Resonant adiabatic invariants:

Asymptotic behavior and applications

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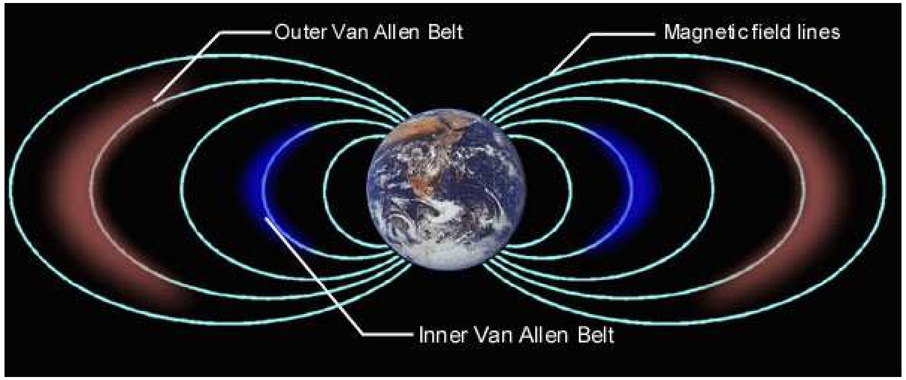
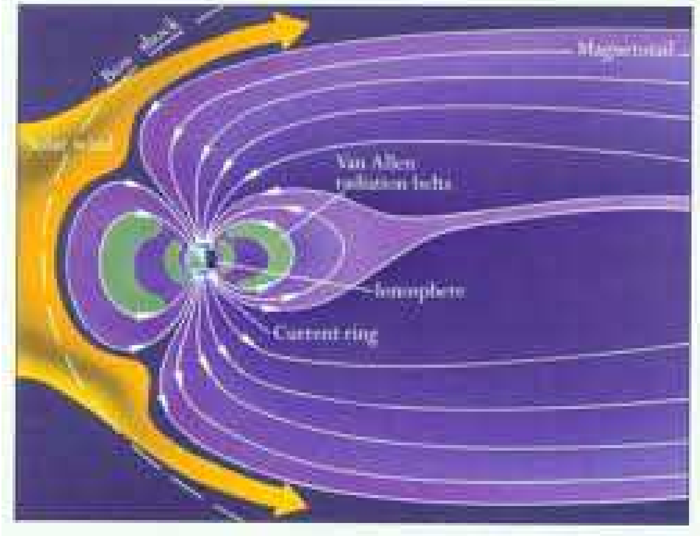
"Adiabatic" condition: one frequency is small

Celestial Mechanics: **Secular** frequencies (of order μ) compared to **mean motion** frequencies (of order unity)

Astrodynamics: **precession** versus **orbital** frequencies

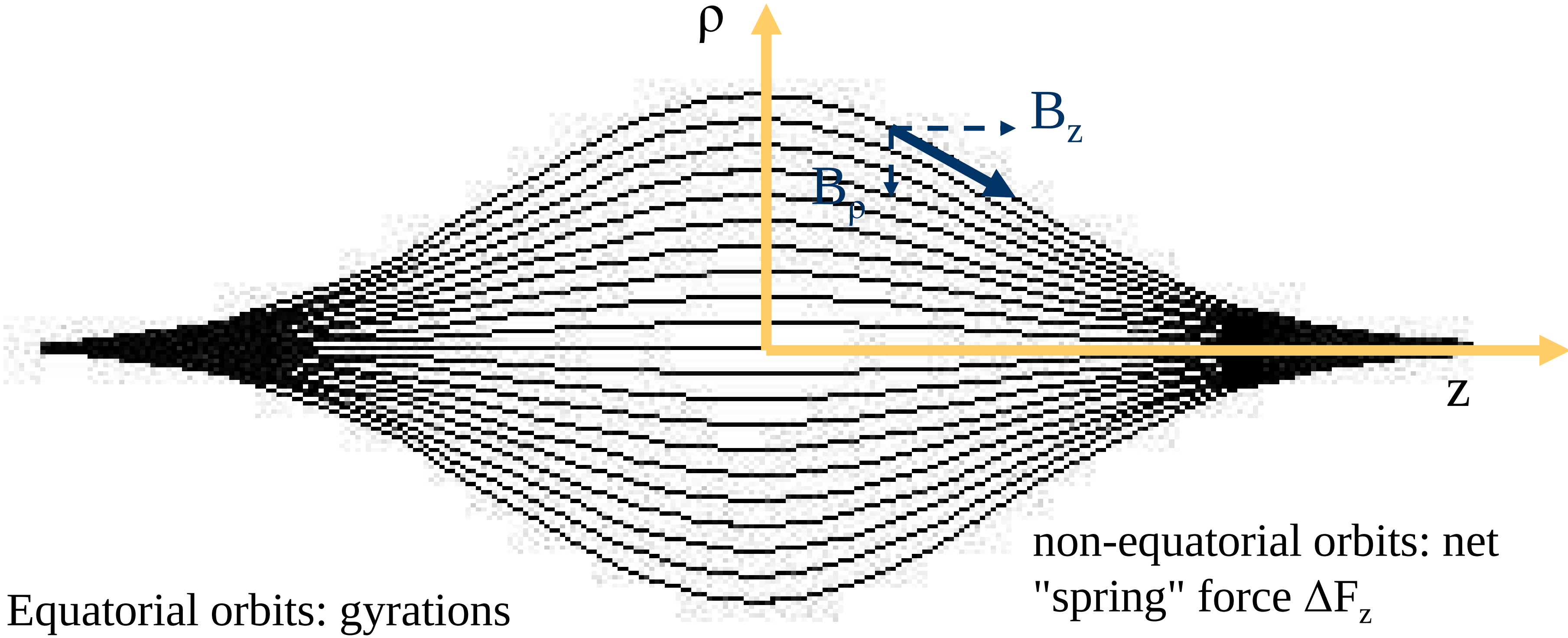
Magnetospheres of stars and planets: "mirror" versus gyration frequencies (e.g. motion of charged particles in the Earth's van Allen zones, or in the current sheet formed in the Earth's magnetotail)



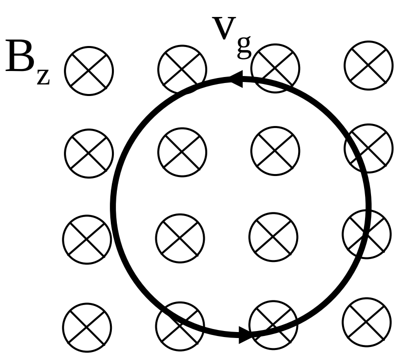


"In the inner belt, particles are trapped in the Earth's nonlinear magnetic field. Particles gyrate and move along field lines. As particles encounter regions of larger density of magnetic field lines, their "longitudinal" velocity is slowed and can be reversed, reflecting the particle. This causes the particles to bounce back and forth between the Earth's poles.[29] Globally, the motion of these trapped particles is chaotic. [30]"

The magnetic bottle paradigm



Equatorial orbits: gyrations

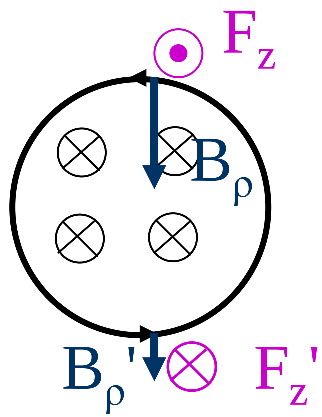


$\omega_c = B_z q/m$

$R_c = m v_g/(B_z q)$

non-equatorial orbits: net "spring" force ΔF_z

$$\omega_2 = \left(\frac{v_g^2}{2B_z} \frac{\partial^2 B_\rho}{\partial\rho\partial z} \right)^{1/2}$$



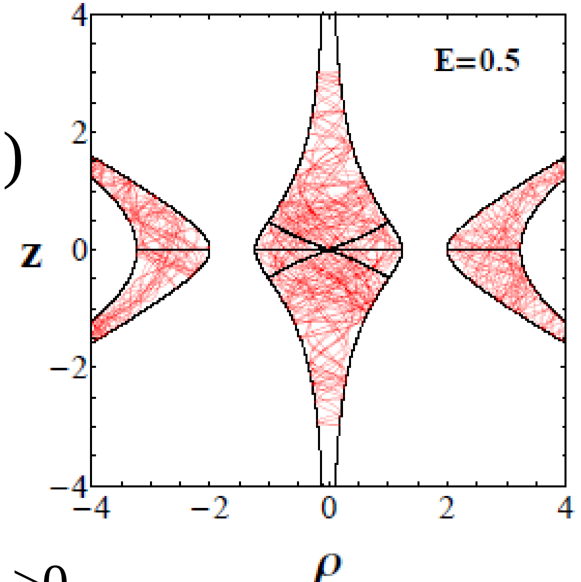
A simple example

$$B_z = B_0 + B_2(z^2 - \rho^2/2)$$

$$B_\rho = -B_2 \rho z$$

$$\vec{B} = \nabla \times \vec{A}$$

vector potential $\quad A_z = A_\rho = 0$

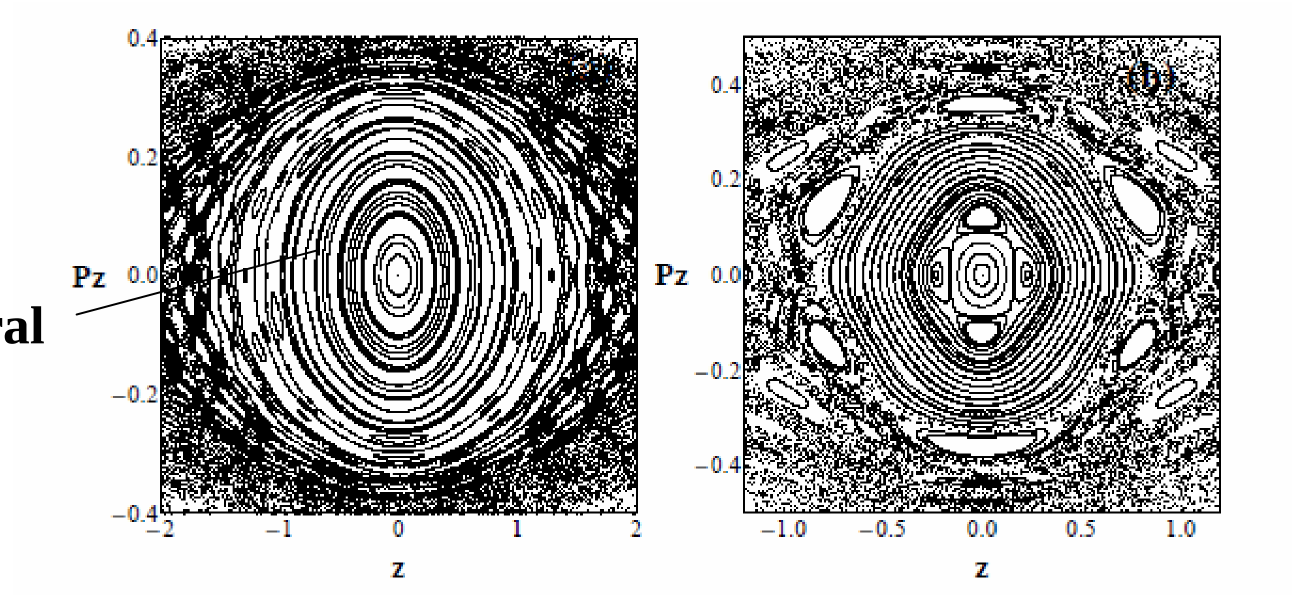


Section: $\rho=0$, $p_\rho>0$

Orbits:
Non-resonant
Resonant
chaotic

Approximate integral

$$\mu = \frac{v_g^2}{2\omega_c}$$



Hamiltonian formulation

$$H = \frac{(\vec{p} + q\vec{A})^2}{2m}$$ (velocity modulus preserved)

For $p_\varphi = 0$ $$H(\rho, z, p_\rho, p_z) = \frac{1}{2}(p_\rho^2 + p_z^2) + V(\rho, z)$$

$$V(\rho, z) = \frac{1}{2}\rho^2 + \frac{1}{2}\rho^2 z^2 - \frac{1}{8}\rho^4 + \frac{1}{8}\rho^2 z^4 - \frac{1}{16}\rho^4 z^2 + \frac{1}{128}\rho^6$$

Basic form

$$H = \frac{p_\rho^2}{2} + \frac{1}{2}\omega_1^2\rho^2 + \frac{p_z^2}{2} + \ldots$$

Non-resonant adiabatic invariants

Kruskal (1962), Contopoulos (1965), Dragt (1965), Bryuno (1971)

Neishtadt (1975), and Lichtenberg and Lieberman (1992)

Canonical formalism - Method of Dragt and Finn (1976)

$$H = \frac{p_\rho^2}{2} + \frac{1}{2}\omega_1^2\rho^2 + \frac{p_z^2}{2} + ...$$

Action - angle canonical pair for the gyration

$$\rho = \sqrt{\frac{2J_1}{\omega_1}} \sin\phi_1, \quad p_\rho = \sqrt{2\omega_1 J_1} \cos\phi_1$$

Series of near-identity canonical transformations $(\phi_1, J_1, z, p_z) \longrightarrow (\theta_1, I_1, \zeta, P_\zeta)$

Hamiltonian in the new variables has the normalized form

$$H(\theta_1, I_1, \zeta, P_\zeta) = Z(I_1, \zeta, P_\zeta) + R(\theta_1, I_1, \zeta, P_\zeta)$$

Normal form structure

$$Z(I_1, \zeta, P_\zeta) = \omega_1 I_1 + \frac{1}{2}(P_\zeta^2 + \omega_2^2(I_1)\zeta^2) + ...$$

Formal process in polynomial variables

i) *Introduction of complex canonical variables.* Introducing the linear canonical change of variables

$$\rho = \frac{q_1 + ip_1}{\sqrt{2}}, \quad p_\rho = \frac{iq_1 + p_1}{\sqrt{2}}, z = q_2, \quad p_z = p_2$$

the Hamiltonian takes the form $H = H_2 + H_4 + H_6$, with

$$H_2(q_1, p_1, p_2) = i\omega_1 q_1 p_1 + \frac{1}{2} p_2^2$$

ii) *Book-keeping:* We organize the terms in the Hamiltonian in groups of 'different order of smallness'. Formally, we introduce a 'book-keeping' parameter λ, with numerical value $\lambda = 1$, and write the Hamiltonian as

$$H \equiv H^{(0)} = H_0^{(0)} + \lambda H_1^{(0)} + \lambda^2 H_2^{(0)} + ...$$

where the superscript (0) means "no normalization step performed so far" while a subscript i, accompanied by a book-keeping power λ^i, means "i-th order of smallness".

iii) *Choice of resonant module M.* To choose a resonant module means to answer the question "which terms do we keep in the normal form along the normalization process".

monomial terms of the form $q_1^{\kappa_1} p_1^{\lambda_1} q_2^{\kappa_2} p_2^{\lambda_2}$

$$\mathcal{M} = \{\kappa_1 = \lambda_1 \text{ and } (\kappa_2 = 0, \lambda_2 = 2 \text{ if } \kappa_1 + \lambda_1 = 0. \text{ or } \lambda_2 = 0 \text{ if } \kappa_1 + \lambda_1 > 0)\}$$

iv) Normalization algorithm with composition of Lie series

$$H^{(r)} \equiv \exp(L_{\chi_r})H^{(r-1)} = Z_2 + Z_3 + Z_4 + ...Z_r + h^{(r)}_{r+1} + ...$$

Homological equation

$$D_\omega \chi_r - \tilde{H}^{(r-1)}_r = 0$$

where D_ω is the linear operator

$$D_\omega = \{\cdot, H_2\}$$

Main issue: the operator **D_ω is not diagonal**

$$D_\omega q_1^{k_1} p_1^{l_1} q_2^{k_2} p_2^{l_2} = i(k_1 - l_1)\omega_1 q_1^{k_1} p_1^{l_1} q_2^{k_2} p_2^{l_2} + k_2 q_1^{k_1} p_1^{l_1} q_2^{k_2-1} p_2^{l_2+1}$$

Solution: solve the homological equation together for terms **in groups**

$$\mathcal{A}_{kl} = \{a_{kl,n} q_1^k p_1^l q_2^n p_2^{r-k-l-n} : k+l \le r, \\ n = 0, 1, \ldots, r-k-l\}$$

the coefficients $a_{kl,n}$, for all k, l, n with $k+l+n = r$ are known. Let $b_{kl,n}$ be the unknown coefficients of the same monomial terms in the function χ_r. Taking into account equations (16) and (18), for any given values of k and l, the two sets of coefficients are linked by a bidiagonal linear system of equations

$$\begin{pmatrix} m & 1 & & & \\ & m & 2 & & \\ & & \ddots & \ddots & \\ & & & m & r-k-l \\ & & & & m \end{pmatrix} \begin{pmatrix} b_{kl,0} \\ b_{kl,1} \\ \vdots \\ \vdots \\ b_{kl,r-k-l} \end{pmatrix} = \begin{pmatrix} a_{kl,0} \\ a_{kl,1} \\ \vdots \\ \vdots \\ a_{kl,r-k-l} \end{pmatrix} \qquad (20)$$

where $m = i(k-l)\omega_1$

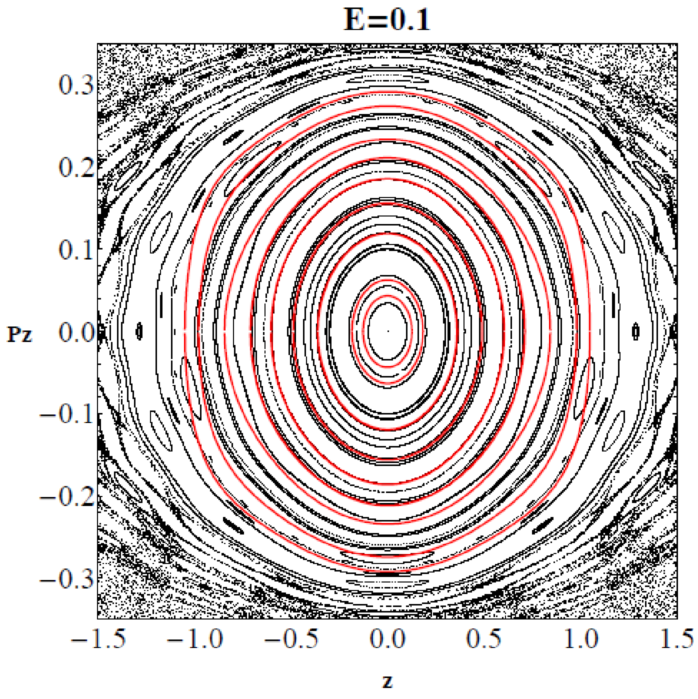
Example

$$
\begin{aligned}
Z_5 \quad = {} & I_1 - 0.1875 I_1^2 - 0.046875 I_1^3 - 0.0256348 I_1^4 \\
& -0.0184021 I_1^5 - 0.0152607 I_1^6 + 0.5 p_2^2 + 0.5 I_1 q_2^2 \\
& +0.15625 I_1^2 q_2^2 + 0.1875 I_1^3 q_2^2 + 0.299194 I_1^4 q_2^2 \\
& +0.551285 I_1^5 q_2^2 - 0.151042 I_1^2 q_2^4 - 0.46224 I_1^3 q_2^4 \\
& -1.29767 I_1^4 q_2^4 + 0.107812 I_1^2 q_2^6 + 0.669227 I_1^3 q_2^6 \\
& -0.0697545 I_1^2 q_2^8
\end{aligned}
$$

where $I_1 = i\omega_1 q_1 p_1$ is a formal integral of motion.

Back-transform I_1 to the original canonical variables

Compute level curves of I_1 on the surface of section



How to deal with resonances?

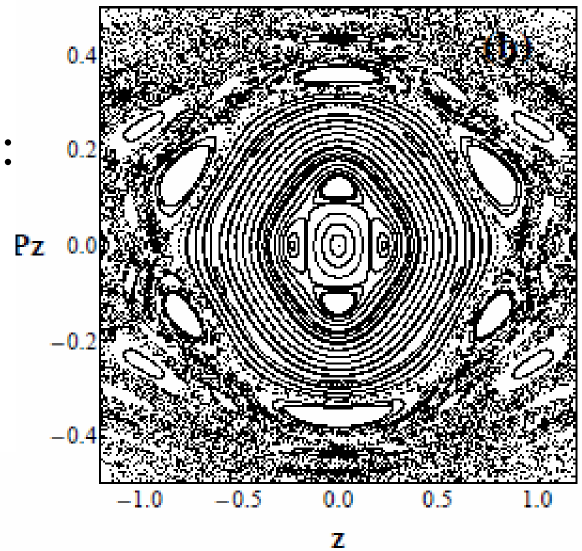
Bifurcation of resonances at $z=P_z=0$

For given value of the energy, one has:

$$Z(I_1^*) = E^*$$

$$\omega_1^* = \frac{\partial Z(I_1)}{\partial I_1}|_{I_1=I_1^*}$$

$$\omega_2^* = \sqrt{2coef(I_1)} \quad (for \quad I_1 = I_1^*)$$



where coef(I_1) is the coefficient in front of q_2^2 in Z.

Compute the value of I_1 where a resonance bifurcates by solving

$$\mathbf{m_1\omega_1(I_1)+ m_2\omega_2(I_1)=0 = 0}$$

Combine "book-keeping" with "detuning" (Pucacco & collaborators)

$$\begin{aligned} H' &= H(q_1, p_1, z, p_z) - i\omega_1 q_1 p_1 + i\omega_1^* q_1 p_1 \\ &+ \tfrac{1}{2}\omega^{*2}_2 z^2 + \lambda(i\omega_1 q_1 p_1 - i\omega_1^* q_1 p_1 - \tfrac{1}{2}\omega^{*2}_2 z^2) \end{aligned}$$

The "kernel" term of the normalization scheme acquires the usual form

$$H_0 = \lambda^0 (i\omega_1^* q_1 p_1 + \frac{1}{2} p_z^2 + \frac{1}{2}\omega^{*2}_2 z^2)$$

Introduce a second pair of complex canonical variables

$$z = \frac{q_2 + ip_2}{\sqrt{2\omega_2^*}} \quad , \quad p_z = \frac{\sqrt{\omega_2^*}(iq_2 + p_2)}{\sqrt{2}}$$

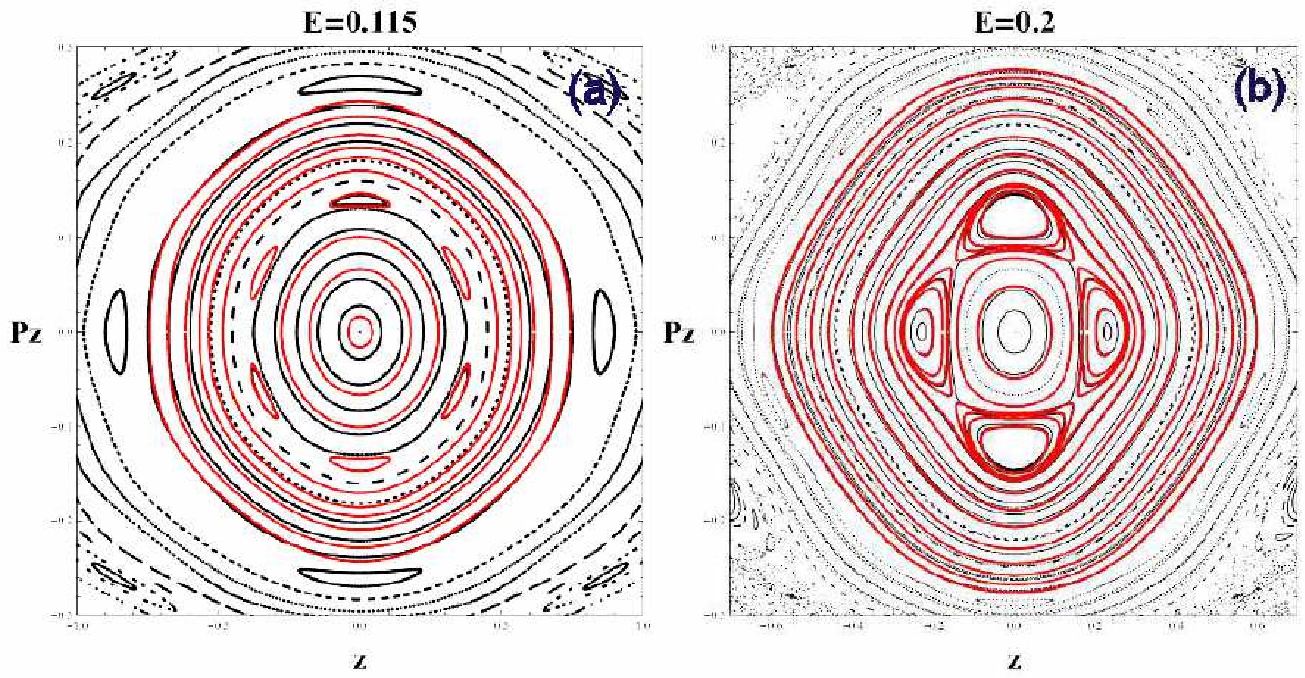
$$H_0(q_1, p_1, q_2, p_2) = i\omega_1^* q_1 p_1 + i\omega_2^* q_2 p_2$$

Choice of **resonant module**: for monomials of the form $q_1^{\kappa_1} p_1^{\lambda_1} q_2^{\kappa_2} p_2^{\lambda_2}$

$$\mathcal{M} = (k_1 - l_1)m_1 + (k_2 - l_2)m_2 = 0$$

Resonant integral: $I_{res} = m_1 i q_1 p_1 + m_2 i q_2 p_2$

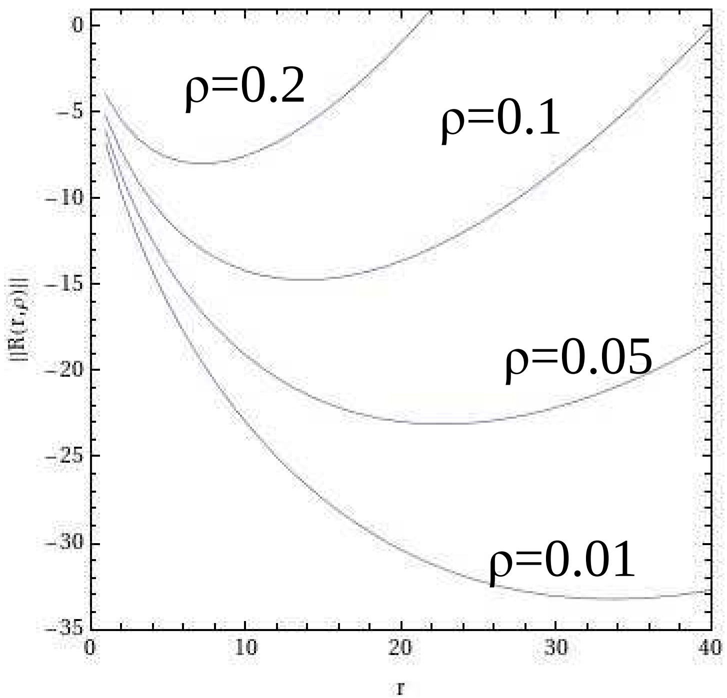
Back transform to the original variables



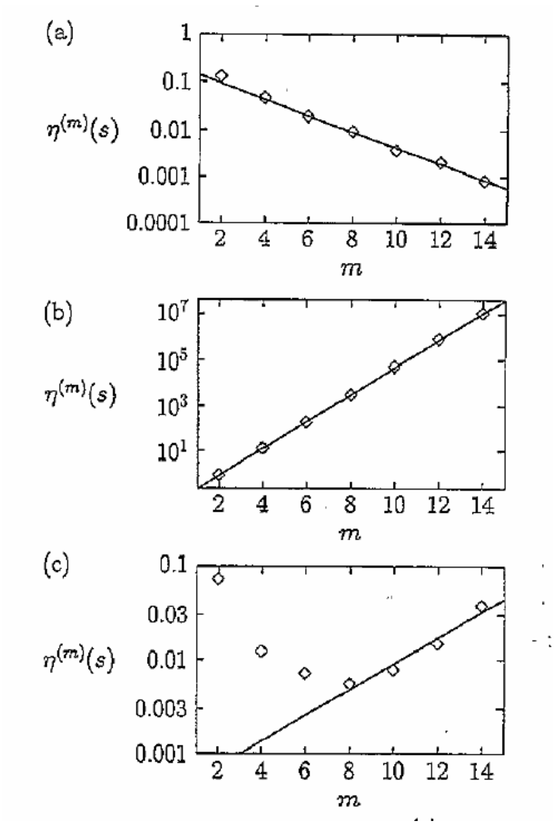
Asymptotic behavior

The formal adiabatic invariant series exhibit an **asymptotic behavior**

Remainder size at different "distances" ρ from the origin

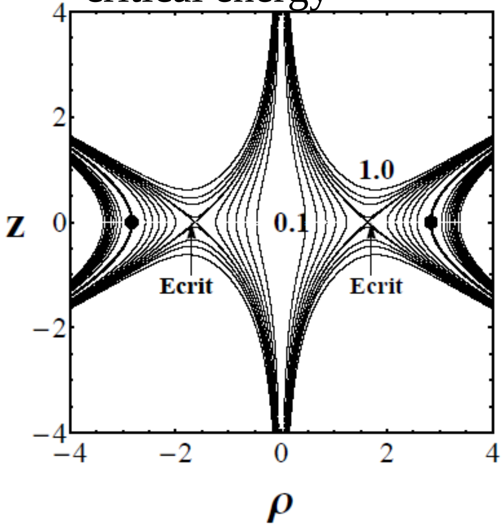


Claims of "convergence" *(Engel et al. 1995)* are just an artifact of not reaching sufficiently high normalization order



"Escape" chaotic dynamics and invariant manifolds

Equipotential lines at the critical energy



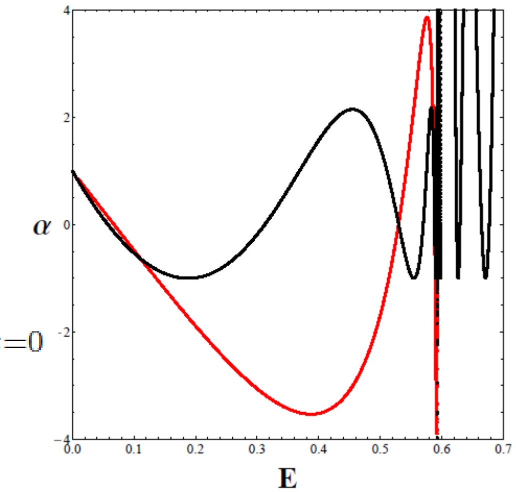
Infinitely many transitions of the central orbit from stability to instability and vice versa

Heggie(1983)

$$\delta = \frac{|E_n - E_{crit}|}{|E_{n+1} - E_{crit}|}$$

$$\delta = exp(\pi\sqrt{|\frac{V_{\rho\rho}}{V_{zz}}|})\ |_{\rho=\rho_{max},z=0}$$

$$\delta \approx 15.19$$



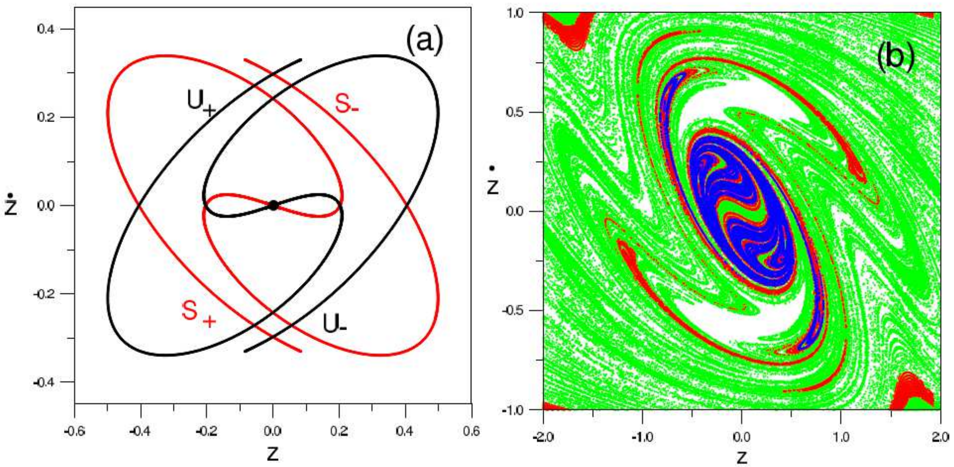


Table I			
n	E_n	$\Delta E_n = \lvert E_n - E_{crit}\rvert$	$\delta = \frac{\Delta E_n}{\Delta E_{n+1}}$
1	0.366882	0.225711	13.03
2	0.5752725	0.017320	15.073
3	0.5914435	0.00114959	15.181
4	0.5925169	0.0000756926	15.176
5	0.592587605	4.98759 x $10^{-}6$	15.200
6	0.592592264575	3.28017 x $10^{-}7$	15.188
7	0.592592570995	2.1597 x $10^{-}8$	-

Conclusions:

1. Computation of non-resonant adiabatic invariants to a **high normalization order**:

   Gives quite **precise** adiabatic invariants
   Computation of the mirror frequency ω_2, as well as of high-order corrections to the gyro-frequency. Computation of the bifurcation energies for resonances
   Reveals the asymptotic character of the formal series

2. **Resonant adiabatic invariants**
   Compute first the non-resonant series
   "Restructure" the Hamiltonian by a combination of "book-keeping" with "detuning"
   Usual procedure allows to obtain the phase portraits in the neighborhood of resonances
   Prediction of the critical energy for the global onset of chaos

3. Dynamics of escapes: correlation with manifold dynamics