Automorphisms of $\mathbb{C}^2$ with an invariant non-recurrent attracting Fatou component biholomorphic to $\mathbb{C} \times \mathbb{C}^*$

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(Joint work with F. Bracci and B. Stensønes)

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$X = \mathbb{P}^1(\mathbb{C})$, and $F(z) = z^2$.

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- Fatou components: $D(0,1)$ and $\mathbb{P}^1(\mathbb{C}) \setminus \overline{D(0,1)}$

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A Fatou component Ω is invariant if $F(\Omega) = \Omega$.

Fatou Components

Theorem (Fatou)

A periodic Fatou component for $F : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ rational map of degree $d \geq 2$ is:

- *either a (super)attracting basin,*
- *or a parabolic basin,*
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Q: If $\Omega \subset \mathbb{C}^2$ is an attracting invariant non-recurrent Fatou component, do we always have $\Omega \simeq \mathbb{C}^2$?

Main result

Theorem (Bracci-R.-Stensønes)

Let $k \geq 2$. There exist holomorphic automorphisms of $\mathbb{C}^k$ having an invariant, non-recurrent, attracting Fatou component biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$.

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Today we focus on the case $k = 2$.

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(ii) Prove that $\Omega \simeq \mathbb{C} \times \mathbb{C}^*$.

(iii) Find $F \in \mathrm{Aut}(\mathbb{C}^2)$ as in (i) and (ii) with Ω being a Fatou component.

(i) Local construction

Consider

$$F_N(z,w) = \left(\lambda z\left(1-\frac{zw}{2}\right), \bar{\lambda} w\left(1-\frac{zw}{2}\right)\right),$$

where $|\lambda| = 1$ is not a root of unity.

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One would expect a “rotational-like” dynamics.

However, since $\lambda\bar{\lambda} = 1$, the eigenvalues of dF_O have *one-dimensional resonances*.

Parabolic dynamics in an elliptic world

Set $u := \pi(z,w) := zw$. Hence

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Idea: adding a tail $O(z^m, w^m)$ with $m >> 1$, the dynamics will not change much.

Parabolic dynamics in an elliptic world

The set

$$B = \{(z, w) \in \mathbb{C}^2 : zw \in S, |z| < |zw|^\beta, |w| < |zw|^\beta\},$$

where $\beta \in (0, \frac{1}{2})$ and S is a small sector in $\mathbb{C}$ with vertex at 0 around the positive real axis, is a local parabolic basin.

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Theorem (Bracci-Zaitsev, 2013)

For any germ of biholomorphism of the form

$$F(z,w) = \left(\lambda z \left(1 - \frac{zw}{2}\right), \overline{\lambda} w \left(1 - \frac{zw}{2}\right)\right) + O(\|(z,w)\|^{\ell})$$

with $\ell \in \mathbb{N}$ sufficiently large, B is a local parabolic basin, i.e. $F(B) \subseteq B$, and $\lim_{n\to\infty} F^{\circ n}(z,w) = (0,0)$ uniformly in $(z,w) \in B$.

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Setting $x = zw, y = w$ the domain has the form

$$\{(x, y) \in \mathbb{C} \times \mathbb{C}^* : x \in S, |x|^{1-\beta} < |y| < |x|^\beta\}.$$

Local Fatou coordinates on B

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The map σ is the limit of

$$\sigma_n(z, w) := \lambda^n \pi_2(F^{\circ n}(z, w)) \exp\left(\frac{1}{2} \sum_{j=0}^{n-1} \frac{1}{\psi(z, w) + j} \right).$$

From local to global

Thanks to results of Forstnerič, we can find $F \in \mathrm{Aut}(\mathbb{C}^2)$ such that

$$F(z,w) = \left(\lambda z\left(1 - \frac{zw}{2}\right), \overline{\lambda} w\left(1 - \frac{zw}{2}\right)\right) + O(\|(z,w)\|^{\ell}),$$

with $\ell > 0$ arbitrary large.

Theorem

If $\ell > 0$ is sufficiently large,

$$\Omega := \bigcup_{n\in\mathbb{N}} F^{-n}(B)$$

is a completely invariant parabolic basin and $\Omega \simeq \mathbb{C} \times \mathbb{C}^*$.

(ii) $\Omega \simeq \mathbb{C} \times \mathbb{C}^*$: Extension of Fatou coordinates

Using dynamics, since $\psi \circ F = \psi + 1$, we can extend ψ to all Ω via

$$g_1(p) := \psi(F^{\circ n}(p)) - n,$$

where $n \in \mathbb{N}$ is such that $F^{\circ n}(p) \in B$.

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Let $H := g_1(B)$ and

$$\Omega_0 := g_1^{-1}(H) = \bigcup_{\zeta \in H} g_1^{-1}(\zeta).$$

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We can also extend σ on Ω_0 (but not on Ω): For any $p \in \Omega_0 = g_1^{-1}(H)$, we set

$$\begin{aligned} g_2(p) &:= \lambda^n \exp\left(\frac{1}{2}\sum_{j=0}^{n-1}\frac{1}{g_1(p)+j}\right)\sigma(F^{\circ n}(p)) \\ &= \lambda^n \exp\left(\frac{1}{2}\sum_{j=0}^{n-1}\frac{1}{\psi(F^{\circ n}(p))+j-n}\right)\sigma(F^{\circ n}(p)), \end{aligned}$$

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where $n \in \mathbb{N}$ is such that $F^{\circ n}(p) \in B$.

Since $g_1(p) \in H = g_1(B)$, we have $\operatorname{Re} g_1(p) > 0$ and the previous formula is well defined.

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T is not defined at $\zeta = 0$. However, since $g_1(\Omega_0) = H$, the map T is well-defined and holomorphic on $G(\Omega_0)$ and satisfies

$$G \circ F = T \circ G.$$

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- To get it we have careful estimates of the speed of convergence of orbits.
- Therefore, $H \times \mathbb{C}^* \subseteq G(\Omega_0)$. Since Ω_0 is not simply connected, we are done.

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- For $n \in \mathbb{N}$, set $\varphi_n : g_1^{-1}(H_n) \to \mathbb{C}^2$ as

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- Therefore, $\varphi_n \colon g_1^{-1}(H_n) \to H_n \times \mathbb{C}^*$ is a fiber preserving biholomorphism.

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- By the functional equation, if $p \in \Omega$ and $F^{\circ n}(p) \in \Omega_0$,

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- Let $\zeta \in H_n \cap H_{n+1}$ and $w \in \mathbb{C}^*$. Then

$$\begin{aligned} \varphi_n \circ \varphi_{n+1}^{-1}(\zeta, w) &= (G \circ F^{\circ n}) \circ (G \circ F^{\circ n})^{-1} T^{-1}(\zeta + n + 1, w) - (n, 0) \\ &= (\zeta, \lambda e^{\frac{1}{2(\zeta+n)}} w). \end{aligned}$$

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- By the functional equation, if $p \in \Omega$ and $F^{\circ n}(p) \in \Omega_0$,

$$G(F^{n+1}(p)) = G(F(F^{\circ n}(p))) = T(G(F^{\circ n}(p))).$$

- Let $\zeta \in H_n \cap H_{n+1}$ and $w \in \mathbb{C}^*$. Then

$$\begin{aligned}\varphi_n \circ \varphi_{n+1}^{-1}(\zeta, w) &= (G \circ F^{\circ n}) \circ (G \circ F^{\circ n})^{-1} T^{-1}(\zeta + n + 1, w) - (n, 0) \\ &= (\zeta, \lambda e^{\frac{1}{2(\zeta+n)}} w).\end{aligned}$$

- This proves that $g_1 : \Omega \to \mathbb{C}$ is a line bundle minus the zero section over $\mathbb{C}$, hence globally trivial.

Is Ω a Fatou component?

Ω is a completely F-invariant open set biholomorphic to $\mathbb{C} \times \mathbb{C}^*$.

Q: How can we show that it is a Fatou component?

Let $(z, w) \in \mathbb{C}^2$ and $(z_n, w_n) := F^{\circ n}(z, w)$. Then $(z, w) \in \Omega$ if and only if $(z_n, w_n) \to (0, 0)$ and

$$|z_n| \sim |w_n|.$$

Q: Is this condition enough to say that Ω is a Fatou component?

Example

$R\colon (z, w) \mapsto (\frac{z}{2}, \frac{w}{2})$. Then $\mathbb{C}^2 \setminus \{zw = 0\}$ is completely R-invariant, the previous condition are satisfied, but it is not a Fatou component!

(iii) Ω is a Fatou component

One more hypothesis:

Theorem

If λ is Brjuno, then Ω is a Fatou component.

(iii) Ω is a Fatou component

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Theorem

If λ is Brjuno, then Ω is a Fatou component.

Key tool: Properties of the Kobayashi distance on $\mathbb{D}^*$ and $\mathbb{B}$.

(iii) Ω is a Fatou component

Pöschel: since λ is *Brjuno*, there exists a local change of coodinates such that

$$F(z,w) = (\lambda z + zwA(z,w), \overline{\lambda} w + zwB(z,w)).$$

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- We can assume Pöchel coordinates are defined on $\mathbb{B}^2$.
- Thus, there exist $p_0 \in \Omega$, $q_0 \in V \setminus \Omega$, and Z a connected open set containing p_0 and q_0 and such that $\overline{Z} \subset V$, and we can assume that the set
$$W := \bigcup_n F^{\circ n}(Z)$$
is forward F-invariant.

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- In Pöschel coordinates, $W \subset \mathbb{B}^2 \setminus \{zw = 0\} =: \mathbb{B}^2_*$.
- For every $\delta > 0$, we can take $p \in Z \cap \Omega$ and $q \in Z \cap (V \setminus \Omega)$ such that $k_W(p, q) < \delta$.

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- For all $n \in \mathbb{N}$, $p_n = (z_n, w_n) := F^{\circ n}(p)$, and $q_n = (x_n, y_n) := F^{\circ n}(q)$

$$k_{\mathbb{B}^2_*}(p_n, q_n) \leq k_W(p_n, q_n) < \delta,$$

and

$$k_{\mathbb{D}^*}(z_n, x_n) < \delta, \quad k_{\mathbb{D}^*}(w_n, y_n) < \delta.$$

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- By the triangle inequality,

$$k_{\mathbb{D}^*}(x_n, w_n) \leq k_{\mathbb{D}^*}(x_n, z_n) + k_{\mathbb{D}^*}(z_n, w_n) < 2\delta.$$

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- Thus

$$\delta > k_{\mathbb{D}^*}(w_n, y_n) \geq k_{\mathbb{D}^*}(x_n, y_n) - k_{\mathbb{D}^*}(x_n, w_n) \geq \log\frac{1-\beta}{\beta} - 2\delta - o(n),$$

  a contradiction.

Questions

Let Ω be an attracting non-recurrent invariant Fatou component for an automorphism of $\mathbb{C}^2$.

Q1: Is $\Omega \simeq \mathbb{C}^2$ or $\Omega \simeq \mathbb{C} \times \mathbb{C}^*$?

Q2: Is (the Kobayashi metric) $\kappa_\Omega \equiv 0$?

Q3: Does there exist a Fatou coordinate ψ on Ω such that (Ω, ψ) is a fiber bundle over $\mathbb{C}$? Positive answers to Q2 and Q3 prove Q1.

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Q4: In the example we constructed, by Pöschel, there exist two Siegel discs for *F* tangent to the axis. Can they be extended to entire Siegel curves for *F*?

Thanks for your attention!