Mapping encounter outcomes onto the *b*-plane

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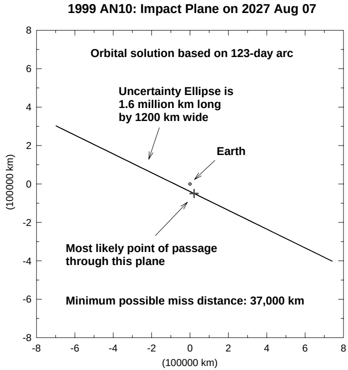
All what you wanted to know about the b-plane, and never dared to ask...

The b-plane of an encounter is the plane containing the planet and perpendicular to the planetocentric unperturbed velocity $\vec{U}$.

The vector from the planet to the point in which $\vec{U}$ crosses the plane is $\vec{b}$.

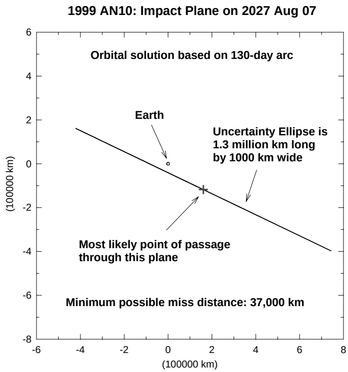
The rest of this talk is about "charting" what happens to a small body as a consequence of crossing the b-plane.

Uncertainty region on the *b*-plane



The uncertainy region, based on a 123 d observed arc, of 1999 AN_{10} projected on the *b*-plane of its Earth encounter on 7 August 2027 (from Chodas 1999).

Uncertainty region on the *b*-plane

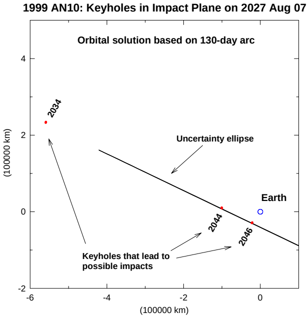


The same region, based on a 130 d observed arc, is smaller, and the nominal solution has moved (from Chodas 1999).

Keyholes

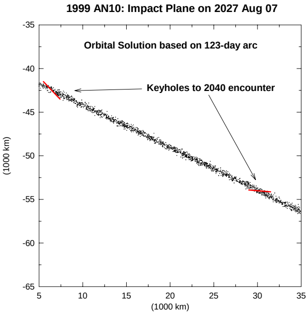
A keyhole (Chodas 1999) is a small region of the *b*-plane of a specific close encounter of an asteroid with the Earth such that, if the asteroid passes through it, it will hit the planet or have a very close encounter with it at a subsequent return.

Keyhole locations



The positions of keyholes in the *b*-plane of the encounter of 7 August 2027 of 1999 AN_{10}, for impacts in 2034, 2044, and 2046 (from Chodas 1999).

Keyhole locations



The positions of keyholes in the *b*-plane of the encounter of 7 August 2027 of 1999 AN_{10}, for a very close encounter in 2040 (from Chodas 1999).

Polar coordinates

Let r be the heliocentric distance, λ the longitude and β the latitude of the small body at time t_*; as functions of the heliocentric orbital elements they are given by:

$$
\begin{aligned}
r &= \frac{a(1-e^2)}{1+e\cos f_*} \\
\lambda &= \Omega + 2\arctan\left[\frac{\sin(\omega+f_*)\cos i}{\cos(\omega+f_*)+\sqrt{1-\sin^2(\omega+f_*)\sin^2 i}}\right] \\
\beta &= \arcsin[\sin(\omega+f_*)\sin i].
\end{aligned}
$$

Close encounter

A close encounter, at time t_*, with a planet orbiting the Sun on a circular orbit of radius a_p in the reference plane, located at longitude λ_p, would take place if:

$$\begin{aligned}
\frac{\Delta r}{a_p} &= \frac{a(1-e^2)}{a_p(1+e\cos f_*)} - 1 \\
\Delta\lambda &= \Omega - \lambda_p \\
&\quad + 2\arctan\left[\frac{\sin(\omega+f_*)\cos i}{\cos(\omega+f_*) + \sqrt{1-\sin^2(\omega+f_*)\sin^2 i}}\right] \\
\Delta\beta &= \arcsin[\sin(\omega+f_*)\sin i]
\end{aligned}$$

were all small.

Close encounter

Excluding the cases in which either $a(1-e) > a_p$ or $a(1+e) < a_p$, that cannot be treated with this theory, there are essentially two typical close encounter situations:

- either the close encounter takes place close to one of the nodes of the small body orbit;
- or $\sin i << 1$, in which case the close encounter can take place even far from both nodes, as discussed in Valsecchi (2006).

Close encounter

We establish an X-Y-Z frame centred on the planet, with the Sun on the negative X-axis, with the Y-axis coinciding with the direction of the planet motion, and the Z-axis parallel to the angular momentum vector of the planet orbit.

The unit of length is a_p, the unit of time is such that the orbital period of the planet is 2π, so that the modulus of the velocity of the planet is 1; in doing so, we ignore the contribution of the mass of the planet to its orbital speed.

Reference frame

A possible definition for the coordinates of the small body at time t_* could be:

$$
\begin{aligned}
X_* &= \frac{r}{a_p} \cos \Delta\lambda \cos \beta - 1 \\
Y_* &= \frac{r}{a_p} \sin \Delta\lambda \cos \beta \\
Z_* &= \frac{r}{a_p} \sin \beta.
\end{aligned}
$$

Reference frame

However:

$$\frac{\Delta r}{a_p} << 1; \qquad \Delta\lambda << 1; \qquad \Delta\beta << 1.$$

Therefore, we keep only the first order terms, so that:

$$\begin{aligned} X_* &= \frac{r}{a_p} - 1 \\ Y_* &= \frac{r}{a_p}\Delta\lambda \\ Z_* &= \frac{r}{a_p}\sin\beta. \end{aligned}$$

Planetocentric motion

We consider the motion near the planet as rectilinear, with constant speed, until small body crosses the b-plane, that is centred on the planet and orthogonal to the incoming asymptote of the planetocentric hyperbolic orbit of the small body.

We then apply, instantaneously, the rotation from the incoming to the outgoing asymptote, and consider the post-b-plane-crossing motion, again, as rectilinear, with constant speed.

Planetocentric motion

The pre-b-plane-crossing motion is given by:

$$\begin{aligned} X(t) &= U_x(t-t_*)+X_* = U\sin\theta\sin\phi(t-t_*)+X_* \\ Y(t) &= U_y(t-t_*)+Y_* = U\cos\theta(t-t_*)+Y_* \\ Z(t) &= U_z(t-t_*)+Z_* = U\sin\theta\cos\phi(t-t_*)+Z_*, \end{aligned}$$

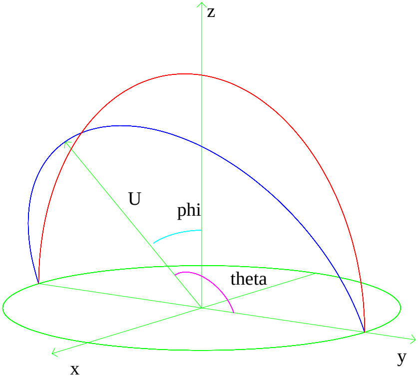
where $X_* = X(t_*)$, $Y_* = Y(t_*)$ and $Z_* = Z(t_*)$ are the planetocentric coordinates of the small body at time t_*, and U_x, U_y, U_z are the components of the unperturbed planetocentric velocity.

Planetocentric motion

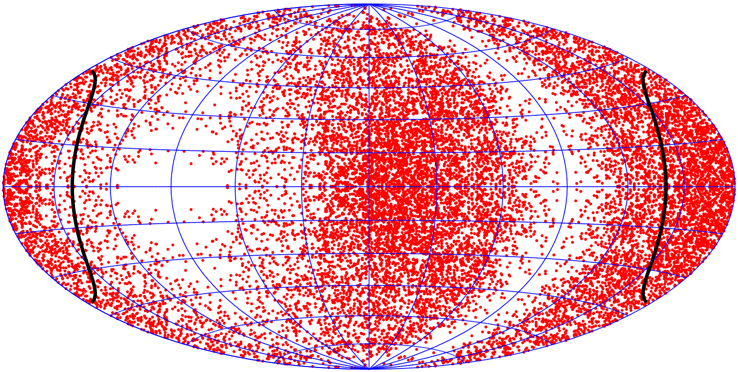
The values of U, θ, ϕ are given by:

$$
\begin{aligned}
U &= \sqrt{3 - \frac{a_p}{a} - 2\sqrt{\frac{a(1-e^2)}{a_p}}\cos i} \\
\cos\theta &= \frac{1 - U^2 - \frac{a_p}{a}}{2U} \\
\sin\theta &= \frac{\sqrt{2 - \frac{a_p}{a} - \frac{a(1-e^2)\cos^2 i}{a_p}}}{U} \\
\sin\phi &= \frac{\sin f_*}{|\sin f_*|} \cdot \frac{\sqrt{2 - \frac{a_p}{a} - \frac{a(1-e^2)}{a_p}}}{U\sin\theta} \\
\cos\phi &= \frac{\cos(\omega + f_*)}{|\cos(\omega + f_*)|} \cdot \frac{\sqrt{\frac{a(1-e^2)}{a_p}}\sin i}{U\sin\theta}.
\end{aligned}
$$

Geometric setup



Impactor radiants



Planetocentric motion

Depending on the close encounter, we can constrain further the choice of t_*; for a close encounter near one of the nodes, t_* can be the time of nodal passage, as in Valsecchi et al. (2003). Thus, either $\omega + f_* = 0$, at the ascending node, or $\omega + f_* = \pi$, at the descending node.

We have then:

$$
\begin{aligned}
X_* &= \frac{a(1-e^2)}{a_p(1 \pm e\cos\omega)} - 1 \\
Y_* &= \frac{a(1-e^2)}{a_p(1 \pm e\cos\omega)} \left[\Omega - \lambda_p + \frac{\pi}{2} \mp \frac{\pi}{2}\right] \\
Z_* &= 0,
\end{aligned}
$$

with the upper sign applying at the ascending node, and the lower sign at the descending one.

Planetocentric motion

Defining t_* such that $\lambda - \lambda_p = 0$, we have:

$$\begin{aligned} X_* &= \frac{a(1-e^2)}{a_p(1+e\cos f_*)} - 1 \\ Y_* &= 0 \\ Z_* &= \frac{a(1-e^2)\sin i \sin(\omega + f_*)}{a_p(1+e\cos f_*)}. \end{aligned}$$

Finally, we can choose t_* such that $r = a_p$, a choice that is valid also for encounters far from the nodes; in this case we have:

$$\begin{aligned} X_* &= 0 \\ Y_* &= \Omega + 2\arctan\left[\frac{\sin(\omega + f_*)\cos i}{1+\cos(\omega + f_*)}\right] - \lambda_p \\ Z_* &= \sin i \sin(\omega + f_*). \end{aligned}$$

From Öpik variables to elements

The computation of orbital elements from $X_*, Y_*, Z_*, U, \theta, \phi$ can be done in the following way; starting from the value of X_*, f_* is given by:

$$\cos f_* = \frac{a(1-e^2) - a_p(1+X_*)}{a_p e(1+X_*)},$$

and the quadrant of f_* can be established from the sign of $\sin\phi$:

$$\sin f_* = \frac{\sin\phi}{|\sin\phi|} \cdot \sqrt{1-\cos^2 f_*}.$$

From Öpik variables to elements

Next, compute ω from Z_*:

$$\sin(\omega + f_*) = \frac{a_p(1 + e\cos f_*)Z_*}{a(1-e^2)\sin i},$$

with the quadrant of $\omega + f_*$ given by the sign of $\cos\phi$:

$$\cos(\omega + f_*) = \frac{\cos\phi}{|\cos\phi|} \cdot \sqrt{1 - \sin^2(\omega + f_*)}.$$

Finally, Y_* gives us Ω:

$$\Omega = \lambda_p - 2\arctan\left[\frac{\sin(\omega + f_*)\cos i}{1 + \cos(\omega + f_*)}\right] + \frac{a_p(1 + e\cos f_*)Y_*}{a(1-e^2)}.$$

The local MOID

To find the local MOID, we consider the motion of the small body; in the expression:

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} U(t-t_*)\sin\theta\sin\phi + X_* \\ U(t-t_*)\cos\theta + Y_* \\ U(t-t_*)\sin\theta\cos\phi + Z_* \end{pmatrix}$$

we eliminate $t - t_*$, using:

$$t - t_* = \frac{Y - Y_*}{U\cos\theta},$$

and obtain:

$$\begin{aligned} X &= (Y - Y_*)\tan\theta\sin\phi + X_* \\ Z &= (Y - Y_*)\tan\theta\cos\phi + Z_*. \end{aligned}$$

The local MOID

Setting $w = Y - Y_*$, the square of the distance from the Y-axis is:

$$\begin{aligned} D_y^2 &= X^2 + Z^2 \\ &= w^2 \tan^2\theta + 2w(X_* \sin\phi + Z_* \cos\phi)\tan\theta + X_*^2 + Z_*^2 \end{aligned}$$

and its derivative with respect to w is:

$$\frac{d(D_y^2)}{dw} = 2w\tan^2\theta + 2(X_* \sin\phi + Z_* \cos\phi)\tan\theta;$$

this derivative is zero at:

$$w_{MOID} = -(X_* \sin\phi + Z_* \cos\phi)\cot\theta.$$

The local MOID

The minimum value of D_y^2 is then:

$$\min D_y^2 = (X_* \cos\phi - Z_* \sin\phi)^2.$$

Therefore, the local MOID as function of X_*, Z_* and ϕ is:

$$\min D_y = |X_* \cos\phi - Z_* \sin\phi|;$$

following Valsecchi et al. (2003), we define the signed local MOID as $X_* \cos\phi - Z_* \sin\phi$.

The coordinates on the b-plane

In a similar way we determine the coordinates in the general case in which at $t = t_*$ the small body is at a generic point (X_*, Y_*, Z_*) not necessarily leading to an encounter at the MOID; we then have:

$$\begin{aligned} X(t) &= U \sin\theta \sin\phi (t - t_*) + X_* \\ Y(t) &= U \cos\theta (t - t_*) + Y_* \\ Z(t) &= U \sin\theta \cos\phi (t - t_*) + Z_* \end{aligned}$$

and we want to minimize the distance from the planet:

$$\begin{aligned} D^2 &= X^2 + Y^2 + Z^2 \\ &= U^2 t^2 + 2U[(X_* \sin\phi + Z_* \cos\phi)\sin\theta + Y_* \cos\theta - U t_*] t \\ &\quad -2U[(X_* \sin\phi + Z_* \cos\phi)\sin\theta + Y_* \cos\theta - U t_*] t_* \\ &\quad +X_*^2 + Y_*^2 + Z_*^2. \end{aligned}$$

The coordinates on the b-plane

We take the derivative with respect to t:

$$\frac{d(D^2)}{dt} = 2U^2 t + 2U[(X_* \sin\phi + Z_* \cos\phi)\sin\theta + Y_* \cos\theta - Ut_*],$$

and find the value $t = t_b$ for which it is zero:

$$t_b = t_* - \frac{(X_* \sin\phi + Z_* \cos\phi)\sin\theta + Y_* \cos\theta}{U}.$$

The coordinates on the b-plane

Thus, one has the minimum approach distance when the small body is in:

$$
\begin{aligned}
X_b &= U \sin\theta \sin\phi (t_b - t_*) + X_* \\
&= X_* - [(X_* \sin\phi + Z_* \cos\phi) \sin\theta + Y_* \cos\theta] \sin\theta \sin\phi \\
Y_b &= U \cos\theta (t_b - t_*) + Y_* \\
&= Y_* - [(X_* \sin\phi + Z_* \cos\phi) \sin\theta + Y_* \cos\theta] \cos\theta \\
Z_b &= U \sin\theta \cos\phi (t_b - t_*) + Z_* \\
&= Z_* - [(X_* \sin\phi + Z_* \cos\phi) \sin\theta + Y_* \cos\theta] \sin\theta \cos\phi .
\end{aligned}
$$

The coordinates on the b-plane

We now apply the coordinate transformation from the X-Y-Z frame to the b-plane frame ξ-η-ζ (Valsecchi et al. 2003), obtaining the coordinates on the b-plane:

$$
\begin{aligned}
\xi &= X_b \cos\phi - Z_b \sin\phi \\
&= X_* \cos\phi - Z_* \sin\phi \\
\eta &= (X_b \sin\phi + Z_b \cos\phi)\sin\theta + Y_b \cos\theta \\
&= 0 \\
\zeta &= (X_b \sin\phi + Z_b \cos\phi)\cos\theta - Y_b \sin\theta \\
&= (X_* \sin\phi + Z_* \cos\phi)\cos\theta - Y_* \sin\theta.
\end{aligned}
$$

The coordinates on the b-plane

The coordinates ξ, ζ are the components of the vector $\vec{b}$, of magnitude $b = \sqrt{\xi^2 + \zeta^2}$.

Note that ξ corresponds to the signed local MOID; thus, ζ plays the rôle of a time-related coordinate, that depends of whether the small body arrives “early” or “late” at the approach, while ξ is related to the orbit geometry.

The coordinates on the b-plane

Conversely, we can get X_*, Y_*, Z_* from $U, \theta, \phi, \xi, \zeta, t_b - t_*$:

$$
\begin{aligned}
X_* &= [\zeta \cos\theta - U(t_b - t_*)\sin\theta]\sin\phi + \xi\cos\phi \\
Y_* &= -(\zeta\sin\theta + U(t_b - t_*)\cos\theta) \\
Z_* &= [\zeta \cos\theta - U(t_b - t_*)\sin\theta]\cos\phi - \xi\sin\phi.
\end{aligned}
$$

From elements to encounter variables and back

Thus, we can:

- from $a, e, i, \Omega, \omega, f_*$ at time t_*, when the small body is near the planet, compute $X_*, Y_*, Z_*, U_x, U_y, U_z$;
- from $X_*, Y_*, Z_*, U_x, U_y, U_z$ compute $U, \theta, \phi, \xi, \zeta, t_b$;
- from $U, \theta, \phi, \xi, \zeta, t_b$ go back to $a, e, i, \Omega, \omega, f_*$.

This means that, from the orbital elements, we can derive a complete set or variables, defined in the *b*-plane reference frame, that allows the computation of the encounter outcome, still in the *b*-plane frame; from there, using the inverse relations, we can derive the post-encounter elements.

The encounter

At the time of b-plane crossing, t_b, we rotate the velocity vector by the angle γ, from being parallel to the incoming asymptote of the planetocentric hyperbola, to being parallel to the other asymptote; the position of the small body is shifted to the one corresponding to the minimum unperturbed distance on the new orbit.

The coordinates in the ξ-η-ζ reference frame pass from

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} X_* \cos\phi - Z_* \sin\phi \\ 0 \\ (X_* \sin\phi + Z_* \cos\phi)\cos\theta - Y_* \sin\theta \end{pmatrix}$$

to

$$\begin{pmatrix} \xi_r \\ \eta_r \\ \zeta_r \end{pmatrix} = \begin{pmatrix} \xi \cos\gamma \\ b \sin\gamma \\ \zeta \cos\gamma. \end{pmatrix}$$

The encounter

Following Valsecchi et al. (2003), we define

$$c = \frac{m}{U^2},$$

and use the expressions for $\sin\gamma$ and $\cos\gamma$:

$$\begin{aligned} \cos\gamma &= \frac{b^2 - c^2}{b^2 + c^2} \\ \sin\gamma &= \frac{2bc}{b^2 + c^2} \end{aligned}$$

to rewrite the previous expressions for the components of the rotated vector $\vec{b}$, that we call $\vec{b'}$, in the ξ-η-ζ reference frame

$$\begin{pmatrix} \xi_r \\ \eta_r \\ \zeta_r \end{pmatrix} = \begin{pmatrix} \frac{\xi(b^2-c^2)}{b^2+c^2} \\ \frac{2b^2c}{b^2+c^2} \\ \frac{\zeta(b^2-c^2)}{b^2+c^2} \end{pmatrix}.$$

The encounter

We denote by X'_b, Y'_b, Z'_b the components of $\vec{b'}$ in the X-Y-Z frame; their explicit expressions are the following:

$$
\begin{aligned}
X'_b &= (\eta_r \sin\theta + \zeta_r \cos\theta)\sin\phi + \xi_r \cos\phi \\
&= \frac{2b^2 c \sin\theta \sin\phi + (b^2 - c^2)(\zeta \cos\theta \sin\phi + \xi \cos\phi)}{b^2 + c^2} \\
Y'_b &= \eta_r \cos\theta - \zeta_r \sin\theta \\
&= \frac{2b^2 c \cos\theta - (b^2 - c^2)\zeta \sin\theta}{b^2 + c^2} \\
Z'_b &= (\eta_r \sin\theta + \zeta_r \cos\theta)\cos\phi - \xi_r \sin\phi \\
&= \frac{2b^2 c \sin\theta \cos\phi + (b^2 - c^2)(\zeta \cos\theta \cos\phi - \xi \sin\phi)}{b^2 + c^2}.
\end{aligned}
$$

The encounter

The components of the rotated velocity vector $\vec{U}'$ are given by:

$$
\begin{aligned}
U'_x &= U \sin\theta' \sin\phi' \\
&= U \frac{[(b^2 - c^2)\sin\theta - 2c\zeta\cos\theta]\sin\phi - 2c\xi\cos\phi}{b^2 + c^2} \\
U'_y &= U \cos\theta' \\
&= U \frac{(b^2 - c^2)\cos\theta + 2c\zeta\sin\theta}{b^2 + c^2} \\
U'_z &= U \sin\theta' \cos\phi' \\
&= U \frac{[(b^2 - c^2)\sin\theta - 2c\zeta\cos\theta]\cos\phi + 2c\xi\sin\phi}{b^2 + c^2}.
\end{aligned}
$$

Post-encounter b-plane coordinates and local MOID

Rotating by θ' and ϕ' the components of $\vec{b'}$ in the X-Y-Z frame we get the coordinates in the post-encounter b-plane:

$$
\begin{aligned}
\xi' &= X'_b \cos\phi' - Z'_b \sin\phi' \\
&= \frac{(b^2+c^2)\xi \sin\theta}{\sqrt{[(b^2-c^2)\sin\theta - 2c\zeta\cos\theta]^2 + 4c^2\xi^2}} \\
\eta' &= (X'_b \sin\phi' + Z'_b \cos\phi')\sin\theta' + Y'_b \cos\theta' \\
&= 0 \\
\zeta' &= (X'_b \sin\phi' + Z'_b \cos\phi')\cos\theta' - Y'_b \sin\theta' \\
&= \frac{(b^2-c^2)\zeta \sin\theta - 2b^2 c\cos\theta}{\sqrt{[(b^2-c^2)\sin\theta - 2c\zeta\cos\theta]^2 + 4c^2\xi^2}}.
\end{aligned}
$$

Note that ξ' is the new local MOID.

Post-encounter propagation

The coordinates at a generic time t along the post-encounter trajectory of the small body are:

$$\begin{aligned} X'(t) &= U'_x(t - t_b) + X'_b \\ Y'(t) &= U'_y(t - t_b) + Y'_b \\ Z'(t) &= U'_z(t - t_b) + Z'_b. \end{aligned}$$

These expressions allow us to compute the post-encounter reference time t'_* corresponding to one of three possibilities ($X'_* = X'(t'_*) = 0$, $Y'_* = Y'(t'_*) = 0$, $Z'_* = Z'(t'_*) = 0$).

In summary, the post-encounter Öpik variables U', θ', ϕ' are:

$$
\begin{aligned}
U' &= U \\
\cos\theta' &= \frac{(b^2 - c^2)\cos\theta + 2c\zeta\sin\theta}{b^2 + c^2} \\
\sin\theta' &= \frac{\sqrt{[(b^2 - c^2)\sin\theta - 2c\zeta\cos\theta]^2 + 4c^2\xi^2}}{b^2 + c^2} \\
\cos\phi' &= \frac{[(b^2 - c^2)\sin\theta - 2c\zeta\cos\theta]\cos\phi + 2c\xi\sin\phi}{(b^2 + c^2)\sin\theta'} \\
\sin\phi' &= \frac{[(b^2 - c^2)\sin\theta - 2c\zeta\cos\theta]\sin\phi - 2c\xi\cos\phi}{(b^2 + c^2)\sin\theta'}.
\end{aligned}
$$

The swing-by

The post-encounter Öpik variables ξ', ζ', t'_* are:

$$\begin{aligned}
\xi' &= \frac{\xi \sin\theta}{\sin\theta'} \\
\zeta' &= \frac{(b^2 - c^2)\zeta \sin\theta - 2b^2 c \cos\theta}{(b^2 + c^2)\sin\theta'} \\
t'_* &= t_b + \frac{\xi' \sin\phi' - \zeta' \cos\theta' \cos\phi'}{U \sin\theta' \cos\phi'}.
\end{aligned}$$

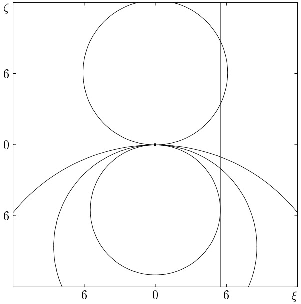
Solving for given θ'

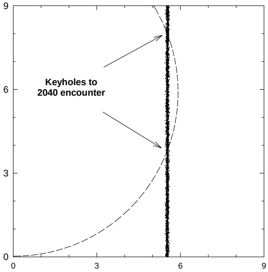
We want to solve for $\theta' = \theta'_*$; rearranging the expression for $\cos\theta'$ we obtain an equation in ξ, ζ that is the equation of a circle of radius $|R|$, centred in $\zeta = D$:

$$\begin{aligned}
0 &= (\xi^2 + \zeta^2 - c^2)\cos\theta + 2c\zeta\sin\theta - (\xi^2 + \zeta^2 + c^2)\cos\theta'_* \\
\xi^2 &= -\zeta^2 + 2D\zeta + R^2 - D^2 \\
D &= \frac{c\sin\theta}{\cos\theta'_* - \cos\theta} \\
R &= \frac{c\sin\theta'_*}{\cos\theta'_* - \cos\theta}.
\end{aligned}$$

Actually, according to Galileo: "...[l'universo] è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche...".

Really, circles?





Top: *b*-plane circles for resonant return in 2040, 2030, 2044, 2046. Bottom: Chodas' plot for 2040, suitably rotated; the circle comes from a best fit.

Solving for given θ' and ϕ'

For given θ'_* the solution, as just seen, lies on a *b*-plane circle; we can define an angle α, such that:

$$\begin{aligned} \xi &= R \sin \alpha \\ \zeta &= D + R \cos \alpha . \end{aligned}$$

In this way, all the post-encounter orbits with given θ' (i.e., with given a') can be obtained as function of α.

Solving for given θ' and ϕ'

Let us recall the expressions for $\cos\phi', \sin\phi'$:

$$\cos\phi' = \frac{[(\xi^2+\zeta^2-c^2)\sin\theta - 2c\zeta\cos\theta]\cos\phi + 2c\xi\sin\phi}{\sqrt{[(\xi^2+\zeta^2-c^2)\sin\theta - 2c\zeta\cos\theta]^2 + 4c^2\xi^2}}$$

$$\sin\phi' = \frac{[(\xi^2+\zeta^2-c^2)\sin\theta - 2c\zeta\cos\theta]\sin\phi - 2c\xi\cos\phi}{\sqrt{[(\xi^2+\zeta^2-c^2)\sin\theta - 2c\zeta\cos\theta]^2 + 4c^2\xi^2}}.$$

We put $\phi' = \phi'_*$, assume that $\theta' = \theta'_*$, substitute the expressions for ξ, ζ as functions of D, R, α, and of $\sin\theta'_*$ as function of $R, D, \sin\theta$.

Solving for given θ' and ϕ'

After some manipulations, we get:

$$
\begin{aligned}
\cos\alpha &= \frac{1}{2DR\{[D - R\cos(\phi'_* - \phi)]\sin\theta - c\cos\theta\}} \\
&\cdot\{2cD^2\cos\theta - [D(R^2 + D^2 - c^2) \\
&-R(R^2 + D^2 + c^2)\cos(\phi'_* - \phi)]\sin\theta\} \\
\sin\alpha &= -\frac{(2DR\cos\alpha + R^2 + D^2 + c^2)\sin\theta\sin(\phi'_* - \phi)}{2cD}.
\end{aligned}
$$

From $\cos\alpha, \sin\alpha$ we compute ξ, ζ, and from them the values of ω, λ_p that the small body must have before the swing-by.

The case of 2009 FD

Asteroid 2009 FD is a not-so-small NEA that could impact the Earth between 2185 and 2196.

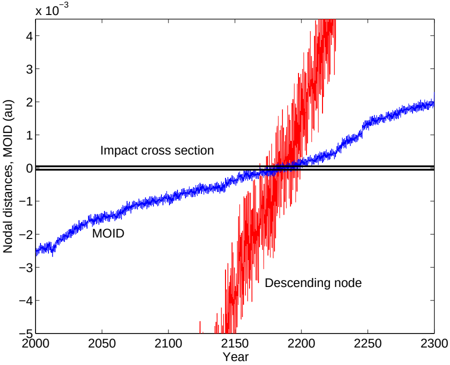
Its orbit is rather well determined, but close Earth encounters between the current epoch and the end of the XXIInd century make its Line of Variations (LoV) projection in the 2185 *b*-plane a very "clean" and interesting case to study.

LeVerrier's LoV

$$
\begin{aligned}
a &= 3{,}14786 + 0{,}01\mu,\\
e &= 0{,}7857161 + 0{,}0007260\mu,\\
\varepsilon &= 356^\circ\,15'\,55'',51 - 12'',67\mu,\\
\varpi &= 356.16.26{,}03 - 27{,}16\mu,\\
\varphi &= 1.34.19{,}53 + 4{,}17\mu,\\
\theta &= 131.53.56{,}00 + 83{,}00\mu.
\end{aligned}
$$

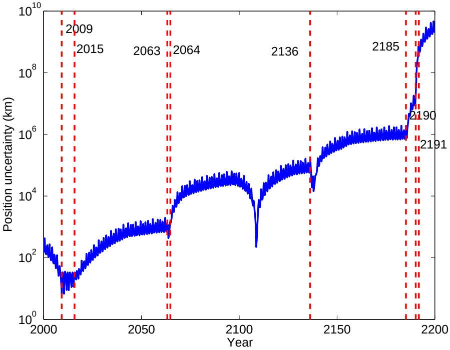
The Line of Variations introduced by LeVerrier for comet Lexell: from top to bottom, semimajor axis, eccentricity, mean longitude at epoch, longitude of perihelion, inclination and longitude of node.

The MOID of 2009 FD



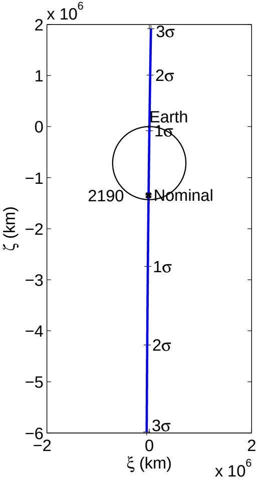
The MOID and the distance at the descending node of 2009 FD until 2300; the MOID allows an Earth impact from 2166 to 2197.

The MOID of 2009 FD



Evolution of the semimajor axis of the positional uncertainty ellipse of 2009 FD; the vertical dashed lines denote Earth approaches within 0.05 au.

The 2185 LoV of 2009 FD

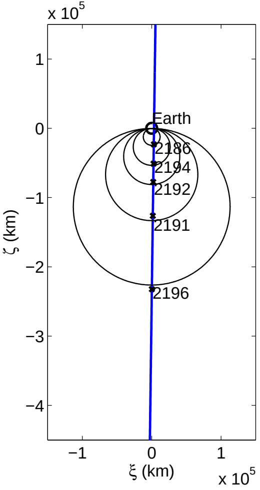


The LoV on the 2185 *b*-plane; the 2190 VI is marked with a cross.

The LoV, $\sigma = -3$ to $+3$, spans more than 7 million km, and straddles the Earth, allowing a range of approach distances, from actual Earth collision, up to rather distant encounters.

The 2185 VI is "almost" a direct impact, with a very low stretching, so has a comparatively large Impact Probability (IP).

The 2185 LoV of 2009 FD



The LoV segment close to the Earth on the 2185 *b*-plane.

The keyholes for impact in (from top to bottom) 2186, 2194, 2192, 2191, and 2196 are marked with crosses; also shown are the *b*-plane circles associated to the corresponding mean motion resonances (1/1, 8/9, 6/7, 5/6, 9/11).

Sensitivity to initial conditions

− 0,3	3,784	0,6632
− 0,2	4,909	0,5251
− 0,1	8,923	0,5349
0,0	+ 60,097	0,9127
+ 0,1	− 58,617	1,0926
0,2	+ 49,751	0,8987
0,3	12,444	0,6532

A small excerpt from LeVerrier's computations: for the values of μ in the left column, the corresponding post-1779 values of semimajor axis and eccentricity of the orbit of comet Lexell.

An analytic estimate of the resonant returns cascade

Let us make an analytic estimate of the range of semimajor axes of the possible post-2185 orbits, using data coming from the pre-2185 encounter orbit of 2009 FD taken from an accurate numerical integration.

Relevant quantities:

- $U = 0.533$;
- $\theta = 97\overset{\circ}{.}7$.
- $c = m_\oplus / U^2 = 0.25\ r_\oplus$, where $r_\oplus$ is the Earth radius;
- $b_\oplus = r_\oplus \sqrt{1 + \frac{2c}{r_\oplus}} = 1.22\, r_\oplus$, the radius of the Earth cross-section on the b-plane.

The resonant cascade

The post-encounter semimajor axis a' of 2009 FD is given by:

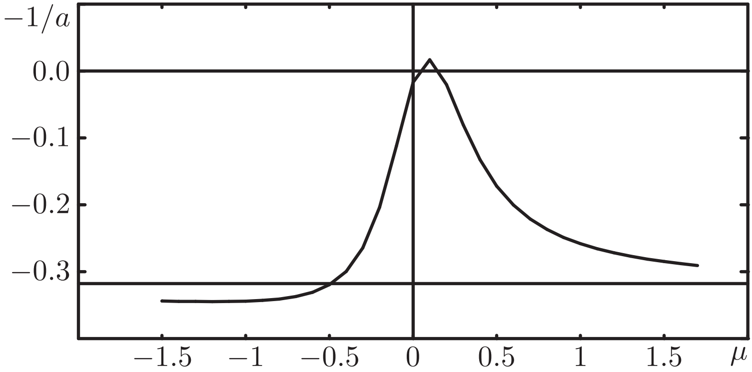
$$a' = \frac{a_p}{1 - U^2 - 2U\cos\theta'}.$$

Note that a' is maximum when $\cos\theta'$ is maximum, and a' is minimum when $\cos\theta'$ is minimum; thus, we consider the expression for $\cos\theta'$ as function of the b-plane coordinates:

$$\cos\theta' = \frac{(\xi^2 + \zeta^2 - c^2)\cos\theta + 2c\zeta\sin\theta}{\xi^2 + \zeta^2 + c^2},$$

and use the "wire" approximation of Valsecchi et al. (2003), i.e. keep ξ constant, like all other quantities in the expression, except ζ.

Sensitivity to initial conditions



The post-1779 values of $-1/a$, in AU^{-1} given by LeVerrier as a function of μ; the lower horizontal line corresponds to the pre-1779 value of $-1/a$.

The resonant cascade

We take the partial derivative with respect to ζ:

$$\frac{\partial \cos\theta'}{\partial\zeta} = \frac{2c[2c\zeta\cos\theta + (\xi^2 - \zeta^2 + c^2)\sin\theta]}{(\xi^2 + \zeta^2 + c^2)^2},$$

and look for the zeroes $\zeta_\pm$ of its numerator:

$$\zeta_\pm = \frac{c\cos\theta \pm \sqrt{c^2 + \xi^2\sin^2\theta}}{\sin\theta}.$$

The resonant cascade

Substituting $c = 0.25\, r_\oplus, |\xi| = 0.52\, r_\oplus, \theta = 97^\circ\!.7$, we get $\zeta_+ = 0.54\, r_\oplus$ and $\zeta_- = -0.61\, r_\oplus$; both values are smaller, in absolute value, than $b_\oplus$, implying that the maximum and minimum possible values for a' are obtained for grazing encounters taking place at $\zeta = \pm\sqrt{b_\oplus^2 - \xi^2} = \pm 1.11\, r_\oplus$.

Thus, the maximum post-encounter a', and the related maximum orbital period P', are:

$$a'_{max} = 2.10 \text{ au} \quad ; \quad P'_{max} = 3.05 \text{ yr},$$

and the minimum post-encounter a', and the related minimum orbital period P', are:

$$a'_{min} = 0.82 \text{ au} \quad ; \quad P'_{min} = 0.74 \text{ yr}.$$

The resonant cascade

This range of post-2185 orbital periods for 2009 FD makes possible a number resonant of returns within 2197, the year after which the secular increase of the MOID precludes the possibility of further collisions with the Earth at the same node.

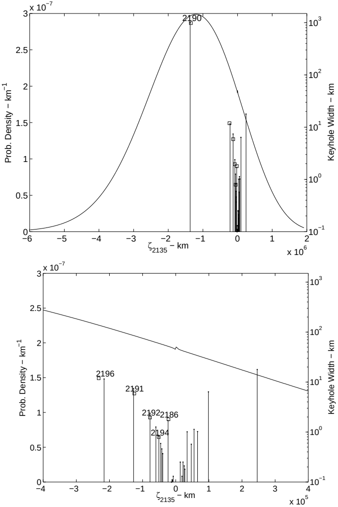
The relevant list of resonances is the Farey sequence with maximum denominator $2197 - 2185 = 12$ comprised between $n/n_p = 1/3.05$ and $n/n_p = 1/0.74$; in practice, between $1/3$ and $4/3$.

There are 43 such resonances, and for 6 of them the impact monitoring software has found the corresponding VI.

The resonant cascade

With the analytical theory of Valsecchi et al. (2003) we compute $\partial\zeta''/\partial\zeta$, the factor by which the ζ-coordinate in the *b*-plane of the second encounter is "stretched" with respect to the ζ-coordinate in the 2185 *b*-plane; this allows us to estimate the maximum size of the corresponding keyhole, that is given by $2b_{\oplus}$ divided by $\partial\zeta''/\partial\zeta$.

The corresponding maximum values of IP, P_{max}, are computed by multiplying the Probability Density Function (PDF) by the maximum keyhole size.



The possible keyholes

2009 FD impact keyholes on the 2185 *b*-plane LoV, computed both numerically and analitically.

The PDF is given by the curve (left scale), the analytically computed keyholes are indicated by vertical lines whose heights give their sizes (right scale).

The 7 actual VIs found numerically are marked with a square.