Wandering domains from the inside

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Dynamical Systems:
From Geometry to Mechanics
February 5-8, 2019

Holomorphic dynamics in $\mathbb{C}$

The complex plane decomposes into two **totally invariant sets**:

- **The Fatou set (or stable set)**: basins of attraction of attracting or parabolic cycles, Siegel discs (irrational rotation domains), ... **[Fatou classification Theorem, 1920]**

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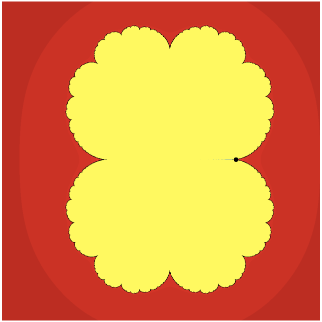
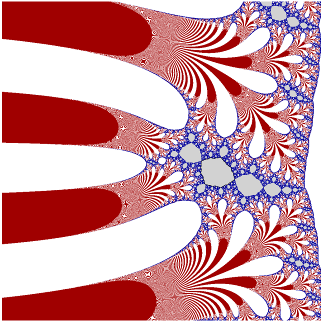
- **The Fatou set (or stable set)**: basins of attraction of attracting or parabolic cycles, Siegel discs (irrational rotation domains), ... **[Fatou classification Theorem, 1920]**
- **The Julia set (or chaotic set)**: the closure of the set of repelling periodic points (boundary between the different stable regions).





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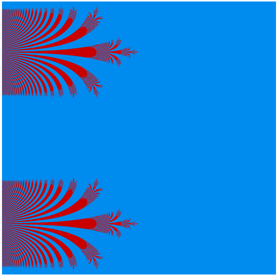
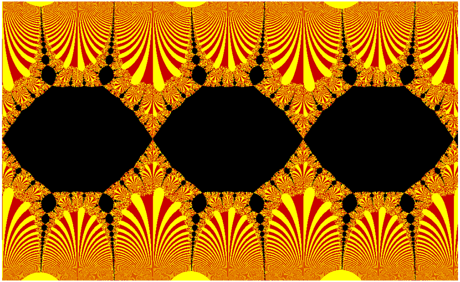
- Transcendental maps may have Fatou components that are not basins of attraction nor rotation domains:
  - U is a **Baker domain** of period 1 if $f^n|_U \to \infty$ loc. unif.
  - U is a **wandering domain** if $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$.



$z + a + b\sin(z)$



$z + 2\pi + \sin(z)$

Wandering domains

Still quite uncharted territory . . .

- They do not exist for rational maps [Sullivan'82] – only for transcendental.
- "Recently" discovered – First example (an infinite product) due to Baker in the 80's (multiply connected, escaping to infinity)
- It is not easy to construct examples – WD are not associated to periodic orbits.
- They do not exist for maps with a finite number of **singular values**.

Singular values

Holomorphic maps are local homeomorphisms everywhere except at the **critical points**

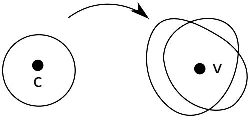
$$Crit(f) = \{c \mid f'(c) = 0\}.$$

Singular values:

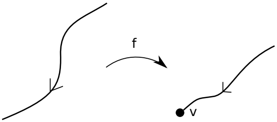
$$S(f) = \{v \in \mathbb{C} \mid \text{not all branches of } f^{-1} \text{ are well defined in a nbd of } v\}.$$

These can be

- **Critical values** $CV = \{v = f(c) | c \in Crit(f)\}$;
- **Asymptotic values** $AV = \{a = \lim_{t\to\infty} f(\gamma(t)); \gamma(t) \to \infty\}$, or
- accumulations of those.



critical value



asymptotic value

Special classes

Some classes of maps are singled out depending on their singular values.

- The **Speisser class or finite type maps**:

$$\mathcal{S} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is finite}\}$$

Example: $z \mapsto \lambda \sin(z)$

Maps in S have **NO WANDERING DOMAINS.**

[Eremenko-Lyubich'87, Goldberg+Keen'89]

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- The **Eremenko-Lyubich class**

$$\mathcal{B} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is bounded}\}$$

  Example: $z \mapsto \lambda \frac{z}{\sin(z)}$.

  Maps in $\mathcal{B}$ have **NO ESCAPING WANDERING DOMAINS**.

  [Eremenko-Lyubich'87]

Types of wandering domains

- $\{f^n\}$ form a normal family on a Wandering domain U.
- All limit functions are constant in $J(f) \cap \overline{P(f)}$ [Baker'02].

$$L(U) = \{a \in \mathbb{C} \cup \infty \mid \exists n_k \to \infty \text{ with } \; f^{n_k} \to a\}$$

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Open question: Do "bounded" domains exist at all?

Oscilating WD in class $\mathcal{B}$

$\to$ a recent result [Bishop'15, Martí-Pete+Shishikura'18]

Examples of wandering domains

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- Quasiconformal surgery [Kisaka-Shishikura'05, Bishop'15, Martí-Pete+Shishikura'18].

State of the art

Postsingular set: $P(f)$ = forward iterates of $S(f)$.

- Examples of WD exist: simply and multiply connected, fast escaping and slowly escaping, bounded (as sets) and unbounded, oscillating, univalent, ...
  [Baker, Rippon+Stallard, Eremenko+Lyubich, F+Henriksen, Sixsmith, ...]

- The relation between limit functions and the singular values is partially understood ($L(U) \in P(f)'$).
  [Baker, Bergweiler *et al*]

- The relation between simply connected WD and $P(f)$ is partially understood. [Rempe-Gillen + Mihailevic-Brandt'16, Baranski+F+Jarque+Karpinska'18]

- **Internal dynamics???**

Lifting of holomorphic maps of $\mathbb{C}^*$: An example

$F(w) = \lambda w^2 e^{-w}$ is semiconjugate under $w = e^z$ to $f(z) = \ln \lambda + 2z - e^z$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ln \lambda + 2z - e^z} & \mathbb{C} \\ e^z \Big\downarrow & & \Big\downarrow e^z \\ \mathbb{C}^* & \xrightarrow{\lambda w^2 e^{-w}} & \mathbb{C}^* \end{array}$$

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BUT ORBITS REMEMBER WHERE THEY CAME FROM!!!

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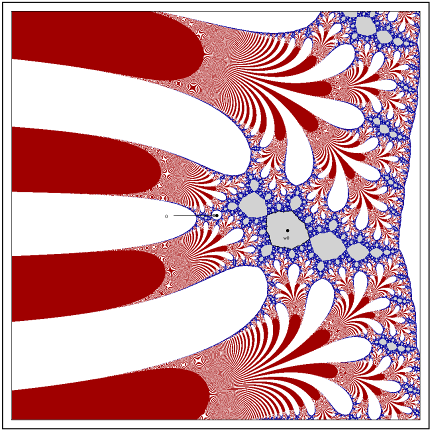
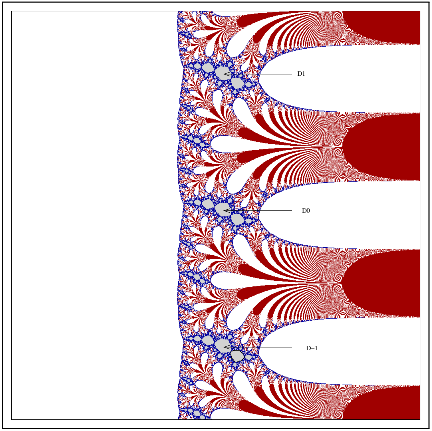


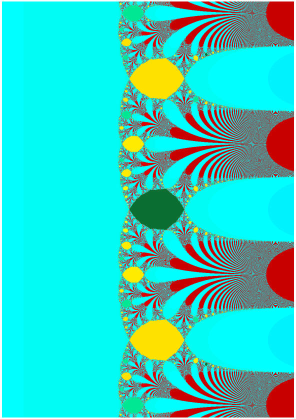


$\lambda_0 w^2 e^{-w}$
Siegel disk (gray).
Basin of 0 (white).

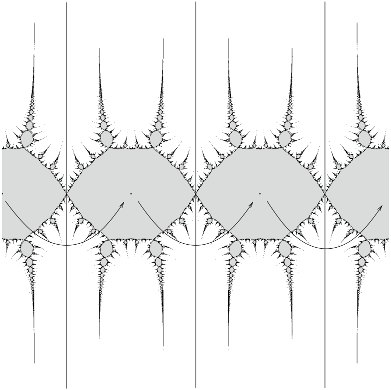
$\ln \lambda_0 + 2z - e^z$
Wandering domain (gray).
Baker domains (white).

Lifting of holomorphic maps of $\mathbb{C}^*$: Examples

Lifts of superattracting basins



$\ln \lambda_1 + 2z - e^z$
Wandering D. (yellow).



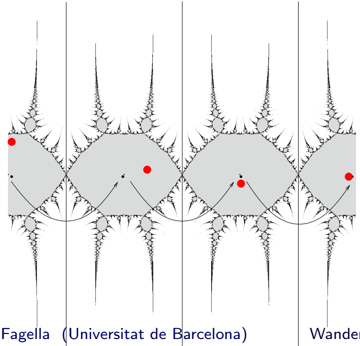
$z + 2\pi + \sin(z)$
WD (black).

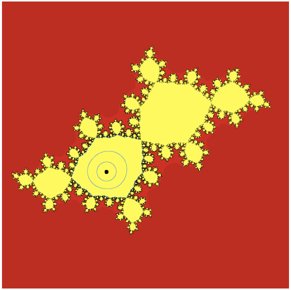
Lifting of holomorphic maps of $\mathbb{C}^*$: Orbits remember

U wandering domain obtained by lifting $V = \exp(U)$

$U_n := f^n(U)$

- V **attracting basin** of a fixed point $p \longrightarrow$ orbits converge to the orbit of $\ln p$, well inside U_n.
- V **parabolic basin** of a fixed point $p \in \partial V \longrightarrow$ orbits converge to the orbit of $\ln p \in \partial U_n$.
- V **Siegel disk** $\longrightarrow$ orbits rotate on the lifts of "invariant curves".





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BUT, dynamics on **multiply connected** wandering domains are quite well understood [Rippon-Stallard]

Internal dynamics

Two prespectives:

- **Orbits move with the wandering domains** (like passengers in a cruise ship follow the ship's trajectory)
- On the other hand there are **intrinsic dynamics relative to each other**, or relative to the domains boundary (like passengers gathering at the buffet for dinner, or going to the ship edges to watch the water).

Internal dynamics: the hyperbolic distance

Intrinsic tool which does not depend on the embedding of the WD in the plane.

- $U_n := f^n(U)$ hyperbolic ($\#\partial U \geq 2$), simply connected.
- $\text{dist}_U(z, w)$ hyperbolic distance between $z, w \in U$.

Schwarz-Pick Lemma

U, V hyperbolic, $f : U \to V$ holomorphic . Then, for all $z, w \in U$,

$$\text{dist}_V(f(z), f(w)) \leq \text{dist}_U(z, w),$$

and "$=$" occurs iff f is an isometry (univalent case).

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Hence $f : U_n \to U_{n+1}$ contracts for all n and

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Different limits for different pairs of z, w???

First classification theorem

Let U be a simply connected, bounded, wandering domain for an entire map f and let $U_n := f^n(U)$. Define the countable set of pairs

$$E = \{(z, w) \in U \times U \mid f^k(z) = f^k(w) \text{ for some } k \in \mathbb{N}\}.$$

Then, exactly one of the following holds as $n \to \infty$, **for all** $(z, w) \notin E$:

(1) U is **(hyperbolically) contracting**, i.e.

$$\mathrm{dist}_{U_n}(f^n(z), f^n(w)) \longrightarrow c(z, w) \equiv 0;$$

(2) U is **(hyperbolically) semi-contracting**, i.e.

$$\mathrm{dist}_{U_n}(f^n(z), f^n(w)) \longrightarrow c(z, w) > 0;$$

(3) U is **(hyperbolically) eventually isometric**, i.e.

$$\exists N > 0 \text{ such that } \forall n \geq N,\ \mathrm{dist}_{U_n}(f^n(z), f^n(w)) = c(z, w) > 0.$$

First classification theorem: Observations

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- Lifts of Siegel disks are **eventually isometric** .
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Could we have several orbits of critical points? (**multiply-supercontracting**?). (Impossible for periodic componentns....)

Moving towards the boundary

Problem: Shape of U_n may degenerate. For example if

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Definition (Convergence to the boundary)

Let Δ_n denote the (euclidean) diameter of the largest disc contained in U_n. We say that the orbit of $z \in U$ *converges to the boundary* (of U_n) if and only if

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Not perfect but quite reasonable.

Second classification theorem

Let U be a simply connected, bounded, wandering domain for an entire map f and let $U_n := f^n(U)$. Then, exactly one of the following holds.

(1) For all $z \in U$

$$\Delta_n \, \lambda_{U_n}(f^n(z)) \underset{n\to\infty}{\longrightarrow} \infty$$

that is, **all orbits converge to the boundary**;

(2) For all points $z \in U$ and every $n_k \to \infty$

$$\Delta_{n_k} \, \lambda_{U_{n_k}}(f^{n_k}(z)) \not\to \infty,$$

that is, **all orbits stay away from the boundary**; or

(3) Neither (1) nor (2), i.e. **all orbits oscillate.**

Convergence to the boundary: Observations

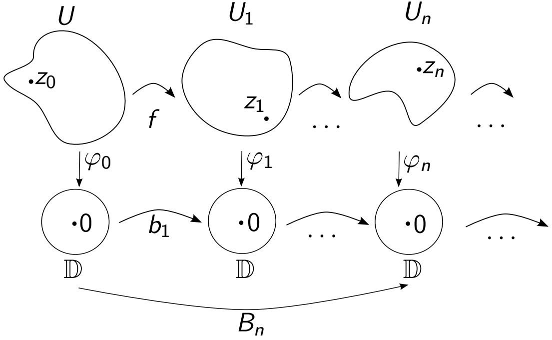
- If U is the lift of a parabolic basin, then U is of type (1) ($\Delta_n = ctant$).
- If U is the lift of a Siegel disk, or an attracting basin, then U is of type (2).
- No llifting example can be of type (3).

Question

In case (1), does there exist a **distinguished point** in the boundary attracting all orbits? (Denjoy-Wolf for this setting?)

A tool

We choose a base point $z_0 \in U$, $z_n := f^n(z_0)$ and choose Riemann maps $\varphi_n : U_n \to \mathbb{D}$ such that $\varphi_n(z_n) = 0$.



The maps $b_n : \mathbb{D} \to \mathbb{D}$ (and hence B_n) are finite Blaschke products.

This can be seen as **Non-autonomous iteration**.

Realization

The classification theorems leave us with a 3×3 table of possibilities.

	$\to \partial$	$\not\to \partial$	oscillating
contracting	Lift of parab. b.	Lift of attrac. b.	?
semi-contracting	?	?	?
ev. isometric	?	Lift of Siegel Disk	?

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ANSWER: YES.

Realization Theorem

Theorem

There exist transcendental entire functions f_i, $i = 1, 2, 3$, having a sequence of bounded, simply connected, escaping wandering domains realizing the following conditions.

(a) Every orbit under f_1 converges to the boundary;

(b) Every orbit under f_2 stays away from the boundary;

(c) Every orbit under f_3 comes arbitrarily close to the boundary but does not converge to it.

Moreover, each of the examples f_i, $i = 1, 2, 3$, can be chosen to be (hyperbolically) attracting, semi-attracting or eventually isometric.

Construction of examples: Approximation theory

Theorem (Extension of Runge's theorem)

Let $\{G_k\}_{k=1}^{\infty}$ *be a sequence of compact subsets of* $\mathbb{C}$ *with the following properties:*

(i) $\mathbb{C} \setminus G_k$ *is connected for every* k*;*

(ii) $G_k \cap G_m = \emptyset$ *for* $k \neq m$*;*

(iii) $\min\{|z|\ z \in G_k\} \to \infty$.

Let $z_{k,i} \in G_k$, $i = 1, \ldots, j$, $\varepsilon_k > 0$ *and the function* ψ *be analytic on* $G = \cup_k G_k$*. Then there exists an entire function* f *satisfying*

$$|f(z) - \psi(z)| < \varepsilon_k, z \in G_k;$$

$$f(z_{k,i}) = \psi(z_{k,i}), \quad f'(z_{k,i}) = \psi'(z_{k,i}), k \in \mathbb{N}.$$

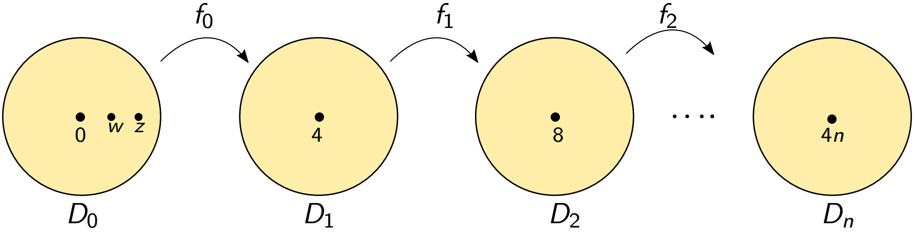
Construction of examples: Approximation theory

We use

- Unit discs D_n centered at $z = 4n$.
- Blaschke products $\beta_n : \mathbb{D} \to \mathbb{D}$ of degree $d_n \geq 1$, moved to the D_n's via translations $T_n(z) = z + 4n$:

$$f_n : T_{n+1} \circ \beta_n \circ T_n^{-1}.$$

- Points $z, w \in D_0$.



Main construction

Theorem

For any choice of β_n, z, w, there exists a transcendental entire function f having a sequence of bounded simply connected escaping wandering domains U_n such that

(i) $\overline{\Delta_n'} := \overline{D(4n, r_n)} \subset U_n \subset D(4n, R_n) := \Delta_n$, where $0 < r_n < 1 < R_n$ and $r_n, R_n \to 1$ as $n \to \infty$;

(ii) $|f(z) - f_n(z)| < o(1)$ uniformly on $\overline{\Delta_n}$.

(iii) $f^n(z) = f_n \circ \cdots \circ f_0(z)$ and $f' = f_n'$ on $f^n(z)$ and $f^n(w)$.

(iv) $f : U_{n-1} \to U_n$ has degree d_n;

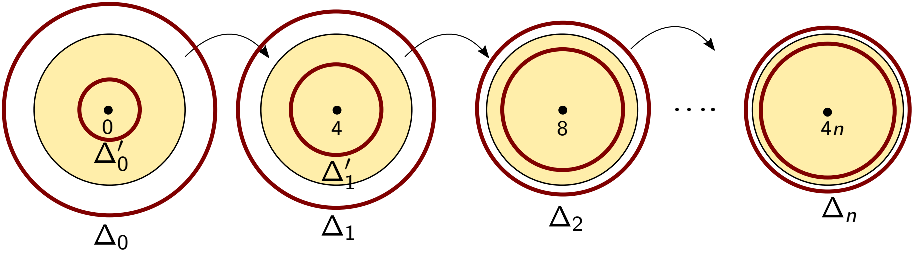
Finally, if $a, b \in \overline{\Delta_1'}$ then the following double inequality is true for the hyperbolic distance

$$k_n \; d_{D_n}(f^n(a), f^n(b)) \leq d_{U_n}(f^n(a), f^n(b)) \leq K_n \; d_{D_n}(f^n(a), f^n(b)),$$

where $k_n < 1 < K_n$ and $k_n, K_n \to 1$ as $n \to \infty$.

Main construction

The wandering domains are sequeezed between Δ_n and Δ'_n (and hence bounded!).



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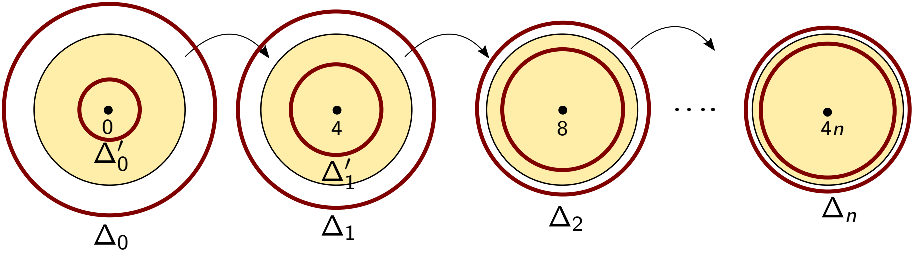

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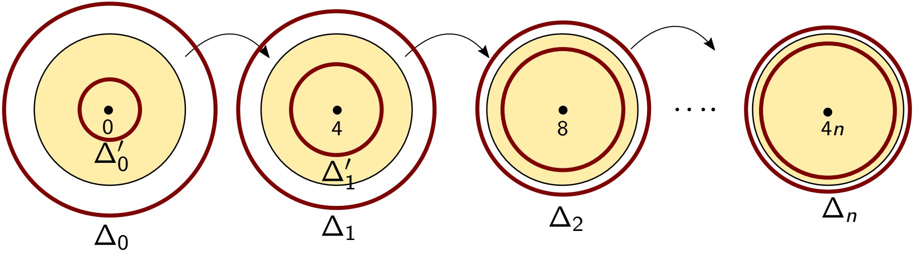

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Obs: We have no information about the global properties of the entire maps f.

Observations and questions

- **Extra bonus:** With this method, we can also construct a wandering orbit of simply connected, bounded, escaping domains with any finite number of orbits of critical points: **multiply super-contracting wandering domains**
- Can we relate the internal dynamics with the global properties of the map? We would need a different method (surgery?) to construct examples with more control on the global results.
- What is the relation between this classification and the postsingular set?
- Possibly, the classification Theorems can be generalized to **unbounded** wandering domains, as long as the degree is finite.

THANK YOU FOR YOUR ATTENTION!

Technical Lemma

Theorem

Let f be a transcendental entire function and suppose that there exist Jordan curves γ_n and Γ_n such that for all $n \geq 0$,

(a) $\gamma_n \subset \operatorname{int}\Gamma_n$;

(b) $\Gamma_n \subset \operatorname{ext}\Gamma_m,\; n \neq m$;

(c) $f(\gamma_n)$ *is surrounded by* γ_{n+1};

(d) $f(\Gamma_n)$ *surrounds* Γ_{n+1};

(e) *there exists* $n_k \to \infty$ *such that for all* k

$$\max\{|z-w| : z \in \Gamma_{n_k}, w \in J(f)\} = o(\operatorname{dist}(\gamma_{n_k}, \Gamma_{n_k})) \;\; \text{as } k \to \infty.$$

Then there exists an orbit of simply connected wandering domains $U_n = f^n(U_0)$ such that $\gamma_n \subset U_n \subset \operatorname{int}\Gamma_n$, for $n \geq 0$.
Moreover, if $f(\gamma_n)$ and $f(\Gamma_n)$ each winds d times round $f^n(z_0)$, for some $z_0 \in \operatorname{int}\gamma_0$, then $f : U_n \to U_{n+1}$ has degree d.