Recent progress on the Viana conjecture

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$f : M \to M$ $C^{1+\alpha}$ surface diffeomorphism. $m =$ Lebesge measure.

Definition (Non-zero Lyapunov exponents)

A set $\Lambda \subseteq M$ with $f(\Lambda) = \Lambda$ has **non-zero Lyapunov exponents** if $\exists$ measurable Df-invariant splitting $T_x M = E^s_x \oplus E^u_x$ such that:

1. $\lim_{n\to\pm\infty} \frac{1}{n} \ln \measuredangle(E^s_{f^n(x)}, E^u_{f^n(x)}) = 0$
2. $\lim_{n\to\pm\infty} \frac{1}{n} \ln \|Df^n_x(e^s)\| < 0 < \lim_{n\to\pm\infty} \frac{1}{n} \ln \|Df^n_x(e^u)\|$

An invariant probability μ is **hyperbolic** if $\mu(\Lambda) = 1$.

By Stable Manifold Theorem, $\exists$ local stable/unstable curves V^s_x, V^u_x.

Definition (Sinai-Ruelle-Bowen measures)

μ is a **Sinai-Ruelle-Bowen** measure if Λ is **fat**: $m(\bigcup_{x\in\Lambda} V^s_x) > 0$.

Conjecture (Viana)

Fat Λ with non-zero Lyapunov exponents $\Rightarrow$ $\exists$ SRB measure

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Definition (Hyperbolic set)

A set $\Lambda \subseteq M$ with $f(\Lambda) = \Lambda$, is (χ, ϵ)**-hyperbolic** if $\exists$ measurable Df-invariant splitting $T_xM = E^s_x \oplus E^u_x$ such that

1. $\measuredangle(E^s_x, E^u_x) \geq C_a(x)$;
2. $\|Df^n_x(e^u)\| \geq C_u(x)e^{\chi n}$ and $\|Df^n_x(e^s)\| \leq C_s(x)e^{-\chi n} \quad \forall\, n \geq 1.$

for measurable positive functions $C_s, C_u, C_a : \Lambda \to (0, \infty)$ satisfying

$$e^{-\epsilon} \leq C(f(x))/C(x) \leq e^{\epsilon} \tag{1}$$

Λ non-zero Lyapunov exponents $\Rightarrow$ (χ, ϵ)-hyperbolic for *every* $\epsilon > 0$.

Definition (Hyperbolic set)

A set $\Lambda \subseteq M$ with $f(\Lambda) = \Lambda$, is (χ, ϵ)**-hyperbolic** if $\exists$ meas. Df-invariant splitting $T_x M = E^s_x \oplus E^u_x$ and meas. positive functions

$$C_s, C_u, C_a \colon \Lambda \to (0, \infty) \quad \text{with} \quad e^{-\epsilon} \le C(f(x))/C(x) \le e^{\epsilon} \quad \text{s.t:} \tag{2}$$

1. $\measuredangle(E^s_x, E^u_x) \ge C_a(x)$;
2. $\|Df^n_x(e^u)\| \ge C_u(x) e^{\chi n}$ and $\|Df^n_x(e^s)\| \le C_s(x) e^{-\chi n} \quad \forall\, n \ge 1$.

Remark

1. If Λ has non-zero Lyap. exponents, it is (χ, ϵ)-hyperbolic $\forall\, \epsilon > 0$;
2. If $\epsilon = 0$, Λ is uniformly hyperbolic
3. If $\epsilon = 0$ for C_a and splitting continuous, Λ is partially hyperbolic

For $\epsilon \ge 0$ sufficiently small, $\exists$ stable/unstable manifolds V^s_x, V^u_x. Their lengths depend on the values of C_a, C_s, C_u at x.

We assume that Λ is (χ, ϵ)-hyperbolic for sufficiently small ϵ.

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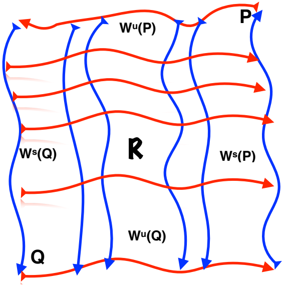
Definition

$\Gamma \subset \Lambda$ is a **rectangle** if $x, y \in \Gamma$ implies $V^s_x \cap V^u_y$ is a single point in Γ. Then

$$\Gamma = C^s \cap C^u = \bigcup_{x \in \Gamma} V^s_x \cap \bigcup_{x \in \Gamma} V^u_x$$



Definition

A rectangle $\Gamma \subseteq \Lambda$ is

1. **nice** if the boundaries are $V^{s/u}_{p/q}$, where p, q periodic points;
2. **recurrent** if every x in Γ returns with positive frequency
3. **fat** if $Leb(\bigcup_{x \in \Gamma} V^s_x) > 0$

Theorem (Climenhaga, L., Pesin)

$\exists\ \Lambda$ *and nice fat recurrent rectangle* $\Gamma \subseteq \Lambda \Leftrightarrow \exists$ *SRB.*

Existing results

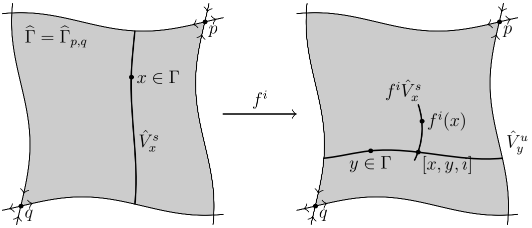
1. Splitting continous, E^u uniform, E^s uniform (*Uniformly Hyp.*)
   Sinai-Ruelle-Bowen *1960's-1970's* (Defin. of **SRB** measures)
2. Splitting continuous, Neutral fixed point
   Katok *1979, Annals of Math*
   Hu *2000, TAMS*
   Alves, Leplaideur *2016, ETDS*
3. Splitting continous, E^u uniform, E^s non-uniform.
   Pesin-Sinai, *(1982) Erg. Th. & Dyn. Syst.*
   Bonatti-Viana *(2000) Israel J. Math.*
4. Splitting continuous, E^s uniform, E^u non-uniform.
   Alves-Bonatti-Viana *(2000) Inv. Math.*
   Alves-Dias-L-Pinheiro *(2015) J. Eur. Math. Soc.*
5. Splitting measurable, E^s, E^u non-uniform
   Benedicks-Young *(1993) Invent. Math.* - Hènon Maps
   Climenhaga-Dolgopyat-Pesin *(2016) Comm. Math. Phys.*

None of the above prove existence under necessary conditions.

Let $\Gamma_0 \subseteq \Lambda$ nice fat recurrent rectangle. Suppose wlog p, q fixed points.

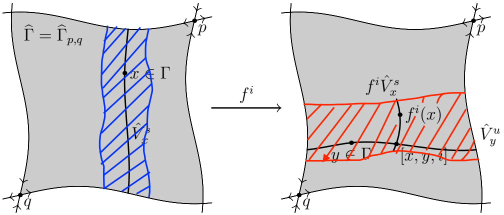
Definition

If $x, y \in \Gamma_0$ and $f^i(V_x^s) \cap V_y^u \neq \emptyset$ then x has an **almost return** to Γ



Theorem (Main Technical Theorem)

Almost return $\Rightarrow \exists$ *hyperbolic branch* $f^i : \widehat{C}^s \to \widehat{C}^u$.



Nice property $\Rightarrow$ for any two such branches, corresponding strips are **nested or disjoint**.

Consider collection of all hyperbolic branches of almost returns:

$$\mathfrak{C} := \{f^i : \widehat{C}^s_{ij} \to \widehat{C}^u_{ij}\}_{ij \in \mathcal{I}}$$

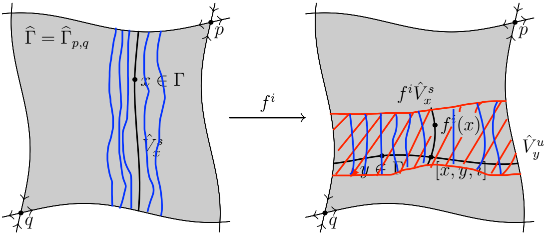
Points of Γ_0 belong to infinitely many C^s_{ij}. Let

$\Gamma :=$ "maximal invariant set" under $\mathfrak{C}$. Then $\Gamma_0 \subseteq \Gamma$.

Proposition

$\Gamma \supset \Gamma_0$ is a nice fat recurrent rectangle with the same collection $\mathfrak{C}$ of hyperbolic branches as Γ_0. Moreover $\Gamma = C^s \cap C^u$ is **saturated**:

1. Every almost return is an actual return;
2. $\forall\ ij \in I$, $C^s_{ij} := f^{-i}(\widehat{C}^u_{ij} \cap C^s) \subseteq C^s$ and $C^u_{ij} := f^i(\widehat{C}^s_{ij} \cap C^u) \subseteq C^u$



Proposition

Let Γ be a nice fat recurrent **saturated** rectangle.
Then the first return map $F = f^\tau : \Gamma \to \Gamma$ defines a Young Tower.

The sets

$$\Gamma^s_{ij} := C^s_{ij} \cap C^u \quad \text{and} \quad \Gamma^u_{ij} := C^u_{ij} \cap C^s$$

are s-subsets and u-subsets of Γ and $f^i(\Gamma^s_{ij}) = \Gamma^u_{ij}$.

Lemma

$\{\Gamma^s_{ij}\}, \{\Gamma^u_{ij}\}$ *are pairwise nested or disjoint.*

Define partial order by inclusion and let $I^* \subset I$ maximal family. Then

- $\mathcal{P} := \{\Gamma^s_{ij}\}_{ij \in I^*}$ is a partition of Γ into s-subsets
- $F|_{\Gamma^s_{ij}} = f^i$ is the first return time to Γ and $F(\Gamma^s_{ij}) = \Gamma^u_{ij}$
- Hyperbolicity of branches $\Rightarrow$ distortion bounds
- Recurrence $\Rightarrow$ integrability of return times

Thus we have a **Young Tower** and therefore an **SRB measure**.