* Riemannian metric. We will denote the associated norm on T_xM by $\|\cdot\|_x$. We will use the same notation $\|\cdot\|_x$ for the dual norm on T_x^*M.
Moreover, the (complete) distance on M associated to the Riemannian metric will be denoted by d.
A function $H : T^*M \to \mathbb{R}, (x,p) \mapsto H(x,p)$, is a **Tonelli Hamiltonian** if it is *at least* C^2 and satisfies the following conditions:

1) **(C^2 Strict Convexity)** At every $(x,p) \in T^*M$, the second partial derivative $\partial^2_{pp}H(x,p)$ is definite > 0. In particular $H(x,p)$ is strictly convex in p.
2) For every $K \geq 0$, we have $\sup_{\|p\|_x \leq K} H(x,p) < +\infty$.
3) **(Superlinearity)** The function H is bounded below on T^*M and $H(x,p)/\|p\|_x \to +\infty$, as $\|p\|_x \to +\infty$ uniformly in $x \in M$.

Tonelli Hamiltonian

A typical example of Tonelli Hamltonian is

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x),$$

where $V : M \to \mathbb{R}$ is C^2.

We now comment the conditions

2) For every $K \geq 0$, we have $\sup_{\|p\|_x \leq K} H(x,p) < +\infty$.
3) **(Superlinearity)** The function H is bounded below on T^*M and $H(x,p)/\|p\|_x \to +\infty$, as $\|p\|_{\to} + \infty$ uniformly in $x \in M$.

When M is compact, condition 2) is automatic, since the set $\{(x,p) \in T^*M \mid \|p\|_x \leq K\}$ is compact.
Moreover, when M is compact, the choice of the Riemannian metric on M is not crucial, since all Riemannian metrics are equivalent.

The results

For a function $U : [0, +\infty[\times M \to \mathbb{R}$, recall that $\mathrm{Sing}^*(U)$ is the set of points $(t, x) \in]0, +\infty[\times M$ where U is not differentiable. Our local result is:

Theorem

If $U : [0, +\infty[\times M \to \mathbb{R}$ is a viscosity solution of the Hamilton-Jacobi equation $\partial_t U + H(x, \partial_x U) = 0$, then $\mathrm{Sing}^*(U)$ is locally contractible.

It is also possible give some indication on the global homotopy type of U, but we will need to introduce the Aubry set and put some condition on U–for example, it works if U is uniformly continuous. More on that latter.

A first version of our work appeared in: **Cannarsa, Piermarco; Cheng, Wei; Fathi, Albert**, *On the topology of the set of singularities of a solution to the Hamilton-Jacobi equation.* C. R. Math. Acad. Sci. Paris 355 (2017), no. 2, 176–180.

Why Tonelli? The Lagrangian!

The important feature of Tonelli Hamiltonians is that they allow to define an action for curves, using the associated Lagrangian which is convex in the speed. This in turn allows to apply Calculus of Variations and to give a "formula" for solutions of the Hamilton-Jacobi equation.

The Lagrangian $L : TM \to \mathbb{R}, (x, v) \mapsto L(x, v)$, associated to the Tonelli Hamiltonian, is defined by

$$L(x, v) = \sup_{p \in T_x^* M} p(v) - H(x, p).$$

The Lagrangian L is also Tonelli: it is C^2, C^2-strictly convex and uniformly superlinear in v.

If $H(x, p) = \frac{1}{2}\|p\|_x^2 + V(x)$, then we have

$$L(x, v) = \frac{1}{2}\|v\|_x^2 - V(x).$$

Action

Definition (Action)

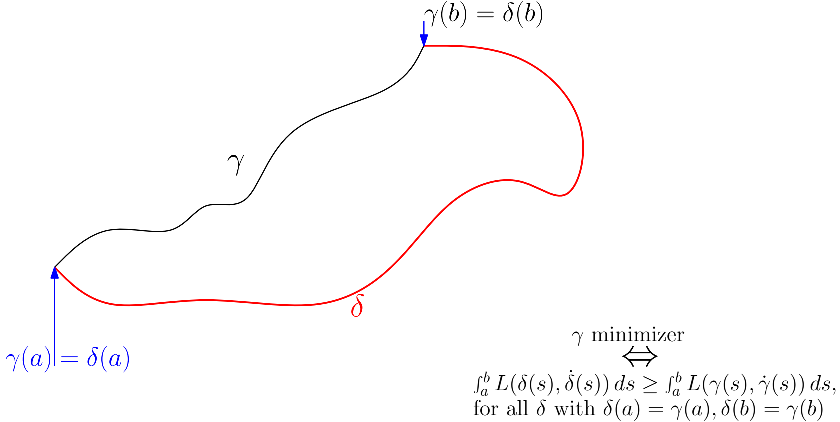
If $\gamma : [a, b] \to M$ is a curve, its *action* (for L) is

$$\mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds.$$

Note that since $\inf_{TM} L > -\infty$, we have
$\mathbb{L}(\gamma) \geq (b-a) \inf_{TM} L > -\infty$.
Tonelli's theorem states that for a Tonelli Lagrangian, given any $a < b \in \mathbb{R}$, and $x, y \in M$, there exists a C^2 curve $\gamma : [a, b] \to M$, with $\gamma(a) = x$ and $\gamma(b) = y$, such that

$$\mathbb{L}(\delta) = \int_a^b L(\delta(s), \dot\delta(s))\, ds \geq \mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds,$$

for every curve $\delta : [a, b] \to M$, with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$.
Such a curve is called a minimizer. Minimizers are, in fact, as smooth as H or L.


$\gamma(b) = \delta(b)$
γ
δ
$\gamma(a) = \delta(a)$
γ minimizer
$\Longleftrightarrow$
$\int_a^b L(\delta(s), \dot{\delta}(s))\,ds \geq \int_a^b L(\gamma(s), \dot{\gamma}(s))\,ds,$
for all δ with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$

The negative Lax-Oleinik semi-group

If $u : M \to [-\infty, +\infty]$ and $t > 0$, we define $T_t^-(u) : M \to [-\infty, +\infty]$ by

$$T_t^-(u)(x) = \inf_\gamma u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\, ds,$$

where the inf is taken over all paths $\gamma : [0, t] \to M$ *with* $\gamma(t) = x$.
We also define $T_0^-(u) = u$.
The family $T_t^-, t \geq 0$ is a semi-group defined on the space $\mathcal{F}(M, [-\infty, +\infty])$ of all $[-\infty, +\infty]$-valued functions defined on M.
This semi-group is called the (negative) Lax-Oleinik semi-group.
For $x, y \in M$ and $t > 0$, we define $h_t(x, y)$ by

$$h_t(x, y) = \inf_\gamma \int_0^t L(\gamma(s), \dot\gamma(s))\, ds,$$

where the inf is taken over all paths $\gamma : [0, t] \to M$ *with* $\gamma(0) = x$ $\gamma(t) = y$.
Therefore, we can also define the Lax-Oleinik semi-group by

$$T_t^-(u)(x) = \inf_{y \in M} u(y) + h_t(y, x).$$

Finitness of $T_t^-(u)$

In fact, we are more interested in *real* valued functions.
Since the action of a curve $\gamma : [0, t] \to M$ is bounded below by $t \inf_{TM} L > -\infty$, we get $h_t(x, y) \geq t \inf_{TM} L$ and

$$T_t^-(u) \geq \inf_M u + t \inf_{TM} L.$$

In particular, if u is bounded away from $-\infty$, so is $T_t^-(u)$. On the other hand it is not difficult to show that $T_t^-(u) < +\infty$, everywhere for $t > 0$, as soon as $u \not\equiv +\infty$.
Therefore, if $\inf_M u > -\infty$ and $u \not\equiv +\infty$, we obtain that $T_t^-(u)$ is finite everywhere.
When M is compact, it is not difficult to see that, if $T_t^-(u)(x)$ is finite for some (t, x) with $t > 0$, then necessarily $\inf_M u > -\infty$ and $u \not\equiv +\infty$, and therefore $T_t^-(u)$ is finite everywhere for $t > 0$.
When M is not compact, a necessary and sufficient condition on u for finiteness everywhere of $T_t^-(u)$, when $t > 0$ is not known.

Lax-Oleinik and Viscosity

If $u : M \to [-\infty, +\infty]$ is a function, its Lax-Oleinik evolution is the function $\hat{u} : [0, +\infty[\times M \to [-\infty, +\infty]$ is defined by

$$\hat{u}(t, x) = T_t^-(u)(x).$$

Connection between viscosity solutions and the Lax-Oleinik evolution is given by:

Theorem

Assume the Lax-Oleinik evolution $\hat{u}$ is finite on $]0, +\infty[\times M$. Then, the function $\hat{u}$ is a locally Lipschitz viscosity solution of the evolution Hamilton-Jacobi equation $\partial_t \hat{u} + H(x, \partial_x \hat{u}) = 0$, on $]0, +\infty[\times M$.
Moreover, this function $\hat{u}$ is locally semi-concave on $]0, +\infty[\times M$.

A locally semi-concave function is a function which is locally in a coordinate system the sum of a concave and a C^∞ function.

Lax-Oleinik and Viscosity

In fact all viscosity solutions are of the form $\hat{u}$. Namely:

Theorem

Assume that the continuous function $U : [0, +\infty[\times M \to \mathbb{R}$ *is a viscosity solution of* $\partial_t \hat{u} + H(x, \partial_x \hat{u}) = 0$ *on* $]0, +\infty[\times M$. *Then* $U = \hat{u}$, *where* $u(x) = U(0, x)$ *for every* $x \in M$.

With these facts, the statement of our local result is:

Theorem

Assume $u : M \to [-\infty, +\infty]$ *is a function whose Lax-Oleinik evolution* $\hat{u}$ *is finite everywhere on* $]0, +\infty[\times M$. *Then* $\mathrm{Sing}^*(\hat{u})$, *the set of points in* $]0, +\infty[\times M$ *where* $\hat{u}$ *is not differentiable, is locally contractible.*

Aubry Set

To give the homotopy type of $\mathrm{Sing}^*(\hat{u})$, we have to introduce the Aubry set $\mathcal{A}(\hat{u}) \subset]0,+\infty[\times M$ of $\hat{u}$. We assume that $\hat{u}$ is finite on $]0,+\infty[\times M$.

The Aubry set $\mathcal{A}(\hat{u})$ is the set of points $(t,x) \in]0,+\infty[\times M$ for which we can find a curve $\gamma :]0,+\infty[\to M$ such that $\gamma(t) = x$ and for every $s, s' \in]0,+\infty[$ with $s < s'$, we have

$$\hat{u}(s',\gamma(s')) - \hat{u}(s,\gamma(s)) = \int_s^{s'} L(\gamma(\tau),\dot{\gamma}(\tau))\, d\tau.$$

The set $\mathcal{A}(\hat{u})$ is a closed subset which is disjoint from $\mathrm{Sing}^*(\hat{u})$. In fact, for every $(t,x) \in]0,+\infty[\times M$, we can find a curve $\gamma :]0,t] \to M$ such that $\gamma(t) = x$ and for every $s, s' \in]0,t]$ with $s < s'$, we have

$$\hat{u}(s',\gamma(s')) - \hat{u}(s,\gamma(s)) = \int_s^{s'} L(\gamma(\tau),\dot{\gamma}(\tau))\, d\tau.$$

Such a curve is a called characteristic.

Global Homotopy Type

The Aubry set $\mathcal{A}(u)$ is the set of point (t, x) for which we can prolong indefinitely a characteristic as a curve which is still a characteristic.

We can now give the

Theorem

If $u : M \to \mathbb{R}$ is the sum of a (globally) Lipschitz function and a bounded function, then $\hat{u}$ is finite everywhere on $]0, +\infty[\times M$. Moreover, the inclusion $\mathrm{Sing}^*(\hat{u}) \subset]0, +\infty[\times M \setminus \mathcal{A}(\hat{u})$ is a homotopy equivalence.

An apparently unrelated consequence

Suppose (M, g) still is a complete Riemannian manifold. We will consider the subset $\mathcal{NU}(M, g)$ of $M \times M$ of pairs (x, y) of points in M which can be joined by 2 *distinct minimizing* geodesics.

Theorem

The set $\mathcal{NU}(M, g)$ is locally contractible.

We can also say something on the global topology of $\mathcal{NU}(M, g)$. To keep things simple we restrict to the case where M is compact. Call Δ_M the diagonal in $M \times M$.

Theorem

If M is compact, the inclusion $\mathcal{NU}(M, g) \subset M \times M \setminus \Delta_M$ is a homotopy equivalence.

In particular, for a compact Riemannian manifold the homotopy type of $\mathcal{NU}(M, g)$ is independent of the Riemannian metric g. Therefore, for every Riemannian metric g on the sphere $\mathbb{S}^k$, the set $\mathcal{NU}(\mathbb{S}^k, g)$ has the homotopy type of $\mathbb{S}^k$.

How do we tie all this together?

Let us consider the simplest Tonelli Lagrangian on the Riemannian manifold (M, g). It is defined by $L(x, v) = \frac{1}{2}\|v\|_x^2$. Its associated Hamiltonian is defined by $H(x, p) = \frac{1}{2}\|p\|_x^2$. In this case, we have

$$h_t(x, y) = \frac{d(x, y)^2}{2t}.$$

In fact, if $\gamma : [0, t] \to M$ is a curve with $\gamma(0) = x$ and $\gamma(t) = y$, we can use the Cauchy-Schwarz inequality to write

$$\begin{aligned} d(x, y)^2 &\leq \left(\int_0^t \|\dot{\gamma}(s)\|_{\gamma(s)} \, ds \right)^2 \\ &\leq \int_0^t \|\dot{\gamma}(s)\|_{\gamma(s)}^2 \, ds \int_0^t 1 \, ds \\ &= 2t\mathbb{L}(\gamma). \end{aligned}$$

On the other hand, if we take as γ a geodesic whose length is $d(x, y)$, its speed is constant. Therefore all of the inequalities above are equalities.

Distance function to a closed subset

If $C \subset M$, its (modified) characteristic fonction $\xi_C : M \to \{0, +\infty\}$ is defined by

$$\xi_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{if } x \notin C. \end{cases}$$

From now on, we assume C non-empty (or equivalently $\xi_C \not\equiv +\infty$).
If we compute $\hat{\xi}_C :]0, +\infty[\times M \to M$, we get

$$\hat{\xi}_C(t, x) = \inf_{y \in M} \xi_C(y) + \frac{d(y, x)^2}{2t} = \inf_{c \in C} \frac{d(c, x)^2}{2t} = \frac{d_C(x)^2}{2t}.$$

Therefore the set of singularities of $\hat{\xi}_C$ in $]0, +\infty[\times M$ is just $]0, +\infty[\times \operatorname{Sing}(d_C^2)$.
Note that d_C^2 is differentiable everywhere on C. Hence $\operatorname{Sing}(d_C^2) = \operatorname{Sing}^*(d_C)$, where $\operatorname{Sing}^*(d_C)$ is the set of points in $\mathbb{R}^k \setminus C$ where d_C is not differentiable.

Distance function to a closed subset

If we apply the general theorem to the Lax-Oleinik evolution $\hat{\xi}$, whose set of singularities is $]0, +\infty[\times \mathrm{Sing}^*(d_C)$, we obtain

Theorem

If C is a closed subset of the complete Riemannian manifold M, then $\mathrm{Sing}^*(d_C)$ is locally contractible.

To avoid explaining what is the Aubry set for this setting, the next result is restricted to M compact. It is a generalization to (compact) Riemannian manifolds of Lieutier's result. This compact case was already proved by
Paolo Albano, Piermarco Cannarsa, K.T. Nguyen & Carlo Sinestrari, *Singular gradient flow of the distance function and homotopy equivalence*, Math. Ann., 356 (2013) 23–43.

Theorem

The inclusion $\mathrm{Sing}^*(d_C) \subset M \setminus C$ is a homotopy equivalence.

This follows from the global homotopy theorem applied to $\hat{\xi}$.

The set $\mathcal{NU}(M, g)$

Let us consider the Riemannian product $(M \times M, g \times g)$. The diagonal $\Delta_M \subset M \times M$ which is a closed subset of $M \times M$. Hence $\hat{\xi}_{\Delta_M}(t, x, y) = d^2_{\Delta_M}(x, y)/2t$.

On the other hand, a simple computation shows that

$$\hat{\xi}_{\Delta_M}(t, x, y) = \frac{d^2(x, y)}{4t}, \text{for } t > 0.$$

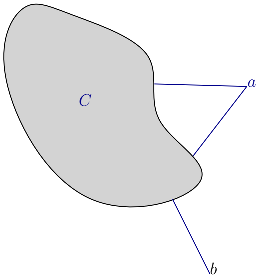
Therefore $d^2_{\Delta_M}(x, y) = d^2(x, y)/2$ and $\mathrm{Sing}^*(d_{\Delta_M}) = \mathrm{Sing}(d^2)$, where $d^2 : M \times M \to \mathbb{R}, (x, y) \mapsto d^2(x, y)$.

Moreover, it is not difficult to show that $d^2 : M \times M \to \mathbb{R}$ is not differentiable at (x, y) if and only if there are at least two distinct minimizing geodesics joining x to y. Therefore, we obtain $\mathcal{NU}(M, g) = \mathrm{Sing}(d^2) = \mathrm{Sing}^*(d_{\Delta_M})$. This implies

Theorem

The set $\mathcal{NU}(M, g)$ is locally contractible.
Moreover, if M is compact, the inclusion $\mathcal{NU}(M, g) \subset M \times M \setminus \Delta_M$ is a homotopy equivalence.

We will now give the proof of the local contractibility theorem and the Lieutier theorem on homotopy equivalence for the distance to a closed subset $C \subset \mathbb{R}^k$ by streamlining our work on singularities to that case.
We recall some well-known facts about the function d_C.
For every $x \in \mathbb{R}^k$, the set $\text{Proj}_C(x) = \{c \in C \mid \|x - c\| = d_C(x)\}$ is a non-empty compact subset of C. It is called the set of projections of x. We will use the notation c_x to denote a point in $\text{Proj}_C(x)$. Such a point c_x will be called a projection of x on A.



$\text{Proj}_C(a)$ contains at least two different points and $\text{Proj}_C(b)$ contains exactly one point.

As is well-known, there is a strong relationship between projections and differentiability of d_C.

Proposition

The function d_C is differentiable at $x \notin C$ if and only if x has a unique projection on C, i.e. $\#\operatorname{Proj}_C(x) = 1$. In that case, the gradient $\nabla_x d_C$ of d_C is given by

$$\nabla_x d_C = \frac{x - c_x}{\|x - c_x\|}, \text{ where } c_x \text{ is the unique projection of } x \text{ on } C.$$

Corollary

If $x \notin C$ and $c_x \in \operatorname{Proj}_C(x)$, the function d_C is differentiable at every point y of the open segment $]c_x, x[= \{(1-s)x + tc_x \mid s \in]0,1[\}$ and $\operatorname{Proj}_C(y) = \{c_x\}$.

The function $\varphi_{y,s}$

If $y \in \mathbb{R}^k$ and $s \in]0, +\infty[$, we define the function $\varphi_{y,s} : \mathbb{R}^k \to \mathbb{R}$ by

$$\varphi_{y,s}(x) = d_C^2(x) - \frac{1+s}{s}\|x - y\|^2.$$

Claim This function $\varphi_{y,s}$ is strictly concave and $\varphi_{y,s}(x) \to -\infty$ as $\|x\| \to +\infty$. This follows from

$$\begin{aligned} \varphi_{y,s}(x) &= d_C^2(x) - \|x - y\|^2 - \frac{1}{s}\|x - y\|^2 \\ &= d_C^2(x) - \|x\|^2 + 2\langle y, x\rangle - \|y\|^2 - \frac{1}{s}\|x - y\|^2. \end{aligned}$$

But the function $x \mapsto d_C^2(x) - \|x\|^2 - 2\langle y, x\rangle - \|y\|^2$ is concave, since we have already seen that the function $x \mapsto d_C^2(x) - \|x\|^2$ is concave.
The claim follows since $\varphi_{y,s}$ is the sum of a concave function and of the function $x \mapsto -\|x - y\|^2/s$.

Therefore, for a given $y \in \mathbb{R}^k$ and a given $s \in]0, +\infty[$, the function

$$\varphi_{y,s}(x) = d_C^2(x) - \frac{1+s}{s}\|x-y\|^2.$$

attains its maximum at a unique point which we will call $F(y,s)$. Note that $\varphi_{y,s}(y) = d_C^2(y)$. Therefore the point $F(y,s)$ satisfies

$$d_C^2(F(y,s) \geq \frac{1+s}{s}\|F(y,s)-y\|^2 + d_C^2(y).$$

This first implies $d_C^2(F(ys)) \geq d_C^2(y)$. Moreover, using $d_C(F(y,s)) \leq d_C(y) + \|F(y,s)-y\|$, we get

$$(d_C(y) + \|F(y,s)-y\|)^2 \geq \frac{1+s}{s}\|F(y,s)-y\|^2 + d_C^2(y).$$

Expanding the square, and carrying out cancellations

$$2\|F(y,s)-y\|d_C(y) \geq \frac{1}{s}\|F(y,s)-y\|^2.$$

Therefore

$$\|F(y,s)-y\| \leq 2sd_C(y).$$

The inequalities that we just obtained

$$d_C^2(F(y,s)) \geq d_C^2(y) \text{ and } \|F(y,s) - y\| \leq 2s d_C(y),$$

yield the following first 3 properties of the function $F : \mathbb{R}^k \times [0, +\infty[\to \mathbb{R}^k$:

1) The map F is continuous and extends continuously to $\mathbb{R}^k \times \{0\}$ by $F(y, 0) = y$.
2) For $c \in C$ and $s \geq 0$, we have $F(c, s) = c$.
3) $d_C(F(y,s)) \geq d_C(y)$, for all $(y, s) \in \mathbb{R}^k \times]0, +\infty[$. Therefore $F(y, s) \notin C$ for every $y \notin C$.

The 4th property needs a slightly more involved argument that we will skip:

4) If d_C is differentiable at $F(y, s)$, then $y \in [c_{F(y,s)}, F(y, s)]$, where $c_{F(y,s)}$ is the unique projection of $F(y, s)$ on C, the function d_C is differentiable at y and
   $d_C(F(y, s)) = (1 + s)d_C(y)$.
   In particular $F(y, s) \in \mathrm{Sing}^*(d_C)$, for all $y \in \mathrm{Sing}^*(d_C)$ and all $s \geq 0$.

Using the properties of F, we can prove

Theorem (Lieutier)

For every bounded connected component $U \subset \mathbb{R}^k \setminus C$, the inclusion $\mathrm{Sing}^*(d_C) \cap U \subset U$ is a homotopy equivalence.

In fact, it follows from property 3) of F

3) $d_C(F(y,s)) \geq d_C(y)$, for all $(y,s) \in \mathbb{R}^k \times]0,+\infty[$. Therefore $F(y,s) \notin C$ for every $y \notin C$.

that the continuous map F is a homotopy from the open set $\mathbb{R}^k \setminus C$ to itself, and therefore for each of its connected components.

If we assume that U is a bounded connected component of $\mathbb{R}^k \setminus C$, then we get that $\sup_U d_C = K < +\infty$. Therefore by property

4) If d_C is differentiable at $F(y,s)$ then d_C is differentiable at y and $d_C(F(y,s)) = (1+s)d_C(y)$. In particular $F(y,s) \in \mathrm{Sing}^*(d_C)$, for all $y \in \mathrm{Sing}^*(d_C)$ and all $s \geq 0$.

we obtain that for every $y \in U$, as soon as $(1+s)d_C(y) > K$, we must have $F(y,s) \in \mathrm{Sing}^*(d_C)$.

Therefore, since $\left(1+\frac{K}{d_C(y)}\right) d_C(y) > K$, if we define the homotopy $G : U \times [0,1] \to U$ by

$$G(y,t) = F\left(y, \frac{Kt}{d_C(y)}\right),$$

it satisfies $G(y,0) = y$ and $G(y,1) \in \mathrm{Sing}^*(d_C) \cap U$, for all $y \in U$. Moreover, we have, again by property 4), the homotopy G satisfies $G(y,t) \in \mathrm{Sing}^*(d_C) \cap U$, for all $y \in \mathrm{Sing}^*(d_C) \cap U$, and all $t \in [0,1]$.

We now sketch the proof of the local result:

Theorem

The set $\mathrm{Sing}^*(d_C)$ *is locally path-connected and even locally contractible.*

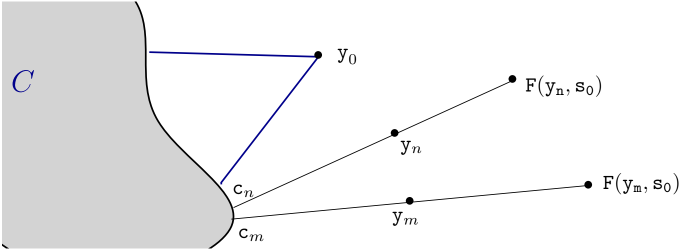
We recall that a topological space X is locally contractible, if for every $x \in X$ and every neighborhood U of x, we can find a neighborhood V of x and a map $\phi : V \times [0,1] \to U$ such that $\phi(y,0) = y$, for all $y \in V$, and $\phi(y,1) = x_0$, for some $x_0 \in U$, and all $y \in V$.

This will follow from the following proposition:

Proposition

For any $y_0 \in \mathrm{Sing}^*(d_C)$, and any $s_0 > 0$, we can find a neighborhood V of y_0 such that $F(y, s_0) \in \mathrm{Sing}^*(d_C)$, for all $y \in V$.

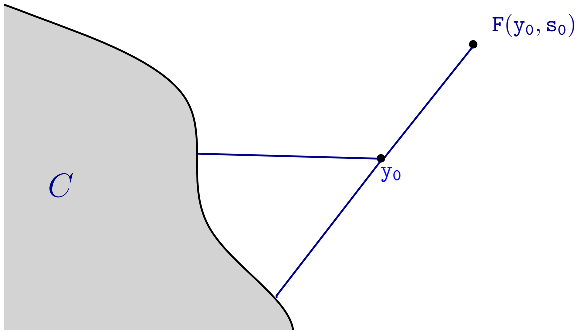
In fact, for y_0 fixed, if the proposition was not true, that would imply that we can find $y_n \to y_0$ such that $F(y_n, s_0) \notin \mathrm{Sing}^*(d_C)$, i.e. d_C is differentiable at $F(y_n, s_0)$. Property 4) of F implies that $y_n \in]c_n, F(y_n, s_n)[$, where c_n is the (unique) projection of $F(y_n, s_n)$ on C, and that c_n is also the unique projection of y_n on C.



From the same Property 4) we also know that

$$(1+s_0)d_C(y_n) = d_C(F(y_n, s_0)) = d_C(y_n) + \|F(y_n, s_0) - y_n\|.$$

Passing to limit we obtain



and $(1+s_0)d_C(y_0) = d_C(F(y_0, s_0)) = d_C(y_0) + \|F(y_0, s_0) - y_0\|$.

It is not difficult to then show that d_C is differentiable at y_0, contradicting the hypothesis $y_0 \in \mathrm{Sing}^*(d_C)$.

To prove the local contractibility theorem, we fix a neighborhood U of y_0 in $\mathbb{R}^k$. By continuity of F, we can find a neighborhood V of y_0 in $\mathbb{R}^k$ ans $s_0 > 0$ such that $F(V \times [0, s_0]) \subset U$. By the proposition above, cutting down $V \subset \mathbb{R}^k$ to a smaller neighborhood, we can also assume $F(V \times \{s_0\}) \subset \mathrm{Sing}^*(d_C)$.
If we choose a small Euclidean ball $B(y_0, \epsilon) \subset V$, we can define a homotopy $G : B(y_0, \epsilon) \times [0, 1] \to U$ by

$$G(y,t) = \begin{cases} F(y, 2ts_0), \text{ for } t \in [0, 1/2], \\ F((2-2t)y + (2t-1)y_0, s_0), \text{ for } t \in [1/2, 1]. \end{cases}$$

by the properties of F and the choices of $B(y_0, \epsilon)$ and s_0, the homotopy G contracts $B(y_0, \epsilon)$ to a point in U, with $G\left[B(y_0, \epsilon) \cap \mathrm{Sing}^*(d_C) \times [0, 1]\right] \subset U \cap \mathrm{Sing}^*(d_C)$.