# Everything is under control

## Optimal control and applications to aerospace problems

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# What is control theory?

## Controllability

Steer a system from an initial configuration to a final configuration.

## Optimal control

Moreover, minimize a given criterion.

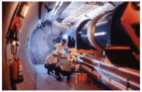
## Stabilization

A trajectory being planned, stabilize it in order to make it robust, insensitive to perturbations.

## Observability

Reconstruct the full state of the system from partial data.

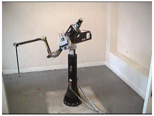
# Control theory and applications

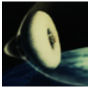


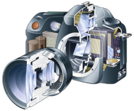
**Application domains of control theory:**


## Mechanics

Vehicles (guidance, dampers, ABS, ESP, …),
Aeronautics, aerospace (shuttle, satellites), robotics

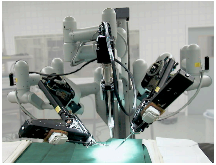


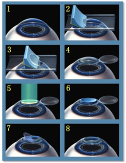




## Biology, medicine

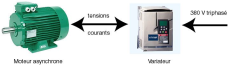
Predator-prey systems, bioreactors, epidemiology,
medicine (peacemakers, laser surgery)

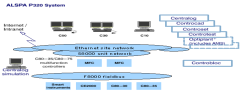





## Electricity, electronics

RLC circuits, thermostats, regulation, refrigeration, computers, internet
and telecommunications in general, photography and digital video



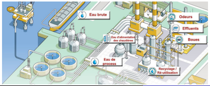



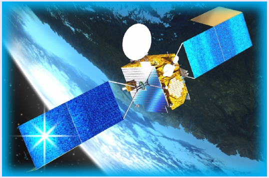

## Economics

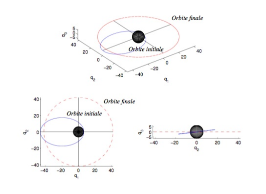
Gain optimization, control of financial flux,
Market prevision

## Chemistry

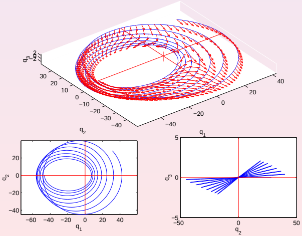
Chemical kinetics, engineering process, petroleum, distillation, petrochemical industry



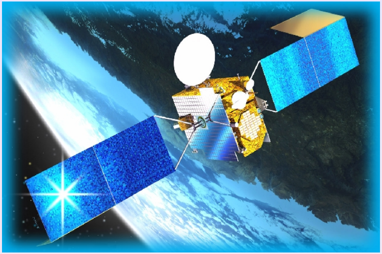





Here we focus on applications of control theory to problems of aerospace.



# The orbit transfer problem with low thrust



## Controlled Kepler equation

$$\ddot{q} = -q\frac{\mu}{r^3} + \frac{F}{m}$$

$q \in \mathbb{R}^3$: position, $r = |q|$, $F$: thrust, $m$ mass:

$$\dot{m} = -\beta|F|$$

## Maximal thrust constraint

$|F| = (u_1^2 + u_2^2 + u_3^2)^{1/2} \leq F_{\max} \simeq 0.1N$

## Orbit transfer

from an initial orbit to a given final orbit.

## Controllability properties studied in

B. Bonnard, J.-B. Caillau, E. Trélat, *Geometric optimal control of elliptic Keplerian orbits*, Discrete Contin. Dyn. Syst. Ser. B **5**, 4 (2005), 929–956.

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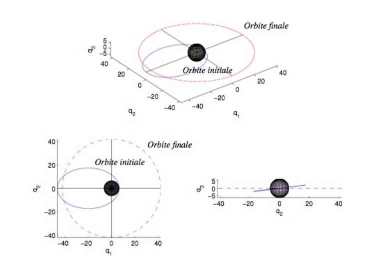


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# Modelization in terms of an optimal control problem

State: $x(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}$

Control: $u(t) = F(t)$

## Optimal control problem

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(t) \in \mathbb{R}^n, \qquad u(t) \in \Omega \subset \mathbb{R}^m,$$

$$x(0) = x_0, \quad x(T) = x_1,$$

$$\min C(T, u), \quad \text{where} \quad C(T, u) = \int_0^T f^0(x(t), u(t))\, dt$$

# Pontryagin Maximum Principle

## Optimal control problem

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$$x(T) = x_1, \quad \min C(T, u), \quad \text{where } C(T, u) = \int_0^T f^0(x(t), u(t))\, dt.$$

## Pontryagin Maximum Principle

Every minimizing trajectory $x(\cdot)$ is the projection of an *extremal* $(x(\cdot), p(\cdot), p^0, u(\cdot))$ solution of

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \qquad H(x, p, p^0, u) = \max_{v \in \Omega} H(x, p, p^0, v),$$

where $H(x, p, p^0, u) = \langle p, f(x, u) \rangle + p^0 f^0(x, u)$.

An extremal is said *normal* whenever $p^0 \neq 0$, and *abnormal* whenever $p^0 = 0$.

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$$u(t) = u(x(t),p(t))$$

$\Big($locally, e.g. under the strict Legendre assumption: $\dfrac{\partial^2 H}{\partial u^2}(x,p,u)$ negative definite$\Big)$

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## Shooting method:

Extremals $(x,p)$ are solutions of

$$\dot{x} = \frac{\partial H}{\partial p}(x,p), \quad x(0) = x_0, \qquad (x(T) = x_1),$$
$$\dot{p} = -\frac{\partial H}{\partial x}(x,p), \quad p(0) = p_0,$$

where the optimal control maximizes the Hamiltonian.

**Exponential mapping**

$$\exp_{x_0}(t, p_0) = x(t, x_0, p_0),$$

(extremal flow)

$\longrightarrow$ Shooting method: determine $p_0$ s.t. $\boxed{\exp_{x_0}(t, p_0) = x_1}$

**Remark**

- PMP = first-order necessary condition for optimality.
- Necessary / sufficient (local) second-order conditions: **conjugate points**.

$\longrightarrow$ test if $\exp_{x_0}(t,\cdot)$ is an immersion at $p_0$.

There exist other numerical approaches to solve optimal control problems:

- direct methods: discretize the whole problem
  $\Rightarrow$ finite-dimensional nonlinear optimization problem with constraints
- Hamilton-Jacobi methods.

The shooting method is called an indirect method.

In the present aerospace applications, the use of shooting methods is priviledged in general because of their very good numerical accuracy.

BUT: difficult to make converge... (Newton method)

To improve their performances and widen their domain of applicability, optimal control tools must be combined with other techniques:

- geometric tools $\Rightarrow$ geometric optimal control
- continuation or homotopy methods
- dynamical systems theory

E. Trélat, *Optimal control and applications to aerospace: some results and challenges*, J. Optim. Theory Appl. (2012).

# Orbit transfer, minimal time

Maximum Principle $\Rightarrow$ the extremals $(x, p)$ are solutions of

$$\dot{x} = \frac{\partial H}{\partial p}, \; x(0) = x_0, \; x(T) = x_1, \quad \dot{p} = -\frac{\partial H}{\partial x}, \; p(0) = p_0,$$

with an optimal control saturating the constraint: $\|u(t)\| = F_{max}$.

$\longrightarrow$ *Shooting method: determine $p_0$ s.t. $x(T) = x_1$,*

*combined with a homotopy on $F_{max} \mapsto p_0(F_{max})$*

Heuristic on $t_f$:

$$t_f(F_{max}) \cdot F_{max} \simeq \text{cste.}$$

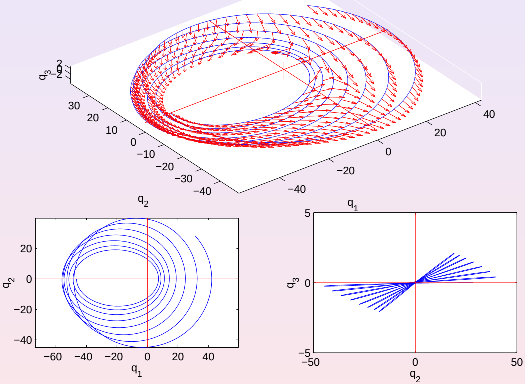
(the optimal trajectories are "straight lines", Bonnard-Caillau 2009)

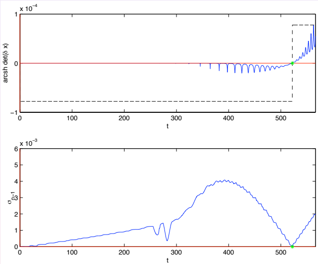
| $F_{max}$ | $t_f$ | Exécution | $F_{max}$ | $t_f$ | Exécution |
|---|---|---|---|---|---|
| 60 | 14.800 | 1 | 1.4 | 606.13 | 33 |
| 24 | 34.716 | 5 | 1 | 853.31 | 44 |
| 12 | 70.249 | 3 | 0.7 | 1214.5 | 64 |
| 9 | 93.272 | 7 | 0.5 | 1699.4 | 234 |
| 6 | 141.22 | 6 | 0.3 | 2870.2 | 223 |
| 3 | 285.77 | 22 | 0.2 | 4265.7 | 226 |
| 2 | 425.61 | 22 | | | |

(Caillau, Gergaud, Haberkorn, Martinon, Noailles, ...)

# Orbit transfer, minimal time

$\mathbf{F_{max}} = \mathbf{6}$ Newton  $P_0 = 11625$ km, $|e_0| = 0.75$, $i_0 = 7^o$, $P_f = 42165$ km




Minimal time: 141.6 hours ($\simeq$ 6 days). First conjugate time: 522.07 hours.

**Main tool used**: continuation (homotopy) method
$\longrightarrow$ continuity of the optimal solution with respect to a parameter $\lambda$

Theoretical framework (sensitivity analysis):

$$\exp_{x_0,\lambda}(T, p_0(\lambda)) = x_1$$

**Local feasibility is ensured:**
in the absence of conjugate points.

$\downarrow$

Numerical test of Jacobi fields.

**Global feasibility is ensured:**
in the absence of abnormal minimizers.

$\downarrow$

this holds true for *generic* systems having more than 3 controls
(Chitour-Jean-Trélat, J. Differential Geom., 2006)

Recent work with EADS Astrium (now Airbus DS):

Minimal consumption transfer for launchers Ariane V and next Ariane VI (third atmospheric phase, strong thrust)

Objective: automatic and instantaneous software.

- continuation on the curvature of the Earth (flat Earth → round Earth)

  M. Cerf, T. Haberkorn, E. Trélat, *Continuation from a flat to a round Earth model in the coplanar orbit transfer problem*, Optimal Appl. Cont. Methods (2012).

- eclipse constraints → state constraints, hybrid systems

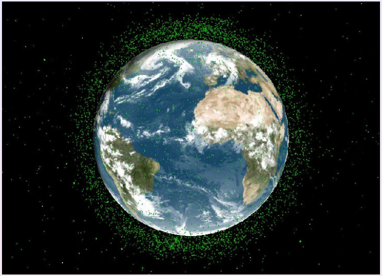
  T. Haberkorn, E. Trélat, *Convergence results for smooth regularizations of hybrid nonlinear optimal control problems*, SIAM J. Control Optim. (2011).

# Optimal control

**A challenge** (urgent!!)

Collecting space debris:

- 22000 debris of more than 10 cm (cataloged)
- 500000 debris between 1 and 10 cm (not cataloged)
- millions of smaller debris



In low orbit

→ difficult mathematical problems combining optimal control, continuous / discrete / combinatorial optimization

(Max Cerf, PhD 2012)

M. Cerf, *Multiple space debris collecting mission - Debris selection and trajectory optimization*, J. Optim. Theory Appl. (2013).

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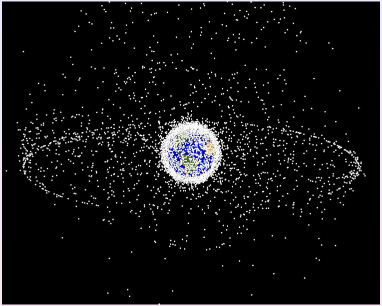

Around the geostationary orbit

→ difficult mathematical problems combining optimal control, continuous / discrete / combinatorial optimization

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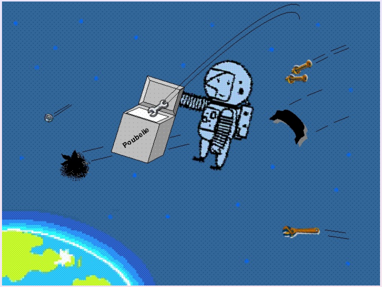

The space garbage collectors

→ difficult mathematical problems combining optimal control, continuous / discrete / combinatorial optimization

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# The circular restricted three-body problem

Dynamics of a body with negligible mass in the gravitational field of two masses $m_1$ and $m_2$ (primaries) having circular orbits:

## Equations of motion in the rotating frame

$$\ddot{x} - 2\dot{y} = \frac{\partial \Phi}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial \Phi}{\partial y}$$

$$\ddot{z} = \frac{\partial \Phi}{\partial z}$$

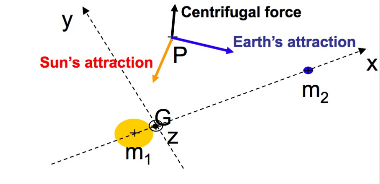
with

$$\Phi(x, y, z) = \frac{x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2},$$

and

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2},$$

$$r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}.$$




## Some references

American team:

Koon, Lo, Marsden, Ross...

Spanish team:

Gomez, Jorba, Llibre, Masdemont, Simo...

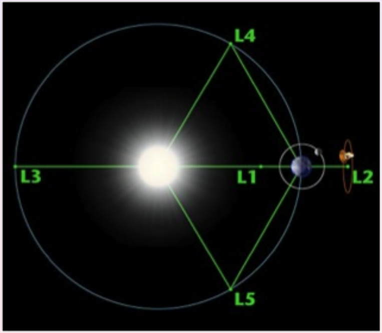
# Lagrange points

Jacobi integral $J = 2\Phi - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ $\longrightarrow$ 5-dimensional energy manifold

Five equilibrium points:

- 3 collinear equilibrium points: $L_1$, $L_2$, $L_3$ (unstable);
- 2 equilateral equilibrium points: $L_4$, $L_5$ (stable).

(see Szebehely 1967)

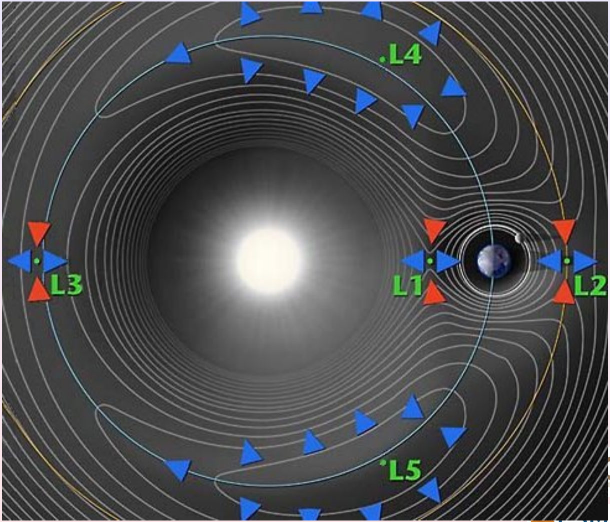



Extension of a Lyapunov theorem (Moser) $\Rightarrow$ same behavior than the linearized system around Lagrange points.

# Lagrange points in the Earth-Sun system

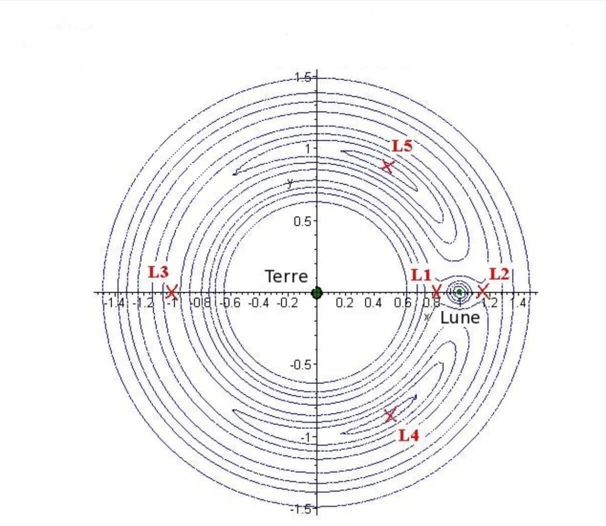
From Moser's theorem:

- $L_1$, $L_2$, $L_3$: unstable.
- $L_4$, $L_5$: stable.



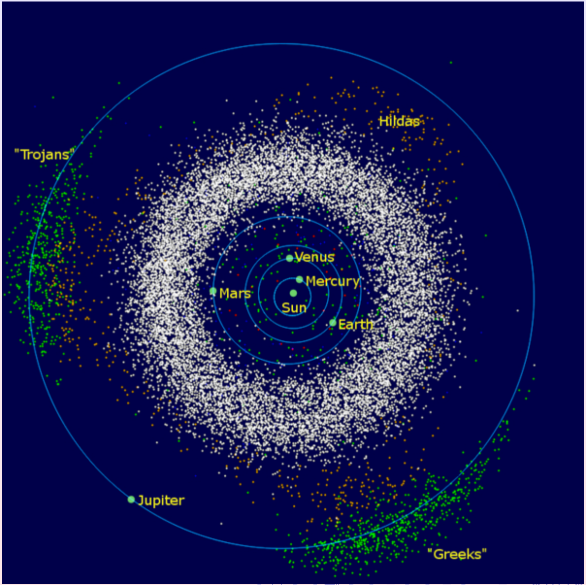
# Lagrange points in the Earth-Moon system

- $L_1, L_2, L_3$: unstable.
- $L_4, L_5$: stable.



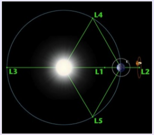
# Examples of objects near Lagrange points

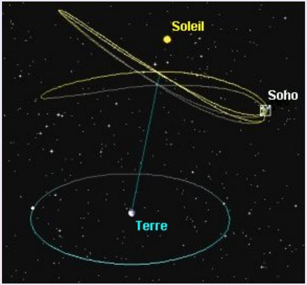
Points L4 and L5 (stable) in the Sun-Jupiter system:
Trojan asteroids



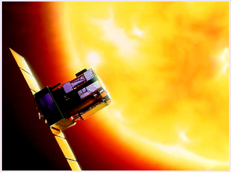
# Examples of objects near Lagrange points

Sun-Earth system:

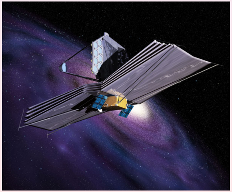



Point L1: SOHO



Point L2: JWST



Point L3: planet X...

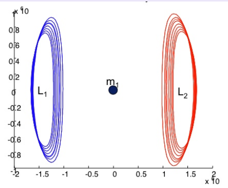
# Periodic orbits

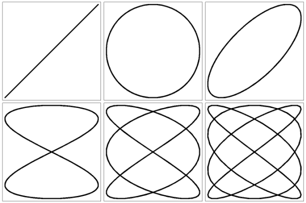
From a Lyapunov-Poincaré theorem, there exist:

- a 2-parameter family of periodic orbits around $L_1$, $L_2$, $L_3$
- a 3-parameter family of periodic orbits around $L_4$, $L_5$

Among them:

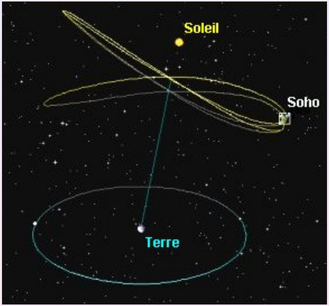
- planar orbits called Lyapunov orbits;
- 3D orbits diffeomorphic to circles called halo orbits;
- other 3D orbits with more complicated shape called Lissajous orbits.



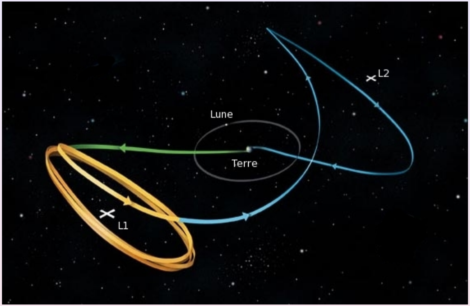



(see Richardson 1980, Gomez Masdemont Simo 1998)

Examples of the use of halo orbits:




Orbit of SOHO around L1



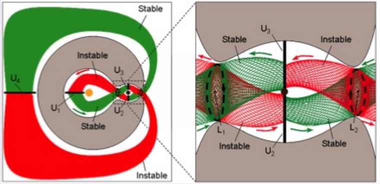

Orbit of the probe Genesis (2001–2004)

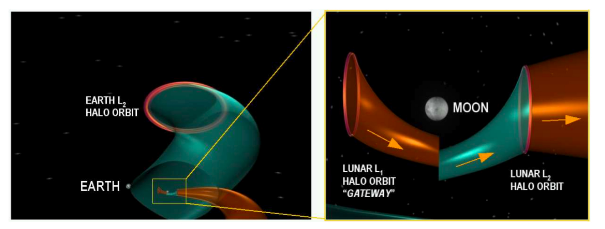
(requires control by stabilization)

# Invariant manifolds

Invariant manifolds (stable and unstable) of periodic orbits:
4-dimensional tubes ($S^3 \times \mathbb{R}$) inside the 5-dimensional energy manifold.
(they play the role of separatrices)

$\longrightarrow$ invariant "tubes", kinds of "gravity currents" $\Rightarrow$ low-cost trajectories



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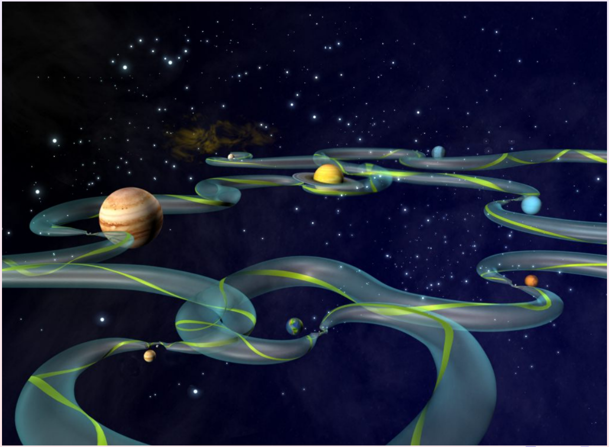

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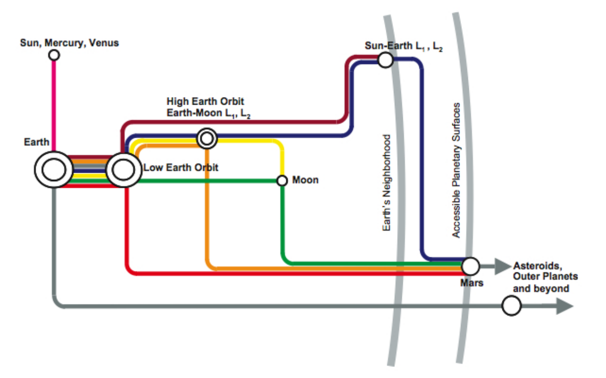
(they play the role of separatrices)

$\longrightarrow$ invariant "tubes", kinds of "gravity currents" $\Rightarrow$ low-cost trajectories




Cartography $\Rightarrow$ design of low-cost interplanetary missions

# Meanwhile...


Back to the Moon

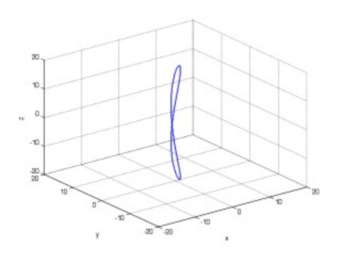
$\Rightarrow$ lunar station: intermediate point for interplanetary missions

**Challenge:** design low-cost trajectories to the Moon and flying over all the surface of the Moon.

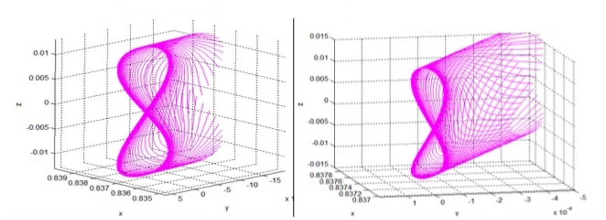
Mathematics used:
dynamical systems theory, differential geometry,
ergodic theory, control, scientific computing, optimization

# Eight Lissajous orbits

(PhD thesis of G. Archambeau, 2008)
Periodic orbits around $L_1$ et $L_2$ (Earth-Moon system) having the shape of an eight:




$\Rightarrow$ Eight-shaped invariant manifolds:

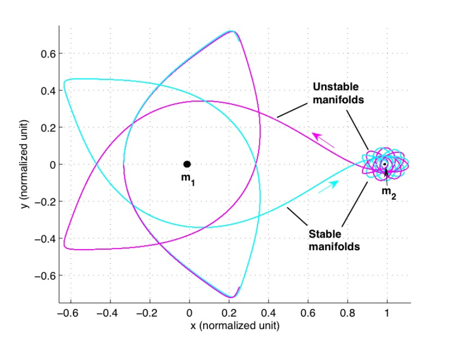


# Invariant manifolds of Eight Lissajous orbits

We observe numerically that they enjoy two nice properties:

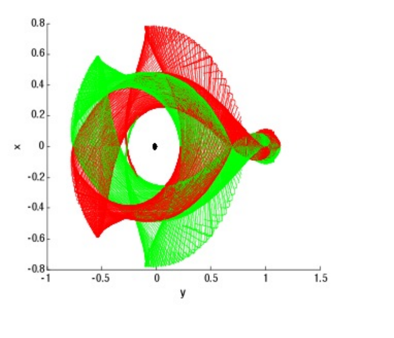
1) Stability in long time of invariant manifolds

Invariant manifolds of an Eight Lissajous orbit:




→ global structure conserved

Invariant manifolds of a halo orbit:




→ chaotic structure in long time

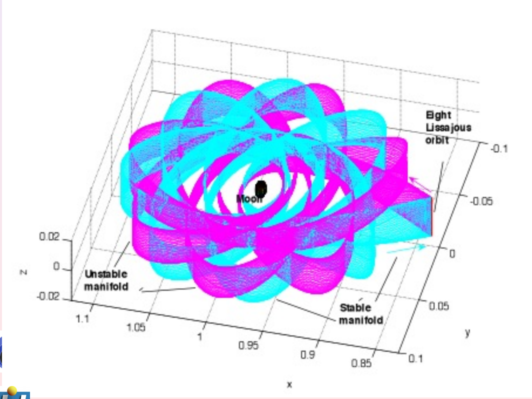
(numerical validation by computation of local Lyapunov exponents)

# Invariant manifolds of Eight Lissajous orbits

We observe numerically that they enjoy two nice properties:

**2) Flying over almost all the surface of the Moon**

Invariant manifolds of an eight-shaped orbit around the Moon:




- oscillations around the Moon
- global stability in long time
- minimal distance to the Moon: 1500 km.

G. Archambeau, P. Augros, E.Trélat,
*Eight Lissajous orbits in the Earth-Moon system*,
MathS in Action (2011).

# Perspectives

Partnership between EADS Astrium (les Mureaux, France) and FSMP (Fondation Sciences Mathématiques de Paris). Kick off in May 2014.

- Planning low-cost ”cargo” missions to the Moon (using gravity currents)
  $\longrightarrow$ Maxime Chupin, ongoing PhD
- Interplanetary missions: compromise between low cost and long transfer time; gravitational effects (swing-by)
- collecting space debris (urgent!)
- optimal design of space vehicles
- optimal placement problems (vehicle design, sensors)
- Inverse problems: reconstructing a thermic, acoustic, electromagnetic environment (coupling ODE’s / PDE’s)
- Robustness problems
- ...

# Invariant manifolds of eight-shaped Lissajous orbits

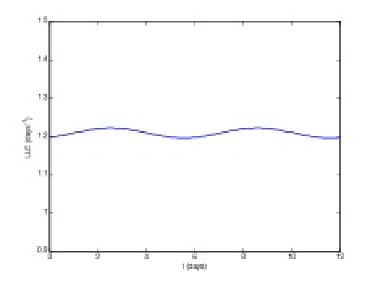
$\Phi(\cdot, t)$: transition matrix along a reference trajectory $x(\cdot)$
$\Delta > 0$.

## Local Lyapunov exponent

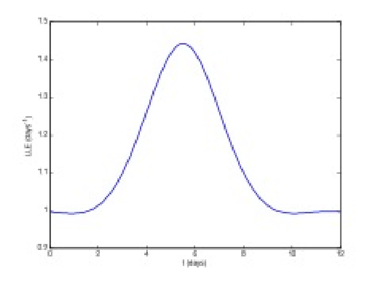
$$\lambda(t, \Delta) = \frac{1}{\Delta} \ln \left( \text{maximal eigenvalue of } \sqrt{\Phi(t+\Delta, t)\Phi^T(t+\Delta, t)} \right)$$

Simulations with $\Delta = 1$ day.

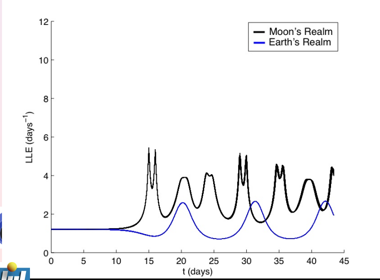
LLE of an eight-shaped Lissajous orbit:



LLE of an halo orbit:



LLE of an invariant manifold of an eight-shaped Lissajous orbit:




LLE of an invariant manifold of an halo orbit: