Orbital Dynamics: an overview of asteroid and artificial satellites motion

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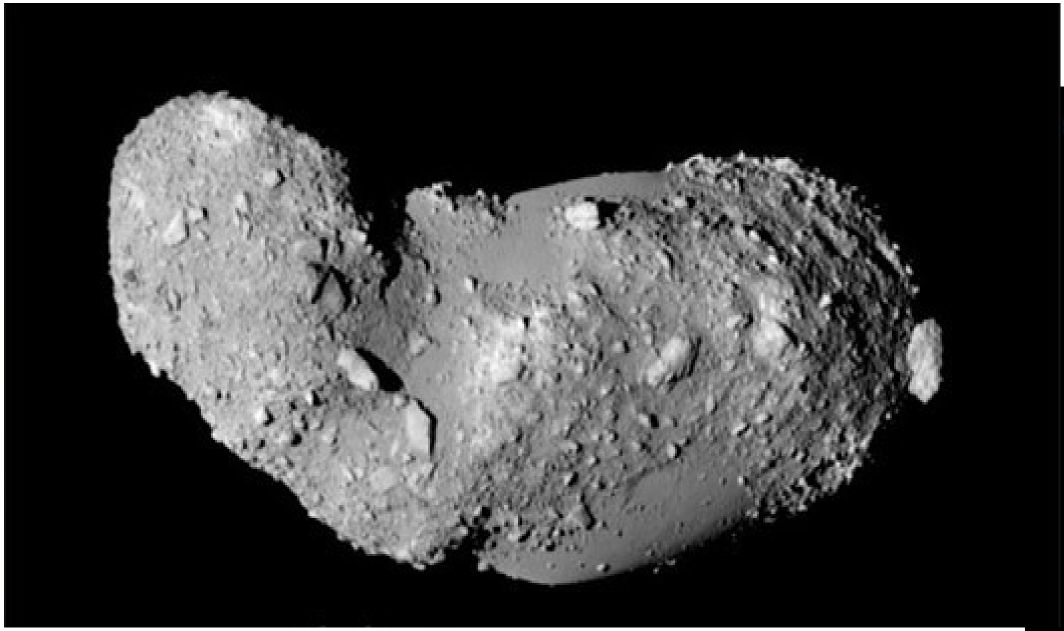
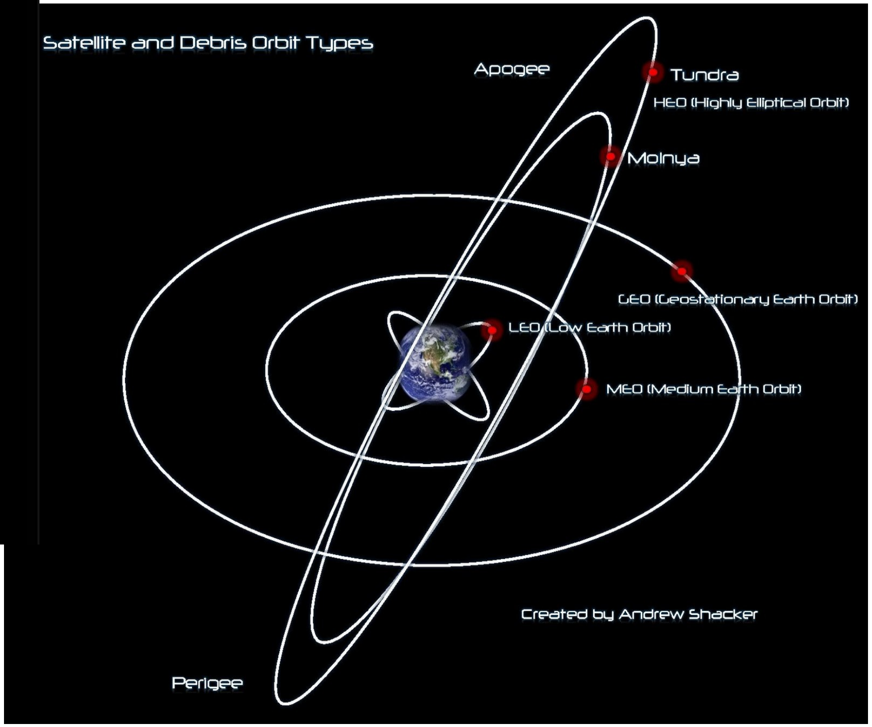
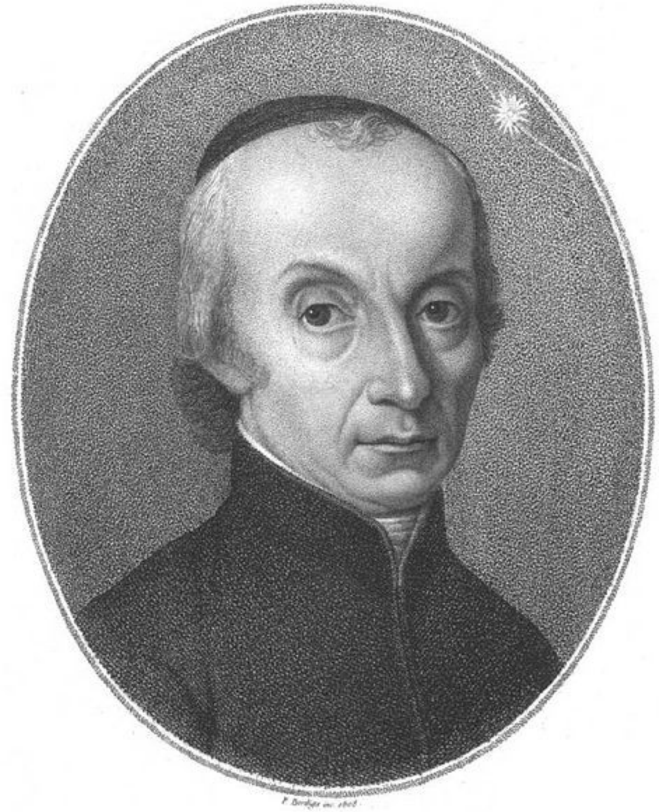




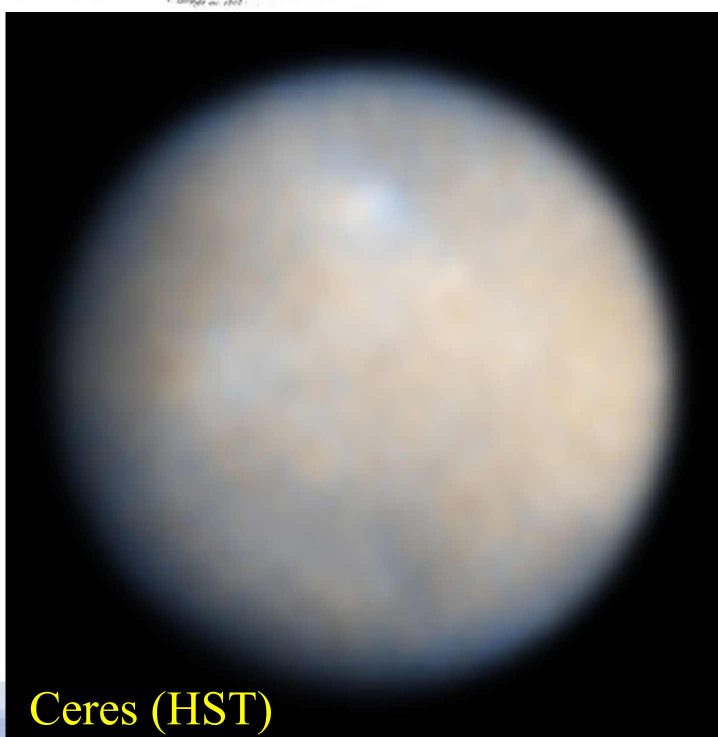
STARDUST School, 9/2014, University of Roma "Tor Vergata"



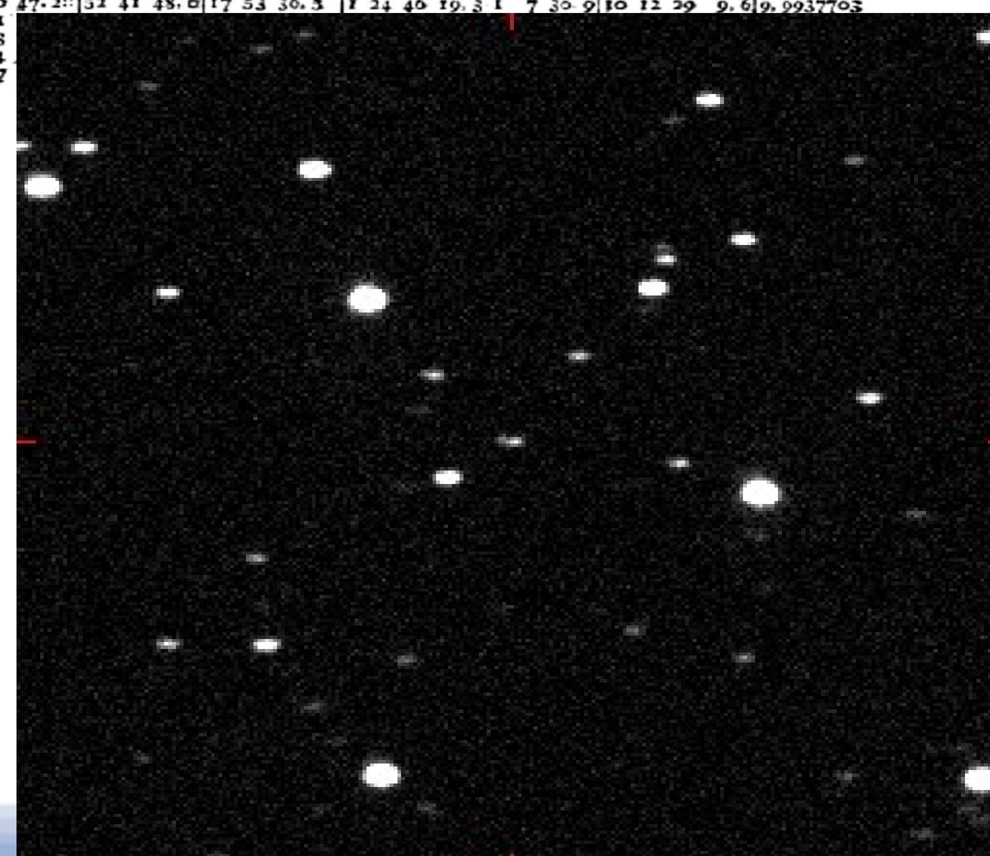
January 1, 1801: Giuseppe Piazzi discovers the first 'asteroid' (now dwarf planet) *Ceres (Δήμητρα)*

Beobachtungen des zu Palermo d. 1 Jan. 1801 von Prof. *Piazzi* neu entdeckten Gestirns.

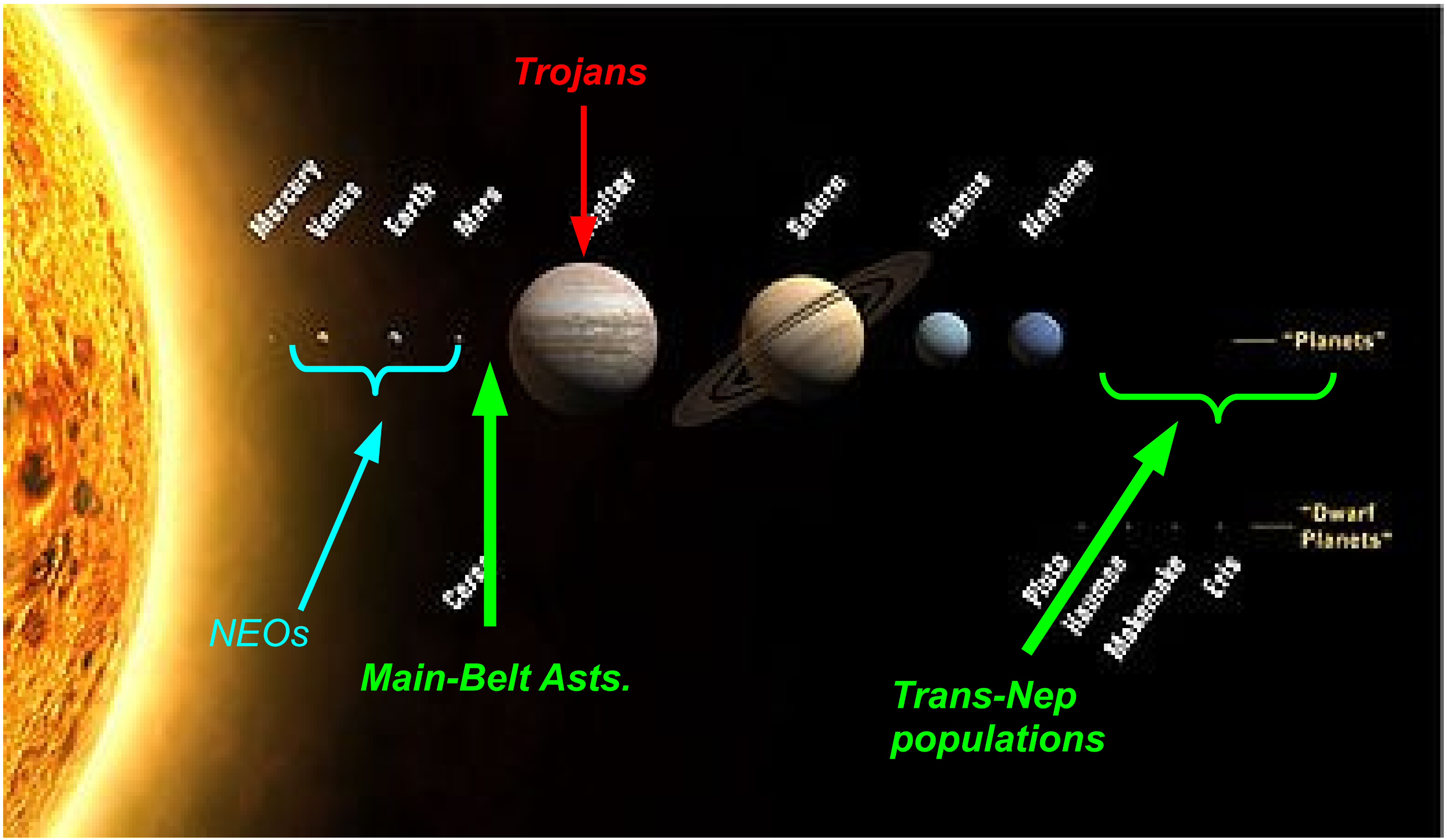
1801	Mittlere Sonnen-Zeit	Gerade Aufstcig in Zeit	Gerade Aufsteigung in Graden	Nördl. Abweich.	Geocentrische Länge	Geocentr. Breite	Ort der Sonne + 20" Aberration	Logar. d. Distanz ⊙ ♁
	St ' "	St ' "	° ' "	° ' "	Z ° ' "	° ' "	Z ° ' "	
Jan. 1	8 43 27,8	3 27 11, 25	51 47 48, 8	15 37 43, 5	1 23 22 58, 3	3 6 42, 1	9 11 1 30, 9	9, 9926156
2	8 39 4,6	3 26 53, 85	51 43 27, 8	15 41 5, 5	1 23 19 44, 3	3 2 24, 9	9 12 2 28, 6	9, 9926317
3	8 34 53,3	3 26 38, 4::	51 39 36, 0	15 44 31, 6	1 23 16 58, 6	2 58 9, 9	9 13 3 26, 6	9, 9926324
4	8 30 42,1	3 26 23, 15	51 35 47, 3	15 47 57, 6	1 23 14 15, 5	2 53 55, 6	9 14 4 24, 9	9, 9926418
10	8 6 15,8	3 25 32, 1::	51 23 1, 5	16 10 32, 0	1 23 7 59, 1	2 29 0, 6	9 20 10 17, 5	9, 9927641
11	8 2 17,5	3 25 29, 73	51 22 26, 0	. . . . . .	. . . . . .	. . . . .	. . . . . .	. . . . .
13	7 54 26,2	3 25 30, 30	51 22 34, 5	16 22 49, 5	1 23 10 27, 6	2 16 59, 7	9 23 12 13, 8	9, 9928490
14	7 50 31,7	3 25 31, 72	51 22 55, 8	16 27 5, 7	1 23 12 1, 2	2 12 56, 7	9 24 14 13, 5	9, 9928809
17	. . . . . .	. . . . . .	. . . . . .	16 40 13, 0	. . . . . .	. . . . .	. . . . . .	. . . . .
18	7 35 11,3	3 25 55, ::	51 28 45, 0	. . . . . .	. . . . . .	. . . . .	. . . . . .	. . . . .
19	7 31 28,5	3 26 8, 15	51 32 2, 3	16 49 16, 1	1 23 25 59, 2	1 53 38, 2	9 29 19 53, 8	9, 9930607
21	7 24 2,7	3 26 34, 27	51 38 34, 1	16 58 35, 9	1 23 34 21, 3	1 46 6, 0	10 1 20 40, 3	9, 9931434
22	7 20 21,7	3 26 49, 42	51 42 21, 3	17 3 18, 5	1 23 39 1, 8	1 42 28, 1	10 2 21 32, 0	9, 9931886
23	7 16 43,5	3 27 6, 90	51 46 43, 5	17 8 5, 5	1 23 44 15, 7	1 38 52, 1	10 3 22 22, 7	9, 9932348
28	6 58 51,3	3 28 54, 55	52 13 38, 3	17 32 54, 1	1 24 15 15, 7	1 21 6, 9	10 8 26 20, 1	9, 9935062
30	6 51 52,9	3 29 48, 14	52 27 2, 1	17 43 11, 0	1 24 30 9, 0	1 14 16, 0	10 10 27 46, 2	9, 9936332
31	6 48 26,4	3 30 17, 25	52 34 18, 8	17 48 21, 5	1 24 38 7, 3	1 10 54, 6	10 11 28 28, 5	9, 9937007
Febr. 1	6 44 59,9	3 30 47, 2::	52 41 48, 0	17 53 36, 5	1 24 46 19, 3	1 7 30, 9	10 12 29 9, 6	9, 9937703
2	6 41 35,8	3 31						
5	6 31 31,5	3 33						
8	6 21 39,2	3 34						
11	6 11 58,2	3 37						



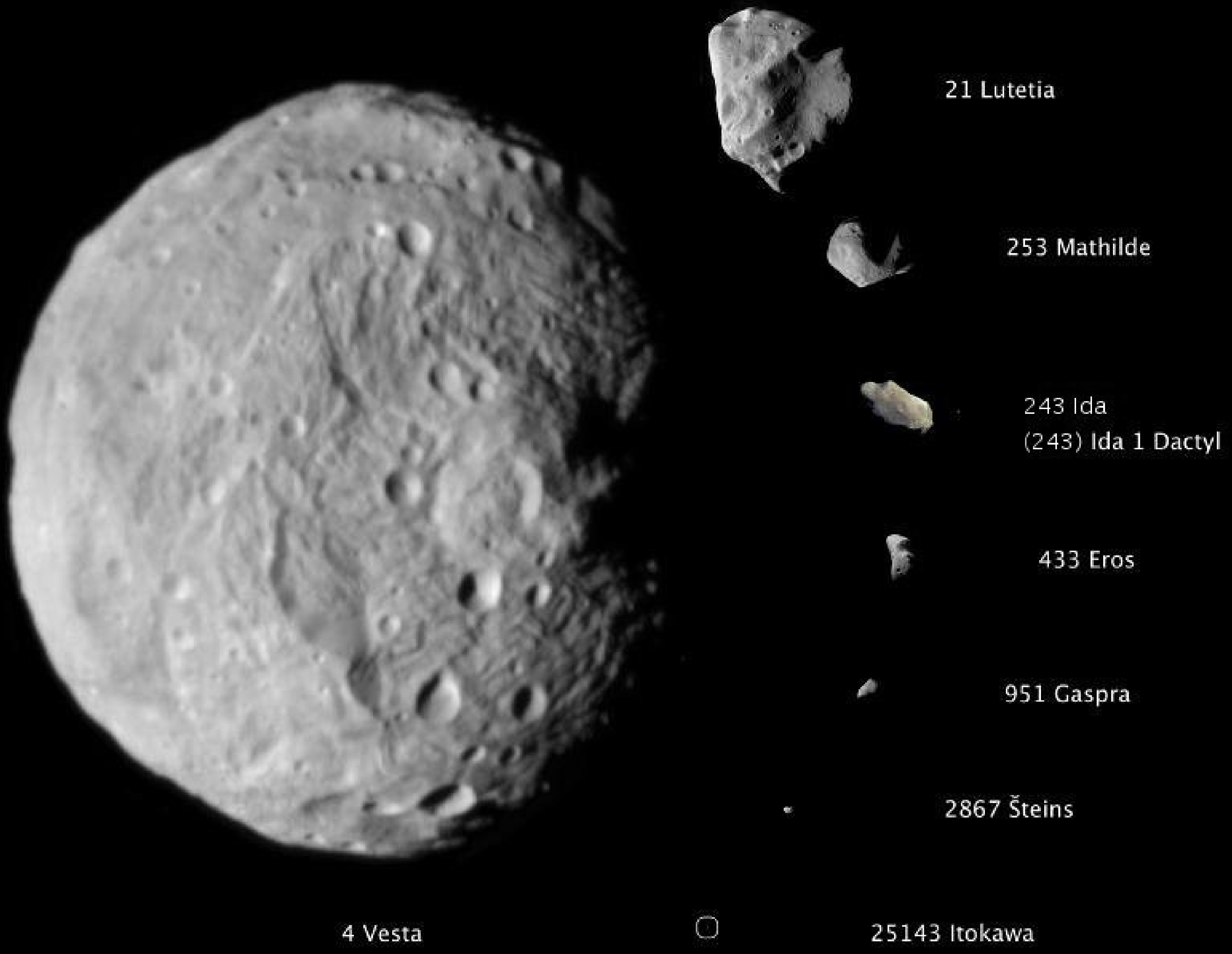
Ceres (HST)



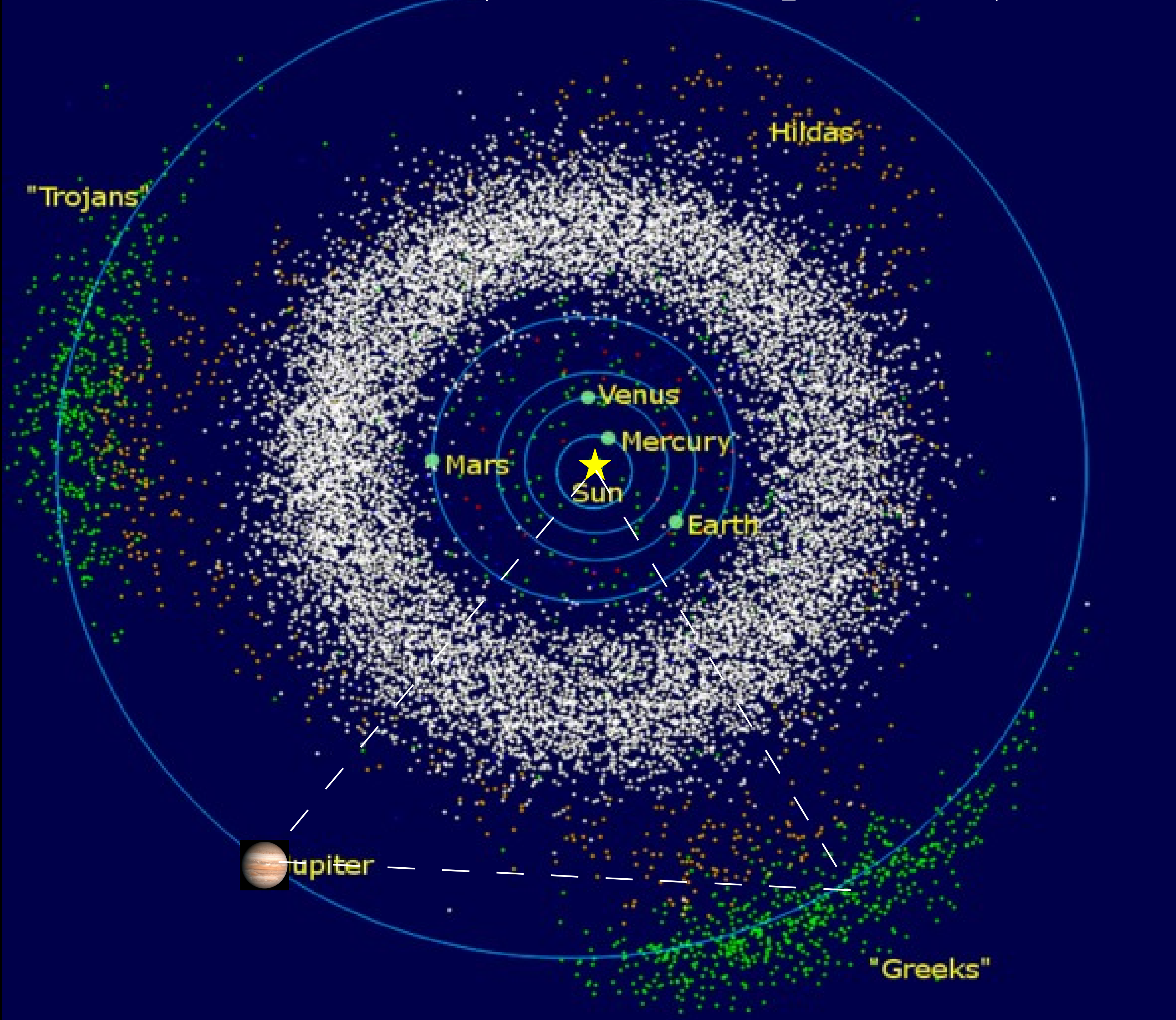
Solar System Architecture...



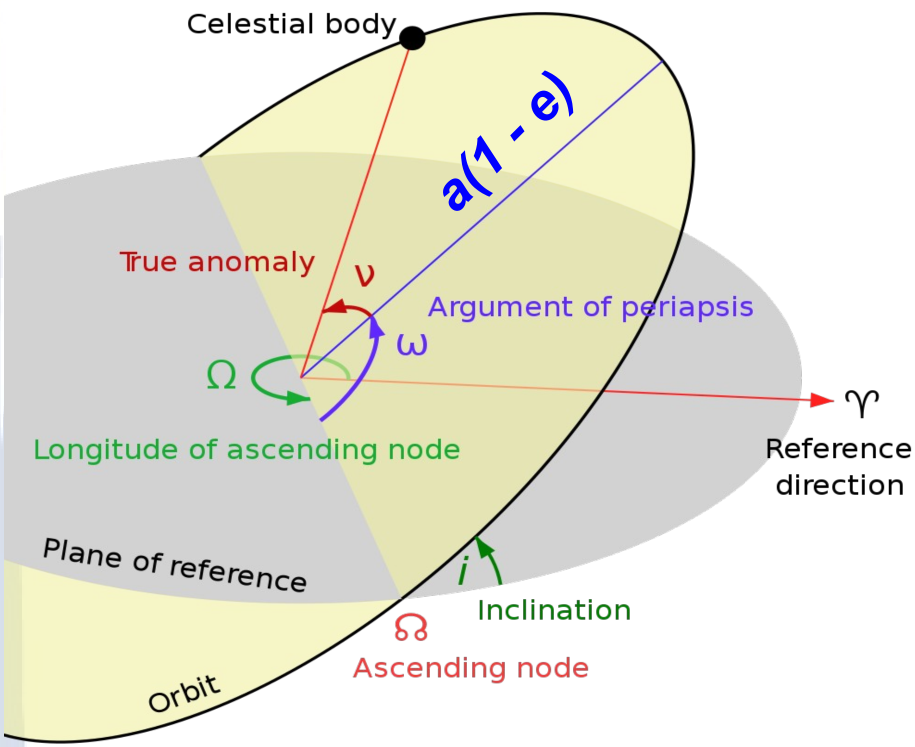
Asteroids' nice pictures...



Asteroids (inner SS top view…)



Orbital elements



Kepler's Laws and Equation

- $h = \sqrt{G M a (1 - e^2)} = const$
- $n^2 a^3 = G M = \mu \quad , \quad n = 2\pi / P$

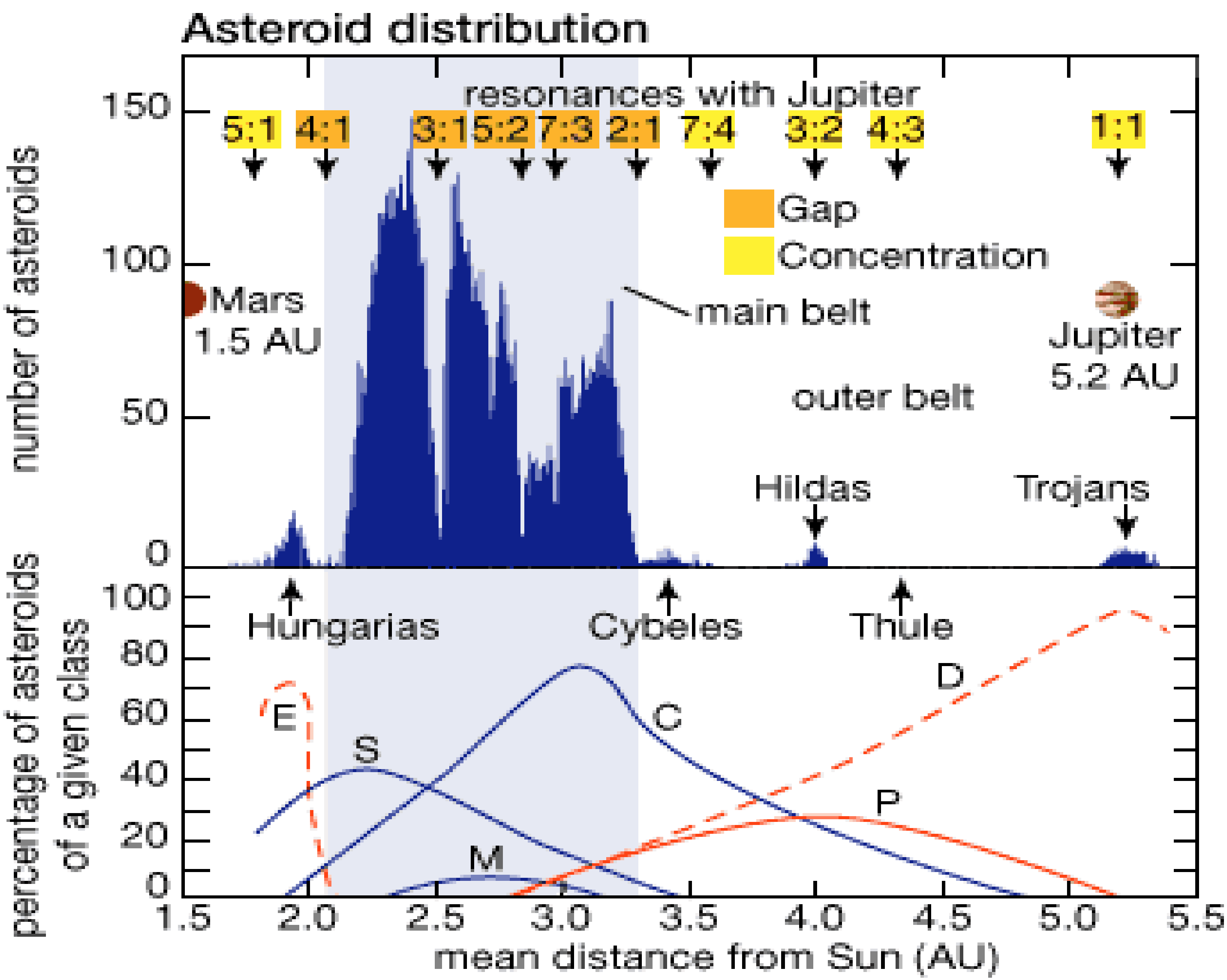
a = semi-major axis
e = eccentricity
i = inclination (rel. to plane of ref.)
Ω = long. of the ascending node
ω = arg. of perihelion
ν = true anomaly

$\boxed{l = n\,(t - t_P) = E - e \sin E}$ mean anomaly (or M)

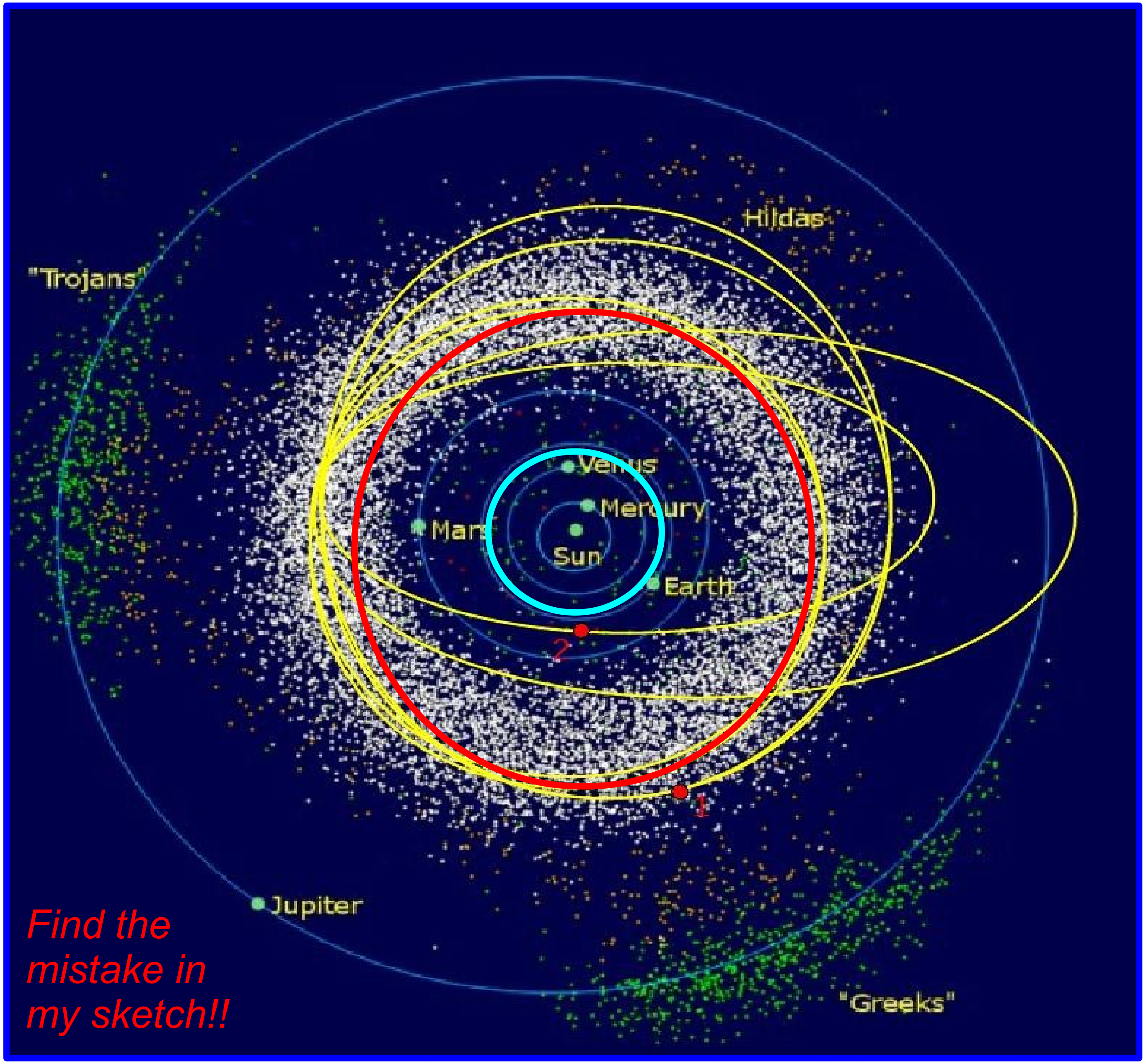
We prefer longitudes, so we define the *mean longitude*, $\lambda = l + \omega + \Omega$, and the *longitude of perihelion*, $\varpi = \omega + \Omega$.

Ignoring gravitational perturbations from other bodies, all ellipses are fixed in inertial space and $[a, e, i, \Omega, \omega, t_P]$ are constants.

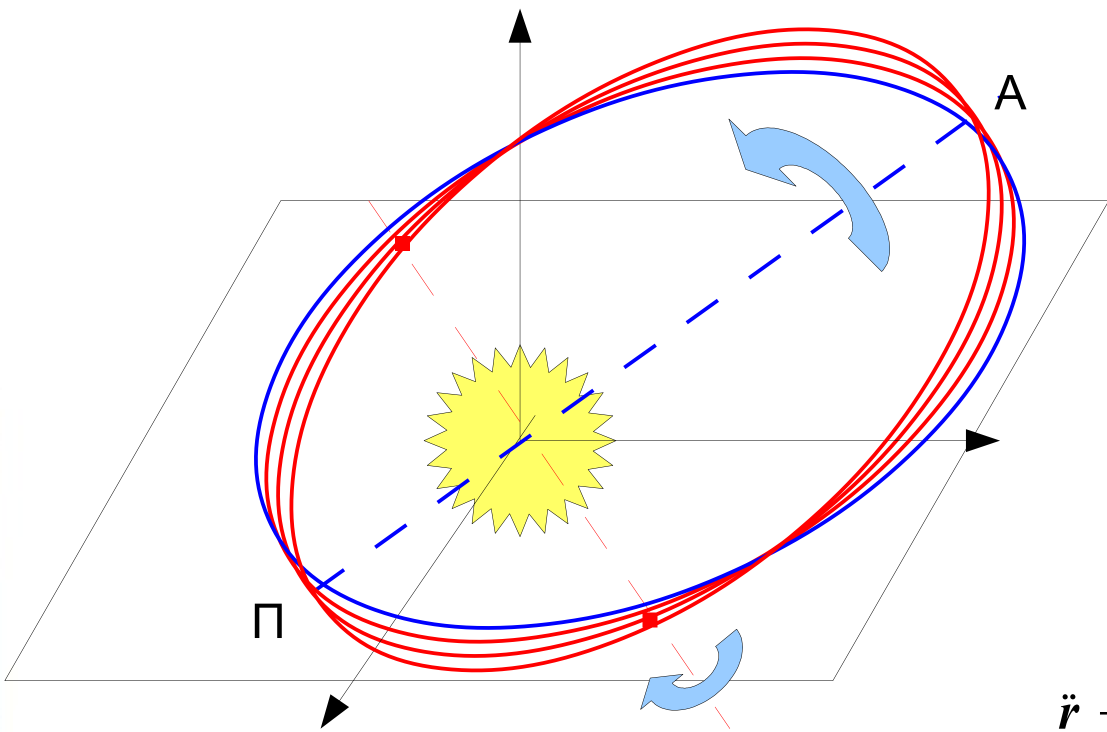
Orbital and Spectral Distribution of Asteroids



NEAs – MB dynamical connection



Dynamics I: Secular precession



2-body: $V = -\frac{G\,M_{Sun}}{r}$

$$\ddot{\boldsymbol{r}} = -\nabla V \rightarrow \boldsymbol{r} = \boldsymbol{f}(t, C_1, \ldots, C_6)$$
$$\boldsymbol{u} = \boldsymbol{g}(t, C_1, \ldots, C_6)$$

with $(C_1, \ldots, C_6) = (a, e, i, \Omega, \omega, t_P)$

Adding a small disturbing force (e.g. Jupiter's gravity)

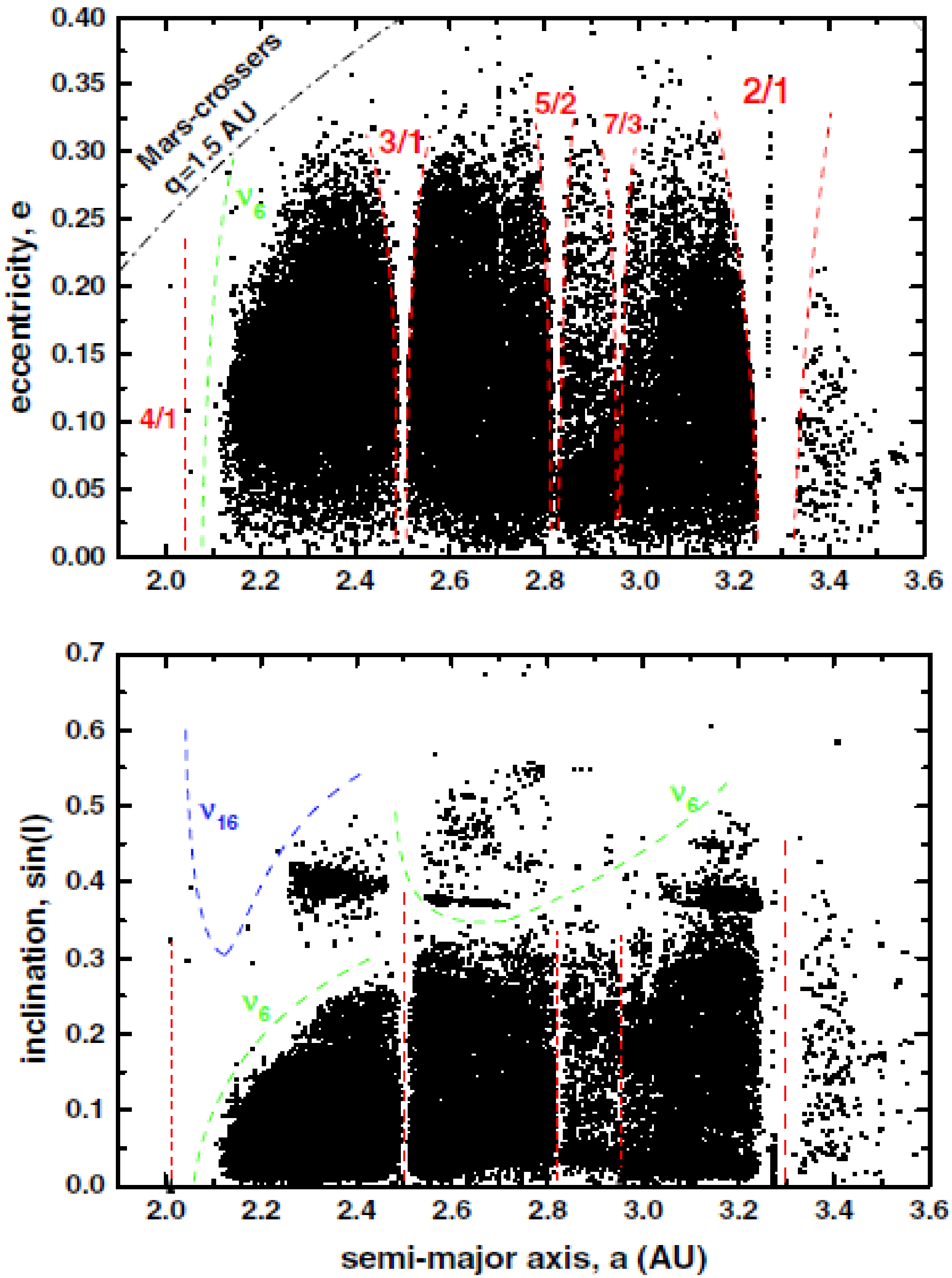
$$\ddot{\boldsymbol{r}} + \nabla V = \varepsilon\, \Delta \boldsymbol{F}$$

$$(C_1, \ldots, C_6) = (C_1(t), \ldots, C_6(t)) \longrightarrow \begin{aligned} \boldsymbol{r} &= \boldsymbol{f}(t, C_1(t), \ldots, C_6(t)) \\ \boldsymbol{u} &= \boldsymbol{g}(t, C_1(t), \ldots, C_6(t)) \end{aligned}$$

In the linear approximation and averaging over the fast (orbital) time-scale, we obtain long-periodic variations

⇨ *secular precession* with periods ~ $O(1/\varepsilon)$

Dynamics II: Resonances



Resonances occur when two (or more) frequencies become commensurate:

- *Mean motion* resonances (MMRs, also 3-B MMRs):

$$n/n_j = p/q$$

- Secular Resonances (SRs):

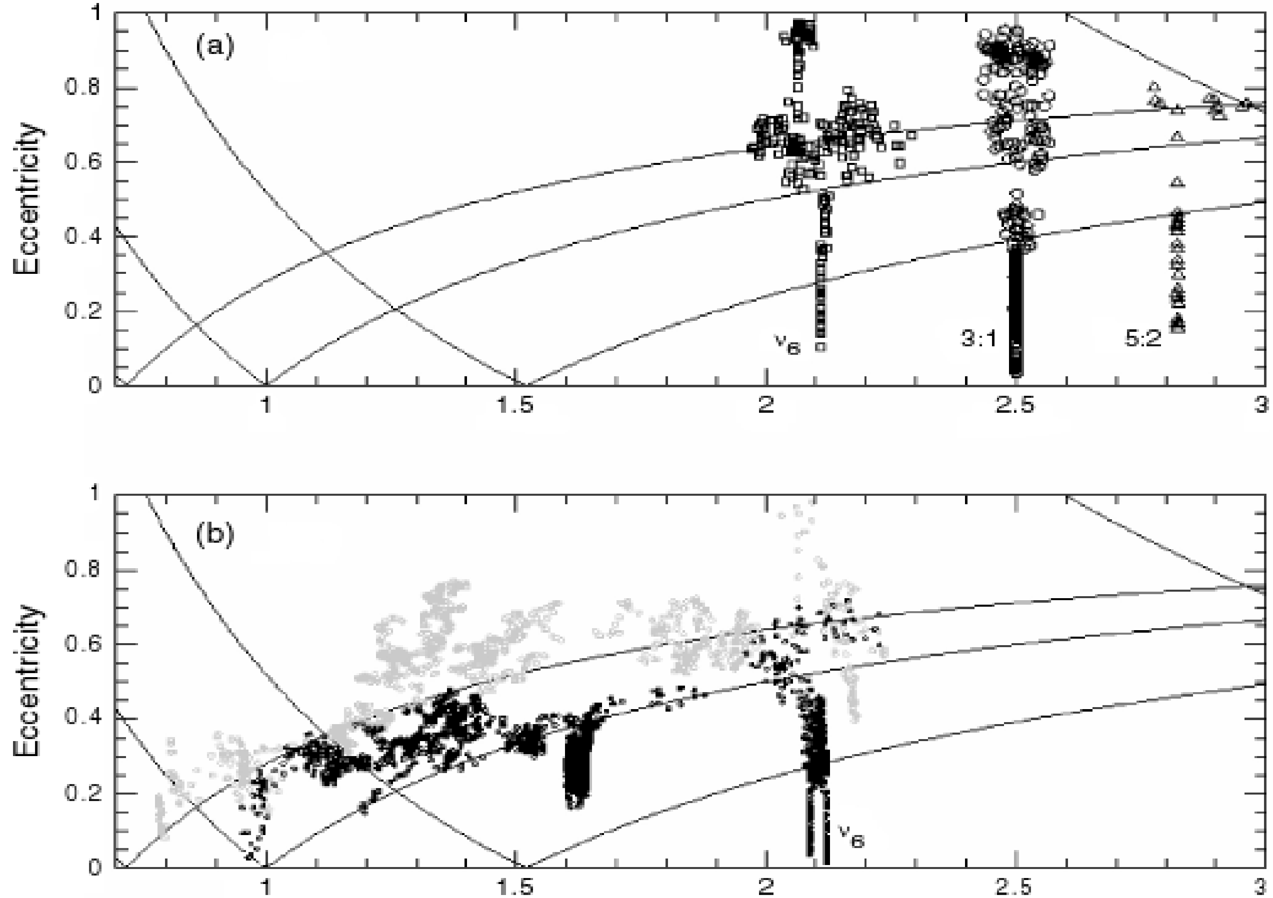
$$\frac{d\varpi}{dt} = \langle \dot{\varpi}_{J,S} \rangle \longrightarrow \nu_{5,6}$$

$$\frac{d\Omega}{dt} = \langle \dot{\Omega}_{S} \rangle \longrightarrow \nu_{16}$$

- Kozai, mixed, secondary...

** SRs can occur inside MMRs*

Dynamics III: Close encounters (NEAs)



MB → NEA via powerful resonances (outer region, t_d<1My)

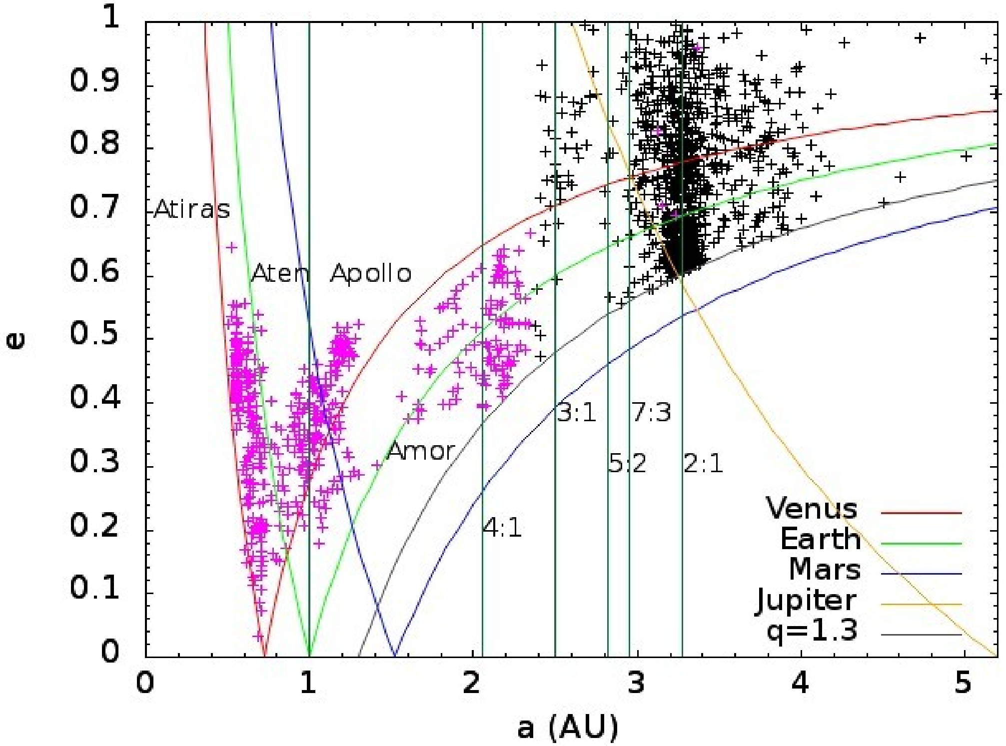
Outer NEA → 'Evovled' NEA with t_d>10 My (close encounters + temporary trapping in SRs, MMRs ...)

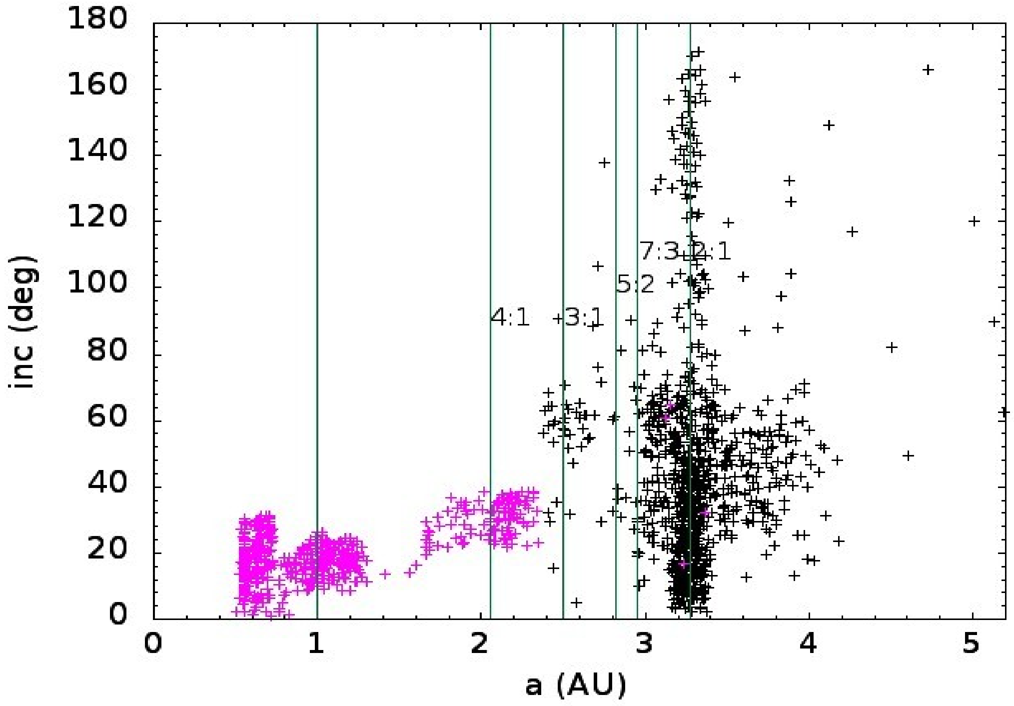
Repeated encounters with a planet ~ conserve the *Tisserand parameter*:

$$\boxed{T = \frac{a_p}{a} + 2\sqrt{(1-e^2)\,\frac{a}{a_p}}\cos i}$$

→ diffusion along T =const (encounters with >1 planets at ~same time break this...

NEAs from the 2:1 MMR





Bodies can be extracted even from the core regions of the 2:1 MMR

< 1% can penetrate the evolved region (a<2 AU) and live >20 My

Mean t_d of 2:1 escapers that become NEAs ~ 1.3 My

Sketch of MMR dynamics

In the restricted 3-body problem ($m_3 \rightarrow 0$) the *Hamiltonian* takes the form:

$$\underbrace{H = \frac{u^2}{2} - \frac{G M_{Sun}}{r}}_{H_{\mathrm{Kep}}} - \underbrace{G m_p \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_p|} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}_p}{r_p^3} \right)}_{}$$

H_{Kep} ***disturbing function :*** $R(\boldsymbol{r}, \boldsymbol{r}_{\mathbf{p}}) \rightarrow R(a, e, i, \Omega, \omega, \lambda, \lambda_{\mathrm{p}})$

$$\boxed{H = -\frac{G M_{Sun}}{2a} - G m_p \sum_{k,q,p,r} A_{k,q,p,r}(a, e, i; e_P) \cos \underbrace{\left(k\lambda - (k+q)\lambda_P + p\varpi + r\Omega\right)}_{\varphi\text{ = critical angle}}}$$

φ = critical angle

and the ***resonant condition*** for period ratio k/(k+q) is:

$$\boxed{\dot{\varphi} \approx \underbrace{k n - (k+q) n_P}_{} + O(m_P / M_{Sun}) \approx 0 \quad , \quad \underline{\mathit{iff}} \quad a_{Res} \approx \left(\frac{k}{k+q} \right)^{2/3} a_P}$$

* για $t=T >> P$ (~100 y), the net effect of each term containing λ's:

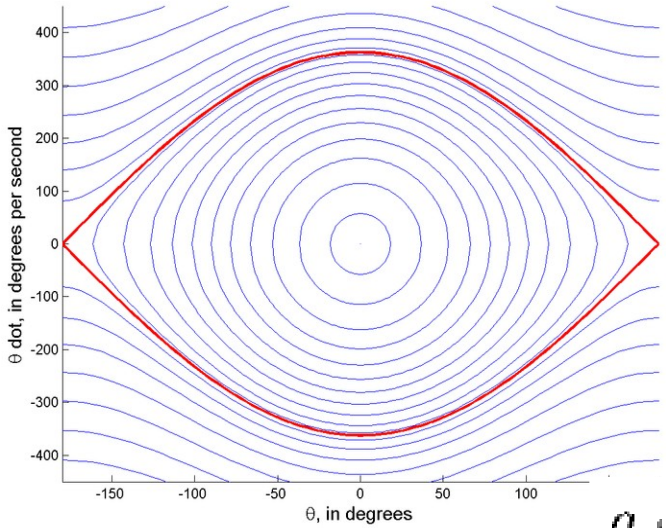
$$\langle A_k \cos \varphi_k \rangle = \frac{1}{T} \int_0^T A_k \cos \varphi_k \, dt \approx 0$$ ***Unless*** **$a \sim a_{Res}$ ⟶ Dominant!!**

Sketch of MMR dynamics:

In 2-D (planar elliptic rTBP) → for each resonance ratio k/(k+q) there are q+1 terms $\varphi_{k,q}$ and

$$H=\frac{1}{2}\beta J_1^2-c\,J_2-\varepsilon\cancel{\sum_p}A_p(J_2)\cos(\theta_1+p\,\theta_2)$$

$$(\,|\beta|\gg|c|\sim O(\varepsilon)\;,\;\;\theta_1=k\,\lambda-(k+q)\lambda_P\;,\;\;\theta_2=\varpi\,)$$



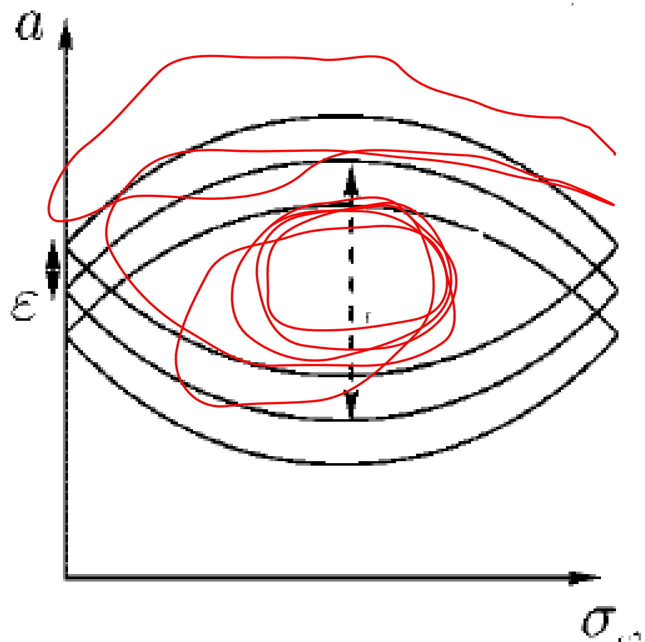
A single term (e.g. p=q) gives $\Phi(J_1,J_2)$=const

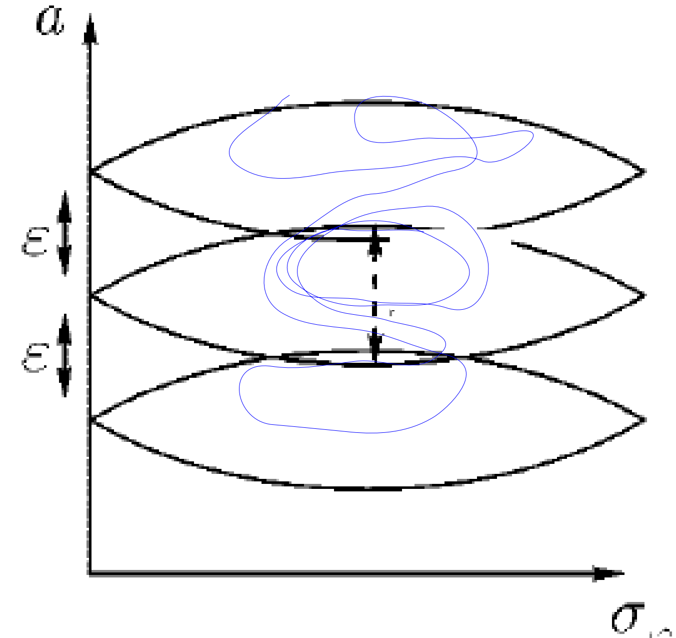
→ *pendulum-like dynamics*

However, there are q+1 terms (and $d\theta_2/dt\sim\varepsilon$)

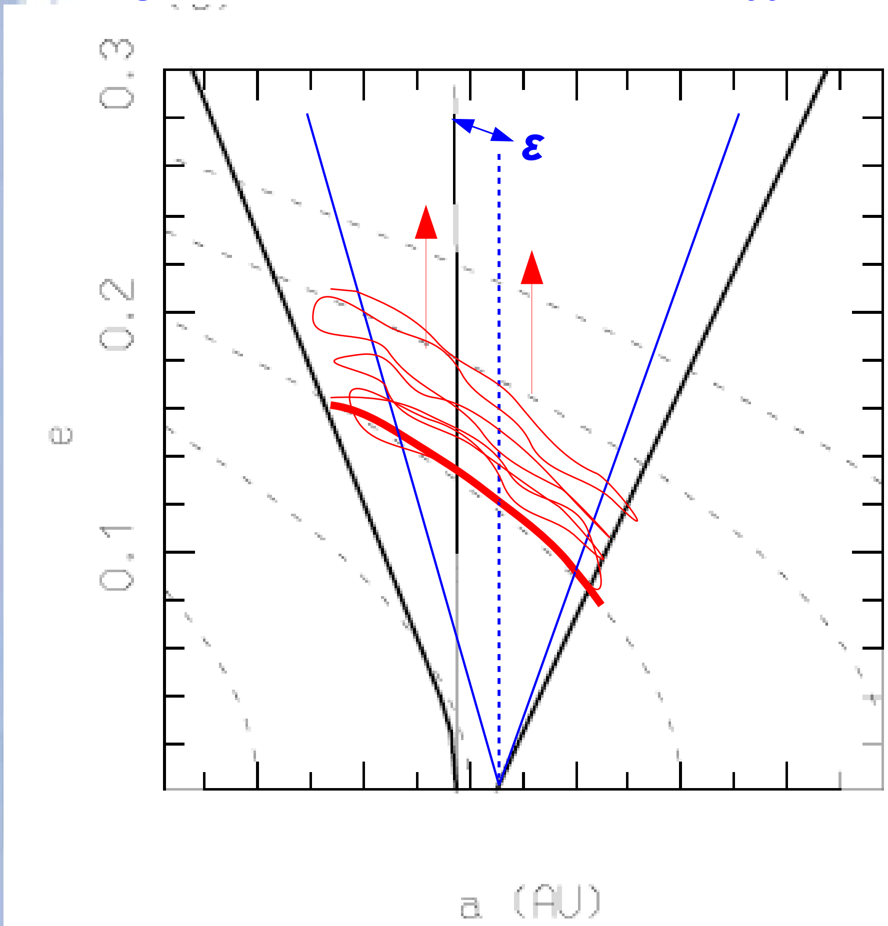
Resonance overlapping

Chaotic motions





Dynamics IV: Chaotic Diffusion



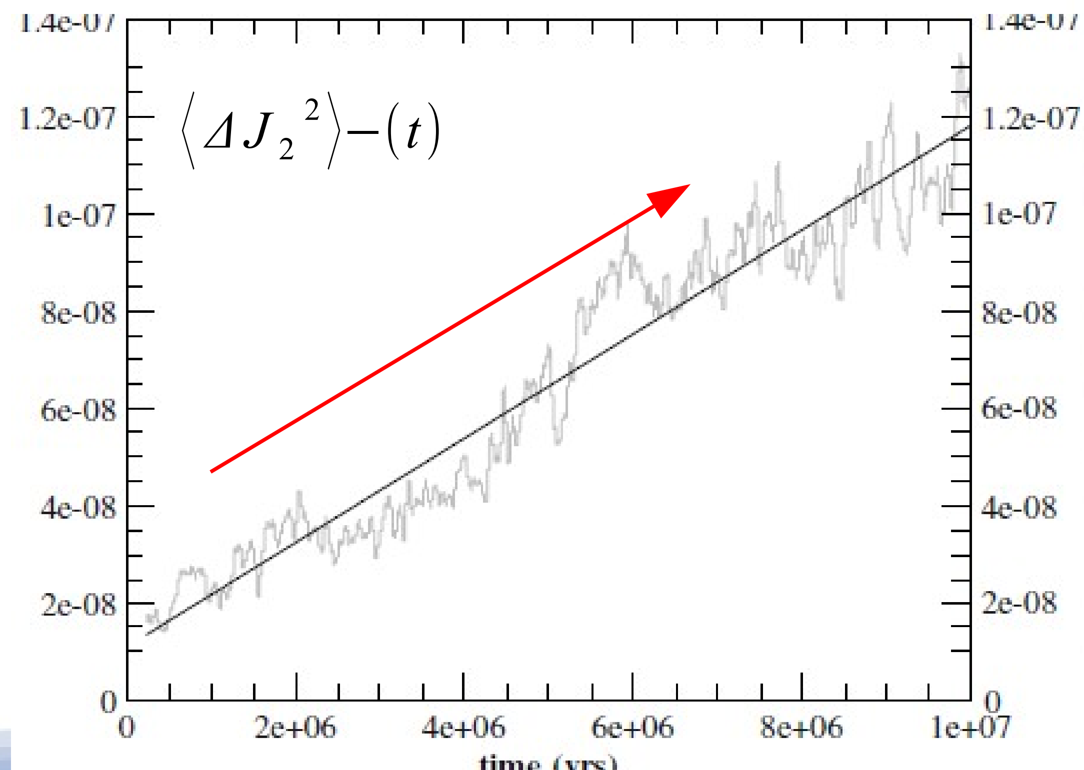
In celestial mechanics we (ab)use this term to describe *irregular, long-term, small-scale variations* of *'proper' elements* that build-up in time

In the single-resonance approximation (e.g. circular rTBP) a 2nd integral of motion exists, and thus

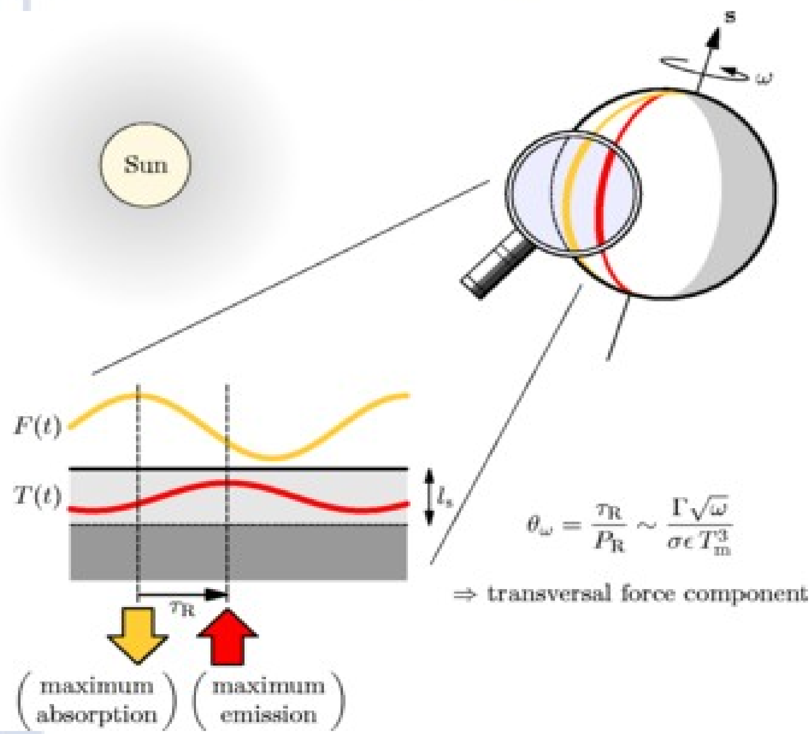
$$\Phi_p(J_1,J_2) = \Phi_p(a,e) = \text{const}$$

This is no longer true in the elliptic rTBP. Depending on MMR, this appears as

Chaotic diffusion



* The Yarkovsky effect



Finite thermal conductivity and dimensions + rotational motion of a body, absorbing solar radiation, → a recoil force that has a tangential component

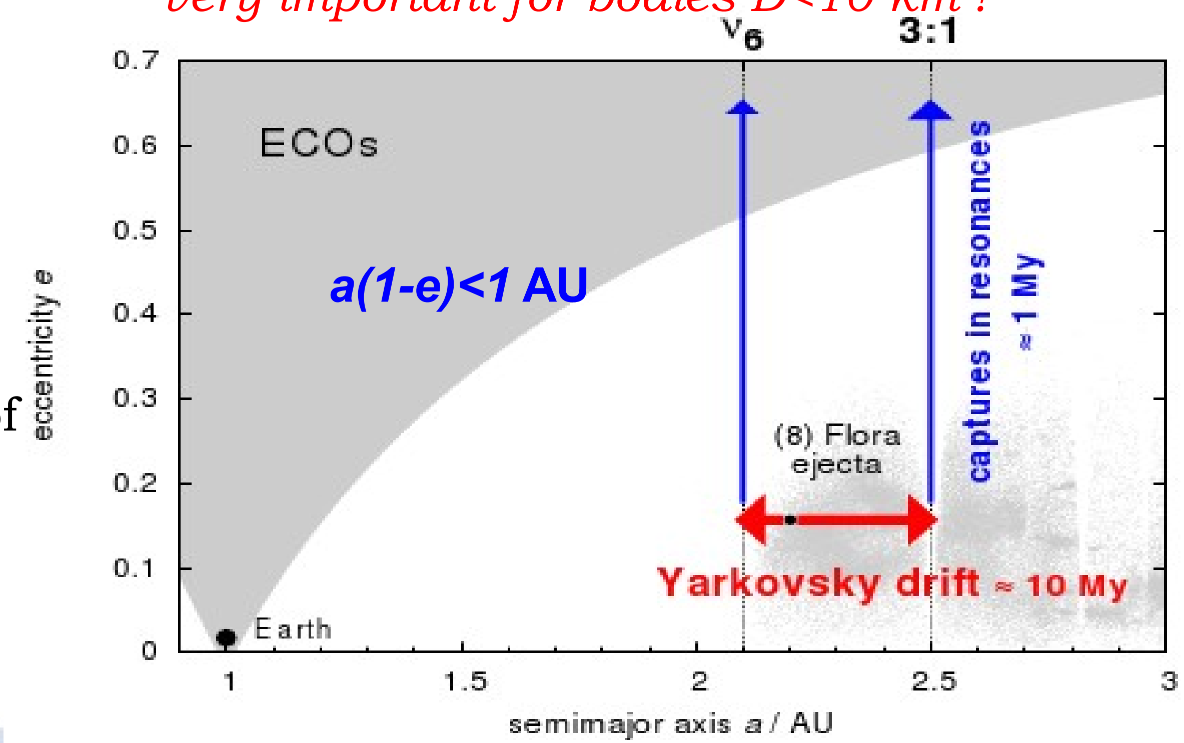
→ $da/dt = f(D, \Theta, \boldsymbol{\omega}) \sim [\ 3 \times 10^{-4}\ / D\ ]$ (AU/My)

For non-spherical shapes (non-zero torque) the rotational state can be strongly affected (YORP)

** very important for bodies D<10 km !*

Supplies “fresh” material into the 'powerful' resonances

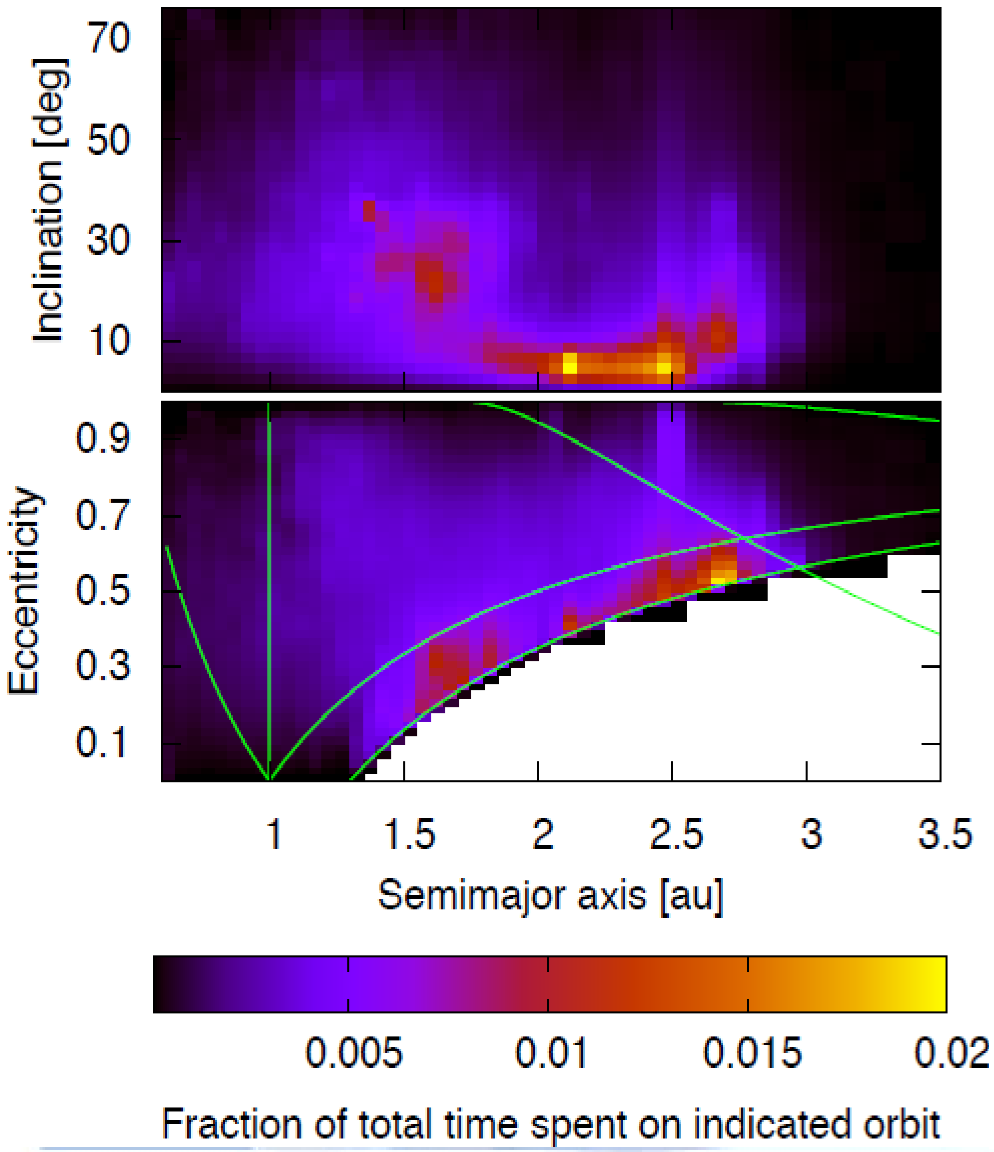
→ continuous production of *NEAs* that leave the Main Belt



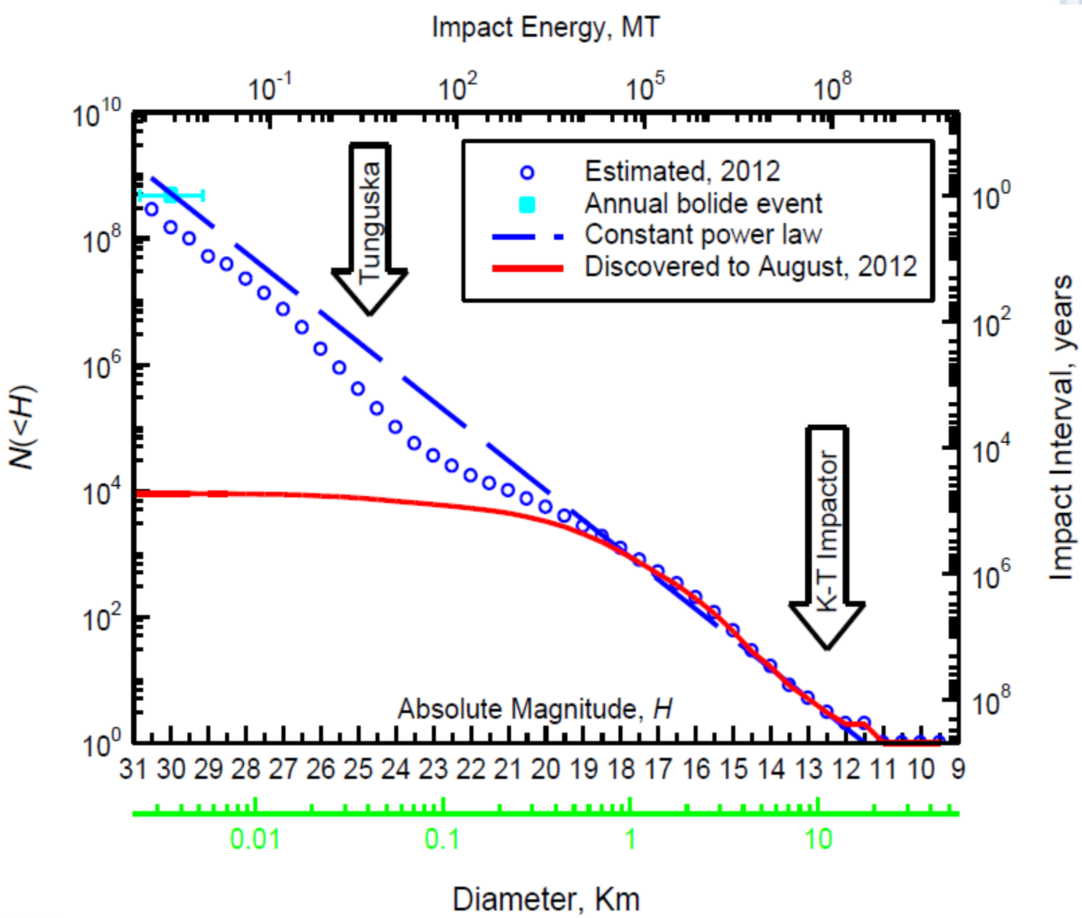
* Models of NEA orbital and size distribution

We can simulate the long-term motion of MBs and keep track of the main *sources* of MCs and NEAs (and their relative contribution)

Compute the mean residence time of orbits in each (a,e,i) cell (q<1.3AU)



+ combine with observations (biases, efficiency etc.)



Ancient Bombardments

There is evidence that the Earth (inner SS) has suffered intense bombardment period(s) during its youth...

3.9 Gy ago the *Late Heavy Bombardment* was ending.

→ Impact rate ~ ***1000x current!!!***

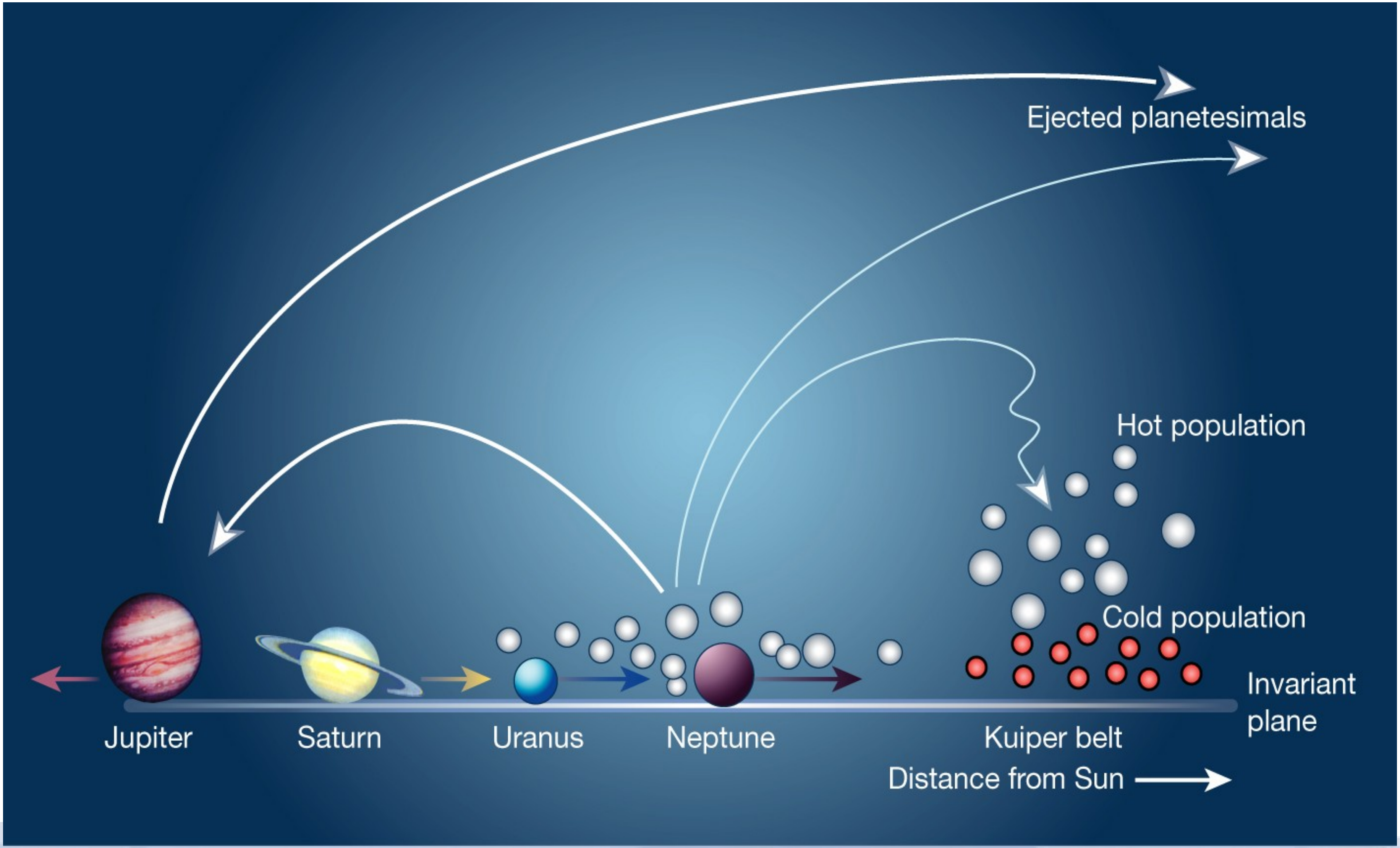
Requires a total mass of small bodies ~1.000x larger than current estimates

→ where was all this mass 'hidden' and why did the cataclysm come so late?

Planet migration believed to be the answer

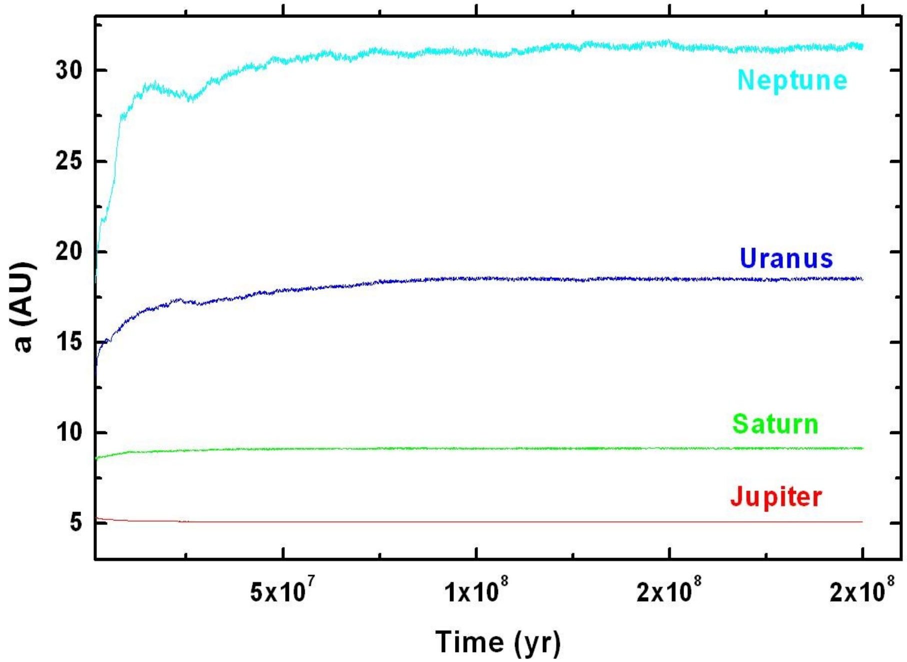
Initial planetary orbits were likely very different (circular, closer to Sun)

Angular momentum exchange with “heavy” belts → *radial migration*

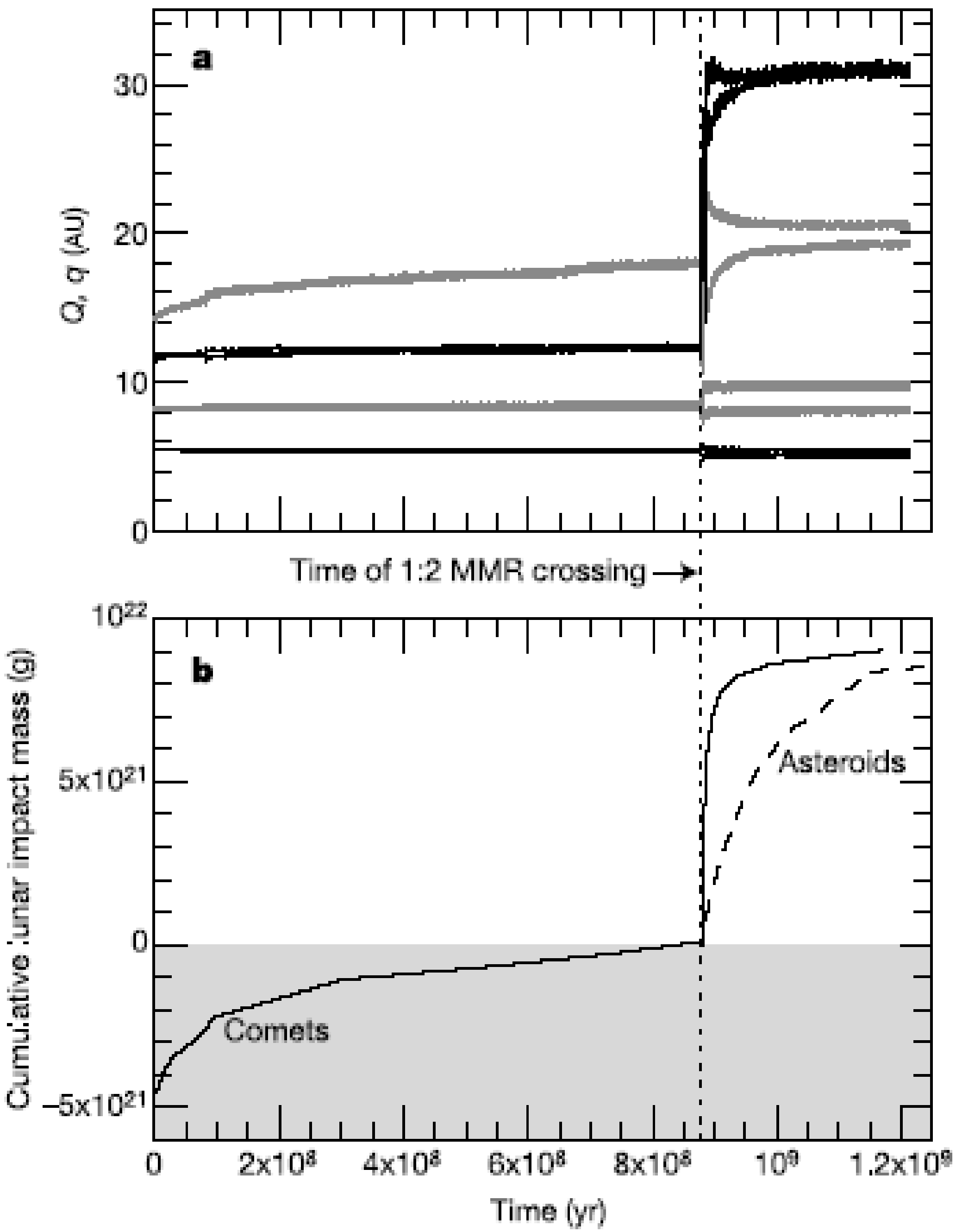


Two migration models

smooth



chaotic

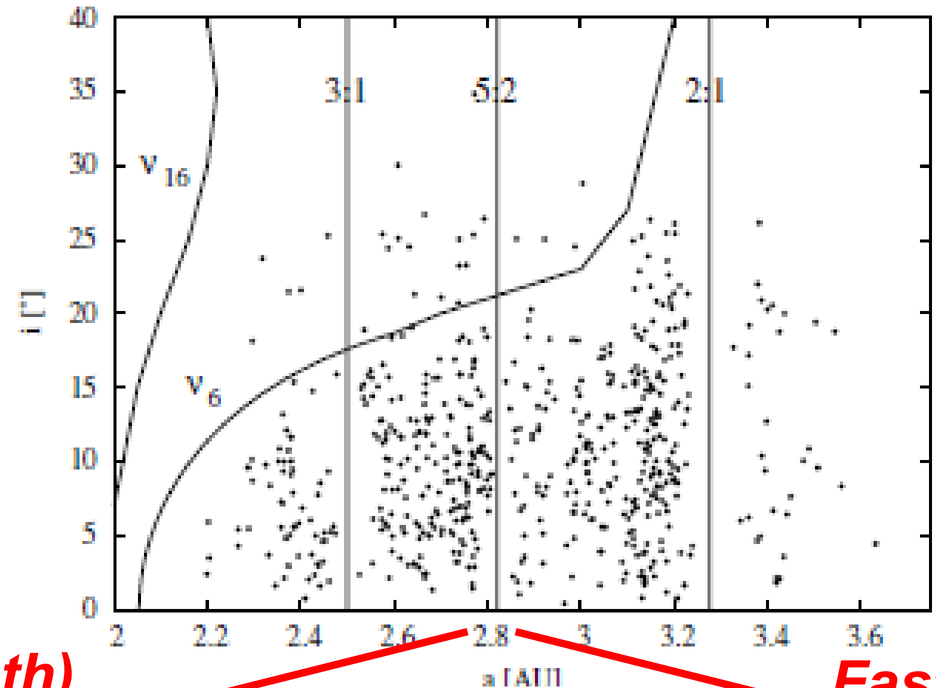


The ***'Nice model'*** explains

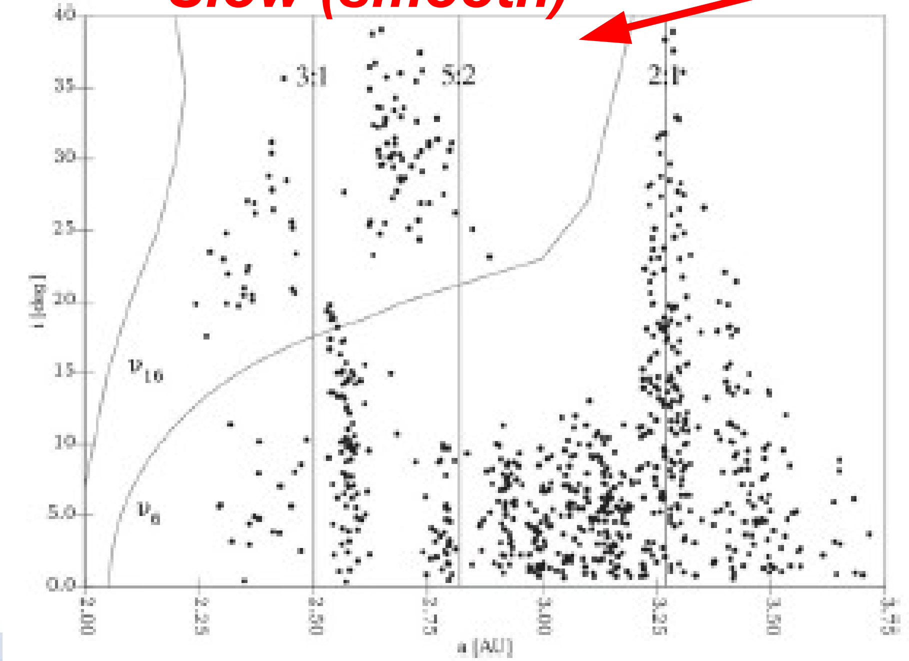
- the current orbits of the planets
- main LHB constraints
- mass loss and orbital KBO distribution

Can the MB structure be a good criterion ?

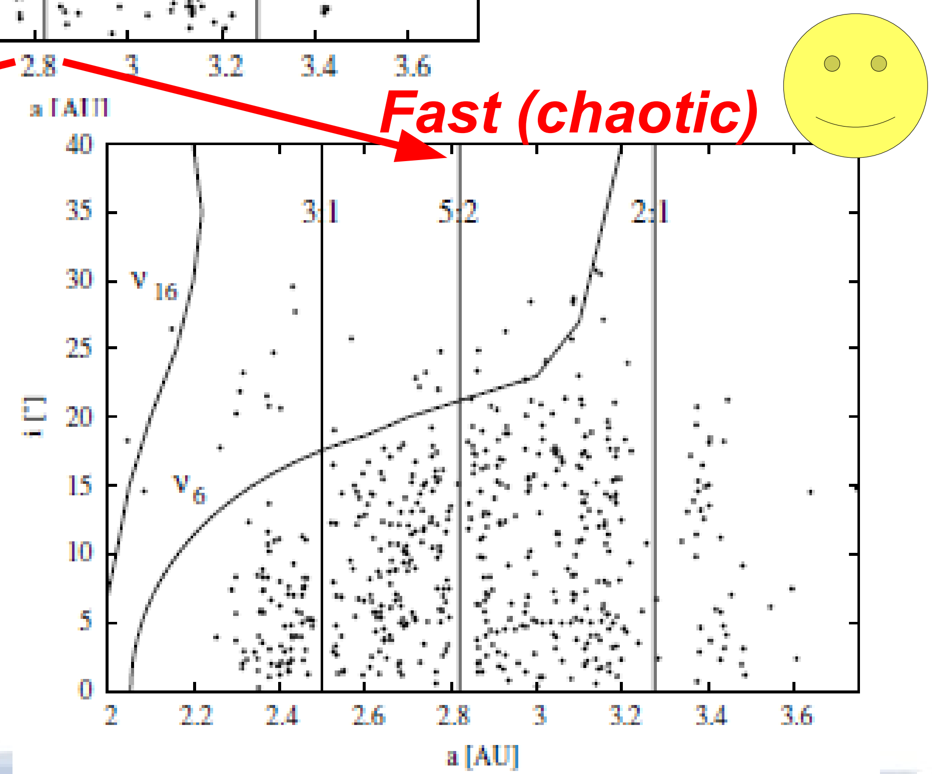
(a,i) distribution of real MB asteroids with *D*>50 km



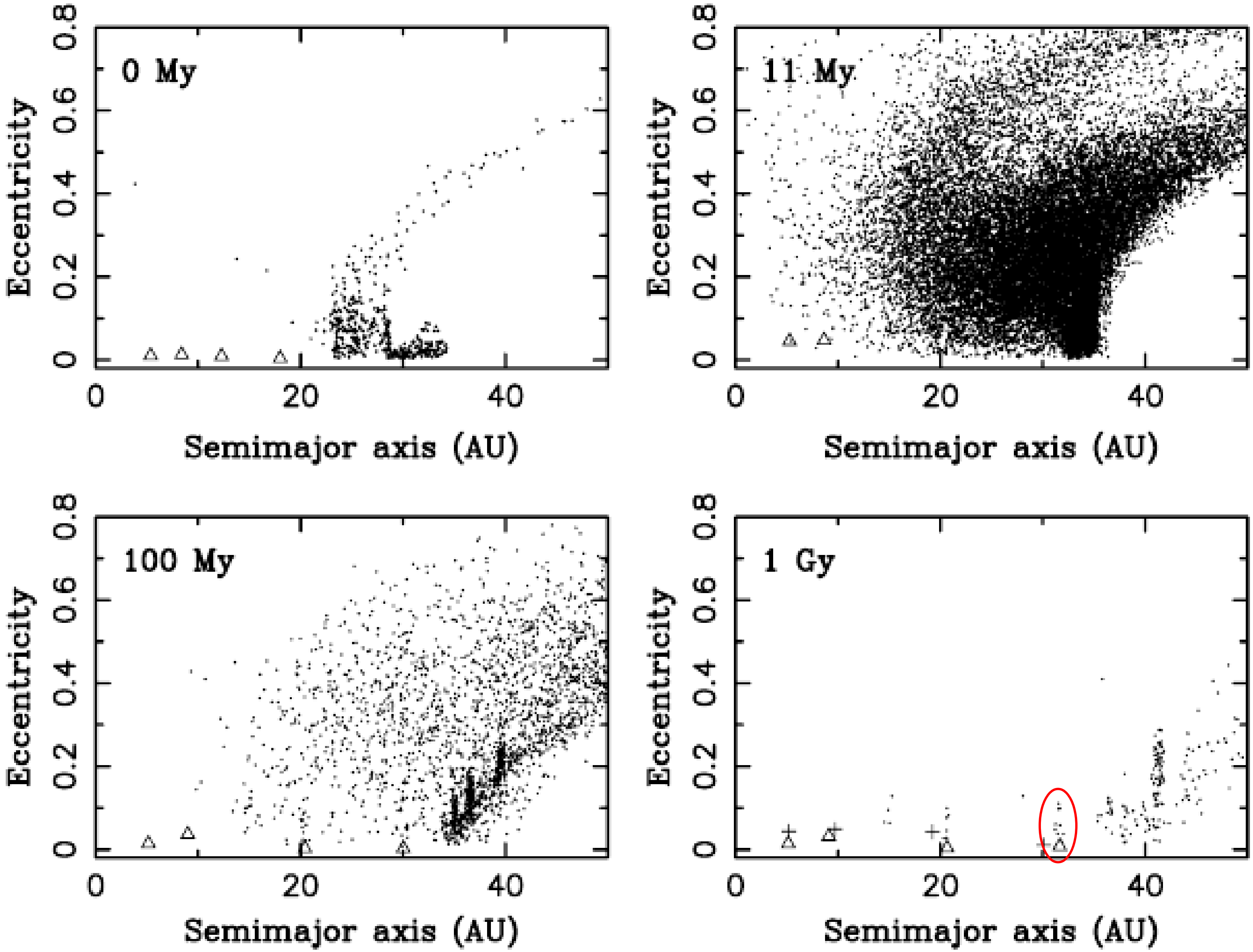
Slow (smooth)

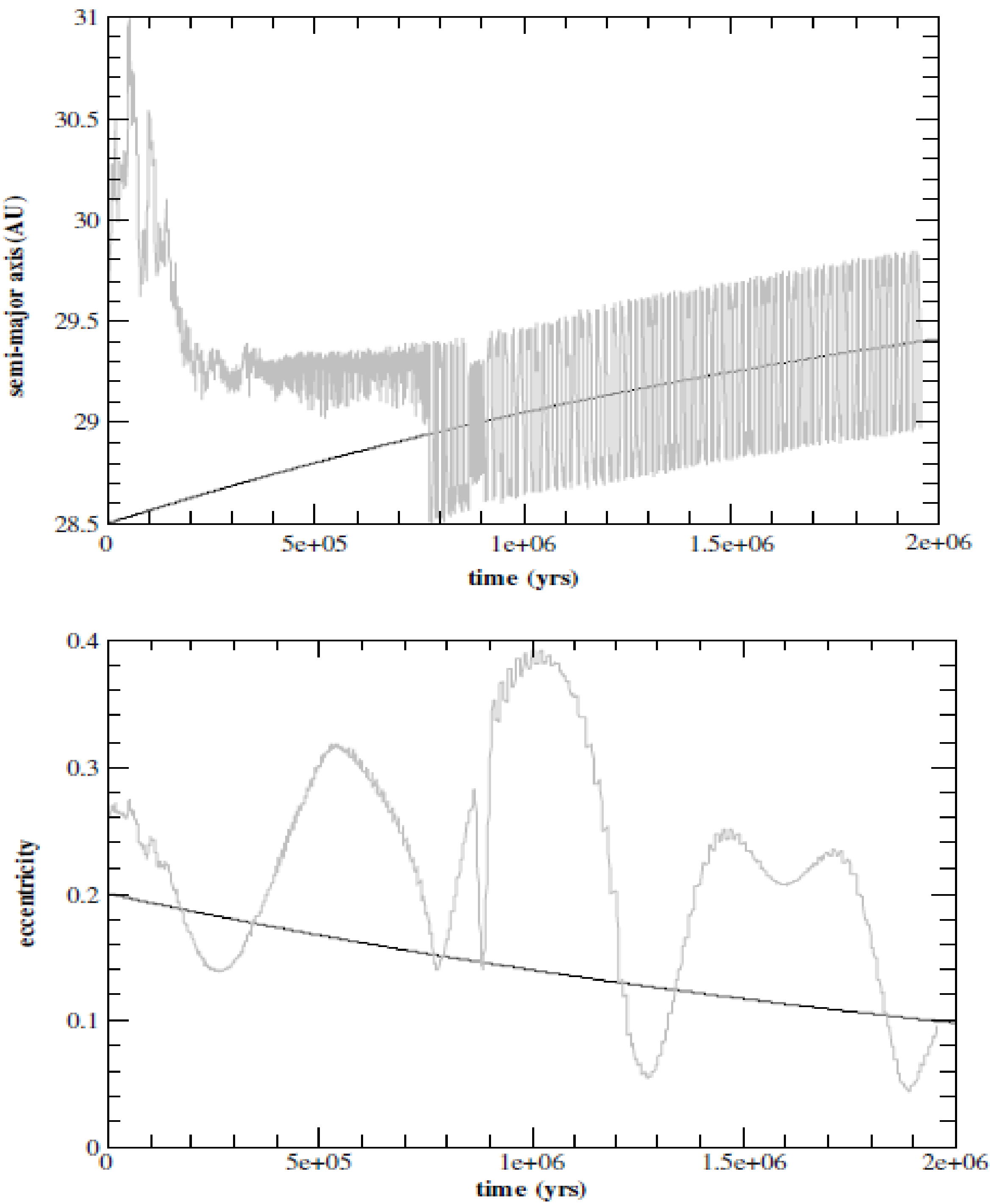


Fast (chaotic)



Outer SS - evolution ($t_0=t_{ins}$ – 10My)





Chaotic capture

- upon "encountering" a resonance (MMR)

- not similar to resonant encounters in the *adiabatic* problem

The planet's eccentricity decreases, until the MMR becomes regular

- works well for Trojans and irregular satellites

Capture of a Neptune Trojan

End of Part I…

Celestial Mechanics: some Theory and Tools

- *2BP and 3BP*
  - Newtonian formalism → Lagrangean perturbation eqs.
  - Hamiltonian formalism

- *The disturbing function*
  - 3BP and Satellite problem
  - examples

- *Canonical transformations - Perturbation theory*
  - Generating functions
  - Lie series method

- *Applications*
  - derive a simpler model for our problem
  - build a symplectic integrator

Perturbed 2BP

We start from the 2BP Newtonian equation for relative motion:

$$\ddot{\boldsymbol{r}} + \frac{\mu}{r^2}\frac{\boldsymbol{r}}{r} = 0, \qquad \boldsymbol{r} = \boldsymbol{f}(C_1, ..., C_6, t), \qquad \dot{\boldsymbol{r}} = \boldsymbol{g}(C_1, ..., C_6, t)$$

$$\boldsymbol{r} \equiv \boldsymbol{r}_{\text{planet}} - \boldsymbol{r}_{\text{sun}}$$

Include a 'small' disturbing force, $\Delta\boldsymbol{F}$, and assume that $C_i = C_i(t)$:

$$\ddot{\boldsymbol{r}} + \frac{\mu}{r^2}\frac{\boldsymbol{r}}{r} = \Delta\boldsymbol{F} \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\partial \boldsymbol{f}}{\partial t} + \sum_i \frac{\partial \boldsymbol{f}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t} = \boldsymbol{g} + \sum_i \frac{\partial \boldsymbol{f}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t}$$

Choosing the following *gauge*: $\boxed{\sum_i \frac{\partial \boldsymbol{f}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t} = 0}$ **(1)**

→ the perturbed orbit is *osculating*

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \frac{\partial \boldsymbol{g}}{\partial t} + \sum_i \frac{\partial \boldsymbol{g}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t} = \frac{\partial^2 \boldsymbol{f}}{\partial^2 t} + \sum_i \frac{\partial \boldsymbol{g}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t}$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial t^2} + \frac{\mu}{r^2}\frac{\boldsymbol{f}}{r} + \sum_i \frac{\partial \boldsymbol{g}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t} = \Delta\boldsymbol{F} \longrightarrow \boxed{\sum_i \frac{\partial \boldsymbol{g}}{\partial C_i}\frac{\mathrm{d}C_i}{\mathrm{d}t} = \Delta\boldsymbol{F}} \quad \textbf{(2)}$$

Perturbed 2BP – Lagrange eqs.

Multiply (1) by -$\partial \boldsymbol{g}/\partial C_n$ and (2) by $\partial \boldsymbol{f}/\partial C_n$ and sum up:

$$\sum_j [C_n\, C_j] \frac{\mathrm{d}C_j}{\mathrm{d}t} = \frac{\partial \boldsymbol{f}}{\partial C_n} \Delta \boldsymbol{F}$$

the *Lagrange brackets* being defined as: $[C_n\, C_j] \equiv \frac{\partial \boldsymbol{f}}{\partial C_n}\frac{\partial \boldsymbol{g}}{\partial C_j} - \frac{\partial \boldsymbol{f}}{\partial C_j}\frac{\partial \boldsymbol{g}}{\partial C_n}$

For a *conservative* force $\Delta \boldsymbol{F} = -\nabla R(\boldsymbol{r})$ and for the usual Keplerian elements:

$$\frac{da}{dt} = -\frac{2}{n\,a}\frac{\partial R}{\partial M} \quad , \quad \frac{dM}{dt} = n + \frac{(1-e^2)}{n\,a^2 e}\frac{\partial R}{\partial e} + \frac{2}{n\,a}\frac{\partial R}{\partial a}$$

$$\frac{de}{dt} = -\frac{(1-e^2)}{n\,a^2 e}\frac{\partial R}{\partial M} + \frac{(1-e^2)^{1/2}}{n\,a^2 e}\frac{\partial R}{\partial \omega} \quad , \quad \frac{d\omega}{dt} = \frac{\cos i}{n\,a^2(1-e^2)^{1/2}\sin i}\frac{\partial R}{\partial i} - \frac{(1-e^2)^{1/2}}{n\,a^2 e}\frac{\partial R}{\partial e}$$

$$\frac{di}{dt} = -\frac{\cos i}{n\,a^2(1-e^2)^{1/2}\sin i}\frac{\partial R}{\partial \omega} + \frac{1}{n\,a^2(1-e^2)^{1/2}\sin i}\frac{\partial R}{\partial \Omega} \quad , \quad \frac{d\Omega}{dt} = -\frac{1}{n\,a^2(1-e^2)^{1/2}\sin i}\frac{\partial R}{\partial i}$$

** beware of different sign conventions for R(r) in various books…*

Perturbed 2BP in Lagrangean / Hamiltonian form

The *principle of d'Alembert* gives the equations of motion for a (un)constrained mechanical system:

$$\sum_{i=1}^{N}\left(\boldsymbol{F}_i - m_i\ddot{\boldsymbol{r}}_i\right)\cdot\delta\boldsymbol{r}_i = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = \sum_{i=1}^{N}\boldsymbol{F}_i\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

- for conservative forces ($\boldsymbol{F}_i = -\nabla_i V(\boldsymbol{r}_i, t)$), we can write the equations using the *Lagrangean, L* :

(1) $$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \quad (j=1,\ldots,n) \quad , \quad L = T - V \quad , \quad T = \frac{1}{2}\sum_i m_i u_i^2$$

The *Hamiltonian, H,* of the system can be defined by the Legendre transform of *L*:

$$H(\boldsymbol{q}_j, \boldsymbol{p}_j) = \sum_{j=1}^{n}\dot{q}_j\, p_j - L(q_j, \dot{q}_j, t) \quad , \quad p_j = \frac{\partial L}{\partial \dot{q}_j}$$

p_j's being the *generalized momenta*. *Hamilton's principle* dictates that:

(2) $$\boxed{\delta\int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt = 0}$$

and *Euler's theorem* in calculus of variations ensures that those orbits that satisfy (2) are in fact the solutions of (1).

Perturbed 2BP equations

In the perturbed 2-body problem, if the perturbation is conservative,

$$\Delta \boldsymbol{F} = -\nabla R(\boldsymbol{r})$$

the functions L and H take the form:

$$L \equiv T - V = (T - U_{Kep}) - R = L_{Kep} - R \qquad \text{and} \qquad H \equiv T(\boldsymbol{p}) + V(\boldsymbol{q}) = H_{Kep} + R$$

With $H_{Kep} \equiv \dfrac{u^2}{2} - \dfrac{\mu}{r}$, and the Lagrange equations are:

$$\boxed{\sum_j [C_n C_j] \frac{dC_j}{dt} = -\frac{\partial R}{\partial C_n}}$$

in any set of (osculating) elements

R is called the *disturbing function**. To use in the above equations we need to know the transformation $q_{\mathrm{j}} \rightarrow C_j$

Check the Lagrange brackets to see that the *Delaunay elements*

[l=M , g=ω , h=Ω , L = $(\mu a)^{1/2}$, G = $L(1\text{-}e^2)^{1/2}$, H =G cos i]

greatly simplify the equations, since: $[l,L]$ = $[\omega,G]$ = $[\Omega,H]$ =1 and all other brackets give zero…

* *can be really disturbing …*

Delaunay elements

The equations become :

$$\frac{dl}{dt} = \frac{\partial R}{\partial L} \quad , \quad \frac{dL}{dt} = -\frac{\partial R}{\partial l}$$
$$\frac{dg}{dt} = \frac{\partial R}{\partial G} \quad , \quad \frac{dG}{dt} = -\frac{\partial R}{\partial g}$$
$$\frac{dh}{dt} = \frac{\partial R}{\partial H} \quad , \quad \frac{dH}{dt} = -\frac{\partial R}{\partial h}$$

clearly they are *canonical elements*, as this system has the *symplectic structure* of a *Hamiltonian system* of canonical equations

the same holds for the *modified Delaunay* (or *Poincaré* elements):

$$\boxed{\begin{array}{ll} \Lambda = L & \lambda = l + g + h \\ \Gamma = L - G & \gamma = -(g+h) = -\varpi \\ Z = G - H & \zeta = -h = -\Omega \end{array}}$$

and, for small (e,i), $\Gamma \sim \Lambda\, e^2/2$ and $Z = \Lambda\, i^2/2$

In these elements, the Hamiltonian reads:

$$\boxed{H = -\frac{\mu^2}{2\Lambda^2} + R(\lambda, \gamma, \zeta, \Lambda, \Gamma, Z, \ldots) = H_{Kep} + R}$$

Hamiltonian formalism

Hamilton's *canonical* equations of motion are given by:

$$\boxed{\dot{q}_j = \frac{\partial H}{\partial p_j} \quad , \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}} \qquad \left(\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \right)$$

For an *autonomous** system H = const = $T+V$ **. Also, any *ignorable* variable ($\partial H/\partial q_j$=0) gives that p_j=const.

Using the *Poisson brackets* $\{f, g\} = \sum_j \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}$

we re-write the equations as: $\dot{q}_j = \{q_j, H\}$, $\dot{p}_j = \{p_j, H\}$ *also* $\{q_i, p_k\} = \delta_{i,k}$

and, for any function $f(q_j,p_j)$: $\frac{df}{dt} = \{f, H\} = \{\ \ , H\}\, f = D_H f$

whose formal solution is the *Lie series of f* under the *t*-flow of *H*:

$$\boxed{f(t) = \exp[t\ D_H]\, f(0) = \left(1 + t\, D_H + \frac{t^2}{2} D_H^2 + \ldots\right) f(0) \equiv S_H^t f}$$

* *a non-autonomous system can be amended by* $\dot{t} = 1$, $-\dot{H} = -\frac{\partial H}{\partial t}$ *to become autonomous in an the extended phase space*

** *forces and constraints (transformation from* $\mathbf{r}_i \rightarrow q_j$*) have to be time-independent as well*

Canonical Transformations

Any time-independent transformation $(q_j,p_j) \rightarrow (Q_j,P_j)$ is *canonical* if it preserves the symplectic form of Hamilton's equations, i.e.

$$\dot{Q}_j = \frac{\partial H'}{\partial P_j} \quad , \quad \dot{P}_j = -\frac{\partial H'}{\partial Q_j}$$

with $H'(Q_j,P_j) = H(\, q_j(Q_j,P_j)\, ,\, p_j(Q_j,P_j)\,)$. Hamilton's principle gives:

$$\delta \int_{t_1}^{t_2} \left(\sum_j \dot{q}_j \; p_j - H \right) dt = 0 = \delta \int_{t_1}^{t_2} \left(\sum_j \dot{Q}_j \; P_j - H' \right) dt$$

From which we can find 4 basic types of *generating functions, F_k*, that define canonical transformations, e.g.

$$\boxed{F_2 = F_2(q_j, P_j) \quad \Rightarrow \quad p_j = \frac{\partial F_2}{\partial q_j} \quad , \quad Q_j = \frac{\partial F_2}{\partial P_j}}$$

Simple geometrical transformations are easily obtained through this (well-known F's). It can be proven that if a generating function $\chi(\boldsymbol{q}', \boldsymbol{p}')$ and a parameter ε exist, such that:

$$\boldsymbol{p} = \boldsymbol{p}' + \int_0^{\varepsilon} \dot{\boldsymbol{p}}'\, dt = \boldsymbol{p}'(\varepsilon) \qquad , \qquad \boldsymbol{q} = \boldsymbol{q}' + \int_0^{\varepsilon} \dot{\boldsymbol{q}}'\, dt = \boldsymbol{q}'(\varepsilon)$$

Then, the Lie series: $\boxed{\begin{array}{l} \boldsymbol{q} = S_\chi^\varepsilon \boldsymbol{q}' \quad , \quad H' = S_\chi^\varepsilon H \\ \boldsymbol{p} = S_\chi^\varepsilon \boldsymbol{p}' \end{array}}$ define a canonical transformation

An interesting example...

Let's make a transformation $(q,p) \rightarrow (Q,P)$ to the Hamiltonian of the *simple pendulum:*

$$H = \frac{p^2}{2} - A\cos q \longrightarrow \begin{cases} \dot{p} = -A\sin q \\ \dot{q} = p \end{cases} \quad (1)$$

Using the generating function $F_2 = q\,p' + \tau H(q, p')$ we get:

$$p = p' + \tau \frac{\partial H}{\partial q} = p' + \tau A \sin q$$

$$\Rightarrow \left.\boxed{\begin{aligned} p' &= p - \tau A \sin q \\ q' &= q + \tau p' \end{aligned}}\right\} \quad (2)$$

$$q' = q + \tau \frac{\partial H}{\partial p'} \Rightarrow$$

To $O(\tau)$, these equations are *a modified Euler method* for integrating (1). Hence (q', p') can be interpreted as the evolution of (q,p) for time $t=\tau$.

Since the transformation (mapping) is by construction symplectic, it constitutes *a symplectic integrator of O(τ)*.

This method has been used in asteroid dynamics to build simple mappings for resonant problems

Note: the pendulum is *integrable* (1 d.o.f autonomous Hamiltonian). The 2-D mapping (2) is the well-known *standard map* (has chaos!)

Constructing Symplectic Integrators

Find a *canonical transformation* ($\boldsymbol{q},\boldsymbol{p}$) → ($\boldsymbol{q}',\boldsymbol{p}'$) that approximates the formal solution ($\boldsymbol{q}(t),\boldsymbol{p}(t)$) for $\delta t=\tau$, up to some order in τ.

$$f(\tau) = \exp[\tau\ D_H]\, f(0) = \left(1 + \tau D_H + \frac{\tau^2}{2} D_H^2 + \ldots\right) f(0) \qquad (1)$$

For $H=T(\boldsymbol{p})+V(\boldsymbol{q})$ the operator is $D_H = D_T + D_V$ and it is easy to see that the application of D_T (or D_V) alone would give an *explicitly* solvable symplectic mapping (as the sub-system has only half the variables...)

$$\dot{q}_j = D_T q_j = \{q_j, T\} = \frac{\partial T}{\partial p_j} \quad \Rightarrow \quad q_j(\tau) = \exp[\tau\ D_T]\, q_j(0)$$

similarly

$$p_j(\tau) = \exp[\tau\ D_V]\ p_j(0)$$

→ the composition of these two mappings ($\exp[\tau D_T]\exp[\tau D_V] f$) is also a symplectic mapping but it does not give (1), as the BCH formula tells us:

$$\exp(\tau Z) = \exp(\tau X)\ \exp(\tau Y) \longrightarrow$$

$$\tau Z = \tau(X+Y) + \frac{\tau^2}{2}[X,Y] + \frac{\tau^3}{12}([X,[X,Y]] + [Y,[Y,X]]) + \frac{\tau^4}{24}[X,[Y,[Y,X]]] + \ldots$$

For two operators that (in general) do not commute and $[X,Y] = X\,Y - Y\,X$.

However, you can show* that for $X=D_T$ and $Y=D_V$, Z corresponds to the exact solution of a Hamiltonian

$$\boxed{\tilde{H} = T + V + \frac{\tau}{2}\{V, T\} + \frac{\tau^2}{12}\left(\{\{T, V\}, V\} + \{\{V, T\}, T\}\right) + \ldots}$$

that is $O(\tau)$ close to H, and whose value is conserved to machine precision.

– Higher-order integrators can be found by finding suitable coefficients so that a multiple composition of elementary mappings:

$$z(\tau) = \exp[\tau(D_T+D_V)]\, z(0) = \left(\prod_{i=1}^{k} \exp(c_i \tau\, D_T) \exp(d_i \tau\, D_V)\right) z(0) \;+\; O(\tau^{n+1})$$

'kills' the commutators up to $O(\tau^n)$, i.e. pushes towards $\tilde{H} = H + O(\tau^n)$

– If the Hamiltonian of a near-integrable problem can take the form
$H = H_0(q_j, p_j) + \varepsilon\, H_1(q_j)$ with H_0 integrable, then:

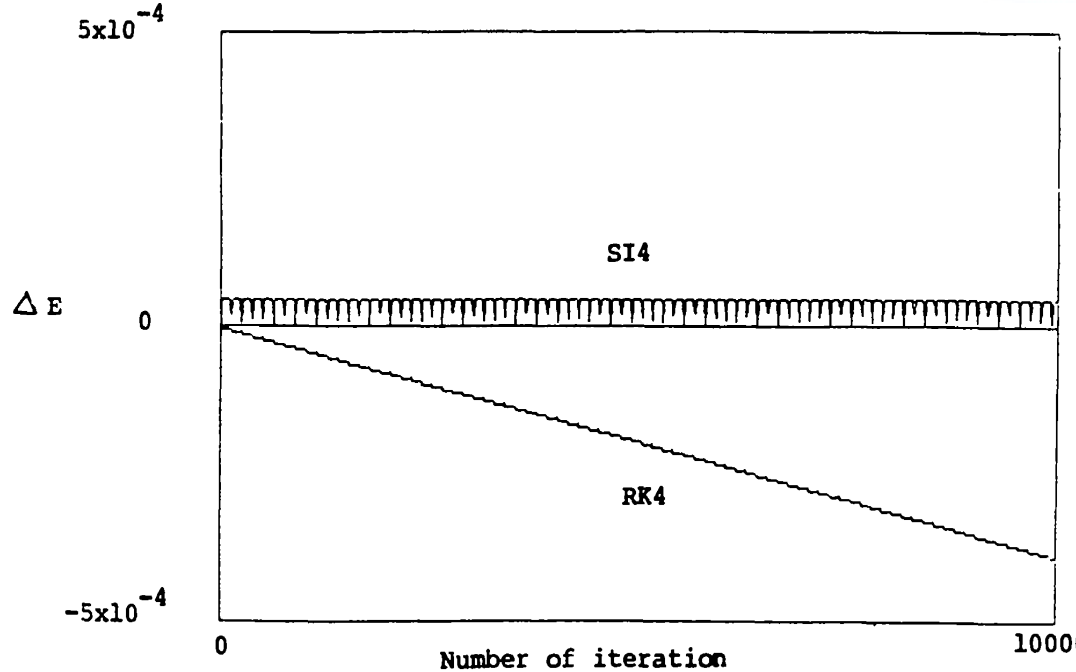
$$\boxed{\tilde{H} = H_0 + \varepsilon\, H_1 + \varepsilon\, \frac{\tau}{2}\{H_1, H_0\} + \varepsilon^2\, \frac{\tau^2}{12}\left(\{\{H_0, H_1\}, H_1\} + \{\{H_1, H_0\}, H_0\}\right) + \varepsilon^3\, \tau^3 \ldots}$$

i.e. the error is much smaller!

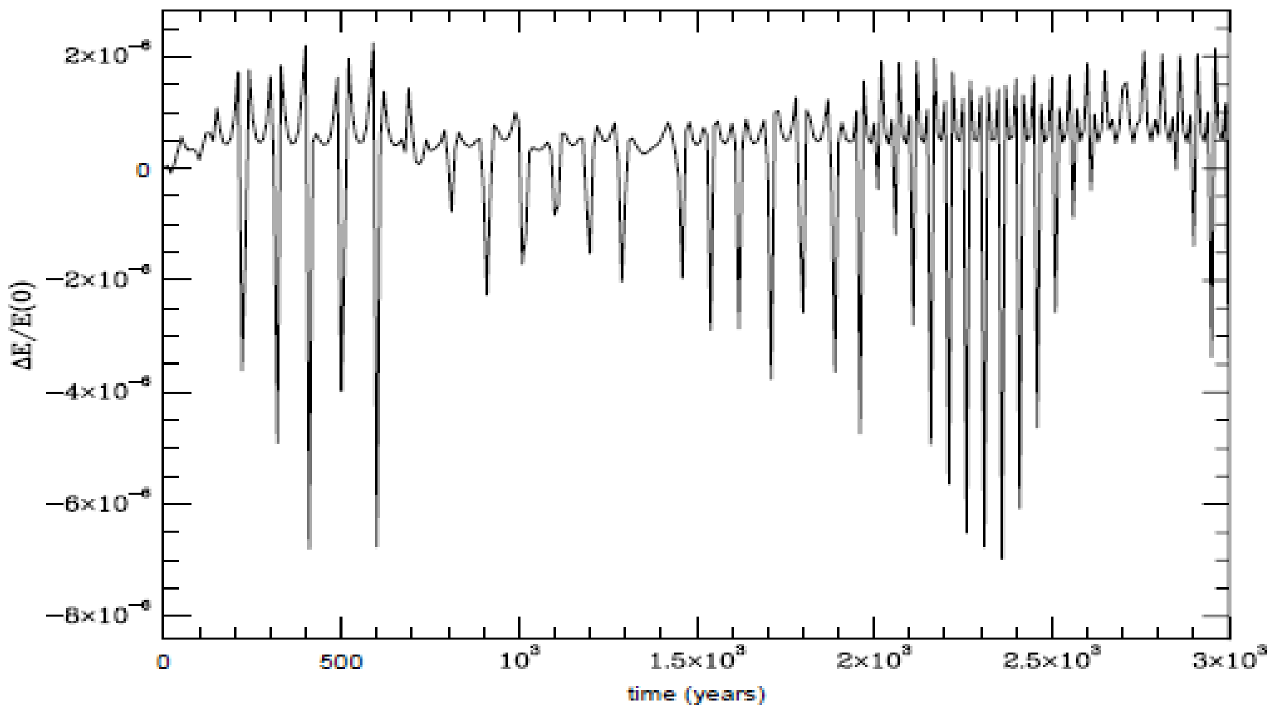
– This is the case with many codes built for solar system dynamics

** start from the BCH formula and apply T, V and [T,V] = (T V – V T) on q and p ...*

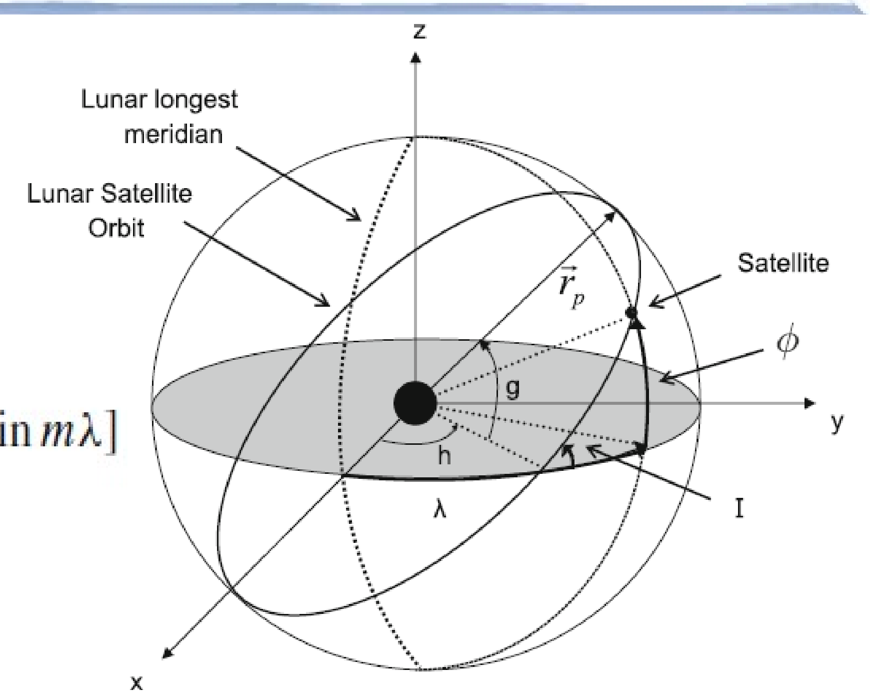
Long-term behavior of the error



The SyMBA integrator



R in the satellite problem



The potential of the central body is:

$$V = -\frac{\mu}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\phi)\,[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]$$

(please..) see e.g. Kaula (1966)

$$V_{lm} = \frac{\mu a_e^l}{a^{l+1}} \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{\infty} G_{lpq}(e) S_{lmpq}(\omega, M, \Omega, \theta) \quad \text{(here, } \theta = \text{GST)}$$

$$S_{lmpq} = \begin{bmatrix} C_{lm} \\ -S_{lm} \end{bmatrix}_{l-m\ \text{odd}}^{l-m\ \text{even}} \cos[(l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)] + \begin{bmatrix} S_{lm} \\ C_{lm} \end{bmatrix}_{l-m\ \text{odd}}^{l-m\ \text{even}} \sin[(l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)]$$

with ***F*** and ***G*** being the '*inclination function*' and '*eccentricity function*' resp. (notorious expansions!)

The 'J$_2$' problem (J$_2$=-C$_{20}$)

The Hamiltonian of the J_2 problem is: $\mathcal{H} = \left(\frac{u^2}{2} - \frac{\mu}{r}\right) + \frac{\epsilon\mu}{r^3} P_{20}(\sin\phi)$

with $\epsilon = J_2 R^2$ and $P_{20}(\sin\phi) = (3\sin^2\phi - 1)/2$ and is 'averaged' w.r.t the 'fast' angle, M=l

It can be re-written in the form: (rotating frame)

$$\boxed{\bar{\mathcal{H}} = -\frac{\mu^2}{2L^2} + \frac{\epsilon\mu^4}{4G^3L^3} - \frac{3\epsilon\mu^4 H^2}{4G^5L^3} - n_M H}$$

..and the equations of motion are:

$$\dot{l} = \partial\bar{\mathcal{H}}/\partial L$$

$$\dot{L} = -\partial\bar{\mathcal{H}}/\partial l = 0$$

$$\dot{g} = \partial\bar{\mathcal{H}}/\partial G = -\frac{3\epsilon\mu^4}{4G^4L^3}\left(1 - 5\frac{H^2}{G^2}\right)$$

$$\dot{G} = -\partial\bar{\mathcal{H}}/\partial g = 0$$

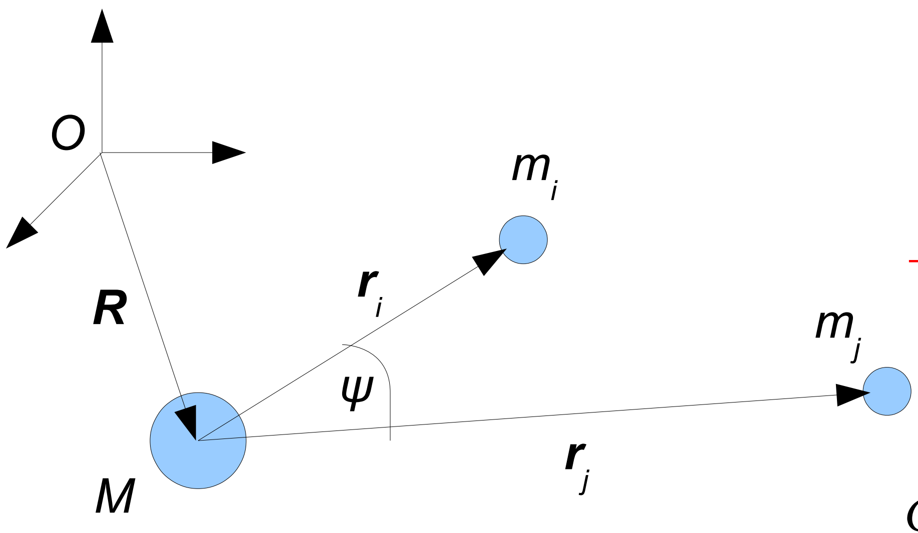
$$\dot{h} = \partial\bar{\mathcal{H}}/\partial H = -n_M - \frac{3\epsilon\mu^4 H}{2G^5L^3}$$

$$\dot{H} = -\partial\bar{\mathcal{H}}/\partial h = 0$$

→ [a , e , i] are constant !

and dg/dt = 0 at the *critical inclination* I_C = 63°,43 *(Molniya & Tundra orbits)*

$\mathcal{R}$ in the '3BP'



$$M\ddot{\boldsymbol{R}} = G M m_i \frac{\boldsymbol{r_i}}{r_i^3} + G M m_j \frac{\boldsymbol{r_j}}{r_j^3}$$

$$m_i \ddot{\boldsymbol{R}}_i = -G M m_i \frac{\boldsymbol{r_i}}{r_i^3} + G m_i m_j \frac{(\boldsymbol{r}_j - \boldsymbol{r}_i)}{r_{ij}^3}$$

$$m_j \ddot{\boldsymbol{R}}_j = -G M m_j \frac{\boldsymbol{r_j}}{r_j^3} + G m_i m_j \frac{(\boldsymbol{r}_i - \boldsymbol{r}_j)}{r_{ij}^3}$$

$$\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{R}}_i - \ddot{\boldsymbol{R}} \qquad \ddot{\boldsymbol{r}}_j = \ddot{\boldsymbol{R}}_j - \ddot{\boldsymbol{R}} \quad \Rightarrow \quad \ddot{r}_i = -\nabla(U_i + R_i)$$

$$U_i = -\frac{G(M+m_i)}{r_i} \quad \textit{and} \quad \boxed{R_i = -\left(\frac{G m_j}{|\boldsymbol{r}_j - \boldsymbol{r}_i|} - G m_j \frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_j^3}\right)}$$

So, in the restricted problem, the Hamiltonian takes the form:

$$\boxed{H = \underbrace{\frac{u^2}{2} - \frac{G M}{r}}_{H_{\text{kep}}} - G m_p \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_p|} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}_p}{r_p^3}\right)}$$

$$\qquad\qquad - \mu_{\text{p}}\, R\,(\boldsymbol{r}, \boldsymbol{r}_{\mathbf{p}})$$

in *heliocentric* coordinates and $R(\boldsymbol{r}, \boldsymbol{r}_{\mathbf{p}}) \rightarrow R(a, e, i, \Omega, \omega, \lambda, \lambda_{\text{p}}, \ldots)$

(see Murray & Dermott 2000, start from: $|\boldsymbol{r} - \boldsymbol{r}_p|^2 = r^2 + r_P^2 - 2 r r_p \cos\psi \Rightarrow \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_P|} = \ldots$ *)*

$\mathcal{R}$ *in the N-planets (+1 small guy) problem*

The Hamiltonian of the test-particle (asteroid), written in elements, reads:

$$\boxed{\begin{aligned} H = -\frac{G\,M_{Sun}}{2a} - \sum_j G\,m_j \sum_{k_j,l_j,n_j,p_j,q_j,r_j} c_{k_j,l_j,n_j,p_j,q_j,r_j}(a,a_j)\; F_j(e,\,e_j)\; G_j(s,s_j) \\ \times\; \cos\big(k_j \underbrace{\lambda + l_j\lambda_j + n_j\varpi + p_j\varpi_j + q_j\Omega + r_j}_{\varphi_{j\ldots}}\Omega_j\big) \end{aligned}}$$

* It's a Fourier series in the angles, $c_{j\ldots}$'s are conveniently expressed with the help of *Laplace coefficients*, while F_j and G_j are power series in e, e_j, s, s_j [$s = \sin(i/2)$]:

$$F_j = \left(e^{\beta}\, e_j^{\beta_j} + \ldots\right) \quad , \quad G_j = \left(s^{\delta}\, s_j^{\delta_j} + \ldots\right)$$

* Not all combinations of angles and not all values of β's and δ's are permissible. Symmetries and analytic properties of $R \rightarrow$ *d' Alembert rules*

The d'Alembert rules

$$H = -\frac{G M_{Sun}}{2a} - \sum_j G m_j \sum_{k_j, l_j, n_j, p_j, q_j, r_j} c_{k_j, l_j, n_j, p_j, q_j, r_j}(a, a_j)\ F_j(e, e_j)\ G_j(s, s_j)$$
$$\times\ \cos(k_j \lambda + l_j \lambda_j + n_j \varpi + p_j \varpi_j + q_j \Omega + r_j \Omega_j)$$

– only cosine terms, real coefficients (inv. under simultaneous change of sign in angles)

– Sum of all integer coefficients in cos = 0 (inv. under rotation around z-axis)

– $\delta+\delta_j$ must be even (inv. under simultaneous change of sign in all inclinations)

– $2\beta-|n_j|$, $2\beta_j-|p_j|$, $2\delta-|q|$ and $2\delta_j-|r_j|$ must be positive and even (for the elimination of apparent singularity at $e,i \to 0$ to be possible by introducing suitable Cartesian coordinates)

* *these are* $\quad x = \sqrt{2\Gamma}\ \sin\gamma \sim e\cos\varpi$, $\quad y = \sqrt{2\Gamma}\ \cos\gamma \sim e\sin\varpi$ $\quad$ *and* $\quad u = \sqrt{2Z}\ \sin\zeta$, $\quad v = \sqrt{2Z}\ \cos\zeta$

Example: the 3:1 MMR

We want to have terms corresponding to $3n' - n = 3\dot{\lambda}' - \dot{\lambda} \approx 0$
→ k =-1, l = 3 and the order of the MMR is l+k = 2

→ sum of the rest of integers should be = 2 . Then, the permissible arguments are:

$$3\lambda' - \lambda - 2\varpi \;\;,\;\; 3\lambda' - \lambda - 2\varpi' \;\;,\;\; 3\lambda' - \lambda - 2\Omega \;\;,\;\; 3\lambda' - \lambda - 2\Omega'$$
$$3\lambda' - \lambda - \varpi - \varpi' \;\;,\;\; 3\lambda' - \lambda - \Omega - \Omega'$$

and their e,i dependence, to lowest degree, is given respectively by:

$$(e^2 + \ldots) \;\;,\;\; (e'^2 + \ldots) \;\;,\;\; (s^2 + \ldots) \;\;,\;\; (s'^2 + \ldots)$$
$$(e\; e' + \ldots) \;\;,\;\; (s\; s' + \ldots)$$

The following arguments *cannot* appear in R :

$$3\lambda' - \lambda - \varpi - \Omega \;\;,\;\; 3\lambda' - \lambda - \varpi' - \Omega', 3\lambda' - \lambda - \varpi' - \Omega \;\;,\;\; 3\lambda' - \lambda - \varpi - \Omega'$$

Since they violate the 3rd rule (even combinations of Ω's)

End of Part II

Celestial Mechanics (cont.)

- *Asteroid long-term dynamics*
  - Canonical derivation of an average Hamiltonian
  - Secular theory and proper elements
  - The MMR problem
  - Resonance overlapping, chaos and diffusion

- *Satellite long-term dynamics*
  - Beyond the J_2 problem
  - Secular motion
  - The 1:1 resonance

Derivation of an average Hamiltonian

We start with a near-integrable Hamiltonian (e.g. perturbed 2bp)
$H(\mathbf{q},\mathbf{p}) = H_0(\mathbf{p}) + \epsilon H_1(\mathbf{q},\mathbf{p})$ and we seek a new set of (q',p'): $H'(q',p')=H(q,p)$

The Lie series of H gives: $H' = H_0 + \epsilon H_1 + \epsilon\{H_0,\chi\} + \epsilon^2\{H_1,\chi\} + \frac{\epsilon^2}{2}\{\{H_0,\chi\},\chi\} + \mathcal{O}(\epsilon^3)$

Let's say we ask for a generating function, χ, such that, to $O(\varepsilon)$ the Hamiltonian is *averaged* over the angle q_i. Then,

$$\bar{H}_1 = \frac{1}{2\pi}\int_0^{2\pi} H_1 dq_i$$ and this will hold iff: $$\boxed{H_1 + \{H_0, \chi\} = \bar{H}_1}$$

If we Fourier-expand H_1 and ask for a similar solution for χ, then:

$$\left.\begin{aligned} H_1(\boldsymbol{q}',\boldsymbol{p}') &= \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}(\boldsymbol{p}')\exp(\imath\,\boldsymbol{k}\cdot\boldsymbol{q}') \\ \chi(\boldsymbol{q}',\boldsymbol{p}') &= \sum_{\boldsymbol{k}} d_{\boldsymbol{k}}(\boldsymbol{p}')\exp(\imath\,\boldsymbol{k}\cdot\boldsymbol{q}') \end{aligned}\right\} \quad \{H_0,\chi\} = -\imath\sum_{\boldsymbol{k}} d_{\boldsymbol{k}}(\boldsymbol{p}')\,\boldsymbol{k}\cdot\boldsymbol{\omega_0}\exp(\imath\,\boldsymbol{k}\cdot\boldsymbol{q}')$$

with $\boldsymbol{\omega_0} = \frac{\partial H_0}{\partial \boldsymbol{p}'}$

and the coefficients are given by:

$$d_0 = 0 \quad , \quad d_{\boldsymbol{k}}(\boldsymbol{p}') = -\imath\frac{c_{\boldsymbol{k}}(\boldsymbol{p}')}{\boldsymbol{k}\cdot\boldsymbol{\omega_0}(\boldsymbol{p}')}$$

If we want to average over all angles this holds for every $\boldsymbol{k}$ and →

$$\boxed{\bar{H}_1 = c_0(\boldsymbol{p}') = \bar{H}_1(\boldsymbol{p}')}$$

Secular Theory (linear)

(3BP) - we ask for χ such that – to *O(ε)* – *H'* is independent (averaged) of both λ and λ'. From the remaining part (*secular*) we keep only the lowest-degree terms:

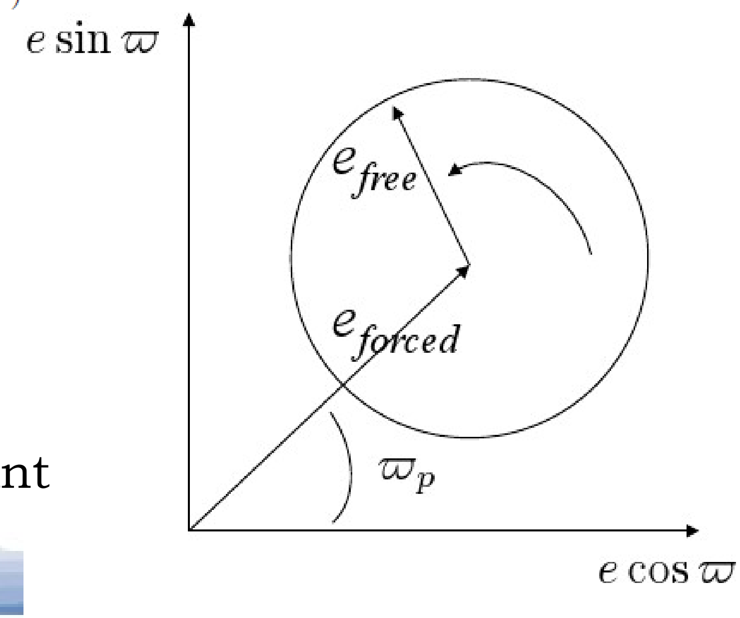
$$\boxed{\mathcal{H}_{\rm sec} = -\frac{\mu_1^2}{2\Lambda^2} + n'\Lambda' - \mu\left[A_1(\Lambda)\,\Gamma + A_2(\Lambda)\,e'\sqrt{\Gamma}\cos(\gamma) + A_3(\Lambda)Z\right]}$$

* γ =*relative pericenter longitude* and Z=conjugate to *mutual node*

Clearly, Λ=const and *Z*=const → constant *a* and $H_{\rm sec}$ reduces to:

$$\mathcal{H}_{\rm sec} = -\mu A_1\,\Gamma - \mu A_2\,e'\sqrt{\Gamma}\cos(\gamma) \xrightarrow{(x=\sqrt{\Gamma}\cos\gamma\ ,\ y=\sqrt{\Gamma}\sin\gamma)} \mathcal{H}_{\rm sec} = \frac{c_1}{2}(x^2+y^2) + c_2\,e'x$$

A fixed point exists: $(\dot{x},\dot{y}) = (0,0) \rightarrow (x_0,y_0) = (-e'c_2/c_1, 0)$



Performing a translation $(X = x - x_0\ ,\ Y = y - y_0)$

$$\mathcal{H}_{\rm sec} = \frac{c_1}{2}(X^2+Y^2) + {\rm const}$$

Switching back to polar coordinates; $\begin{array}{l} X = \sqrt{2\,\Phi}\cos\varphi \\ Y = \sqrt{2\,\Phi}\sin\varphi \end{array}$

$$\Rightarrow \boxed{\mathcal{H}_{\rm sec} = c_1\Phi} \ldots + c_2\,\Psi$$

which describes a harmonic oscillation with constant frequency **g = c_1**.

Secular Theory (linear / N-body)

(N-BP) – We first need to solve the problem for the planets! Same initial steps, more sums in the perturbation. The solution is:

$$\begin{aligned} a_i &= \quad \text{const} \\ e_i \exp(\iota\varpi_i) &= \textstyle\sum_j e_{i,j} \exp \iota(g_j t + \beta_{i,j}) \\ i_i \exp(\iota\Omega_i) &= \textstyle\sum_j i_{i,j} \exp \iota(s_j t + \delta_{i,j}) \end{aligned}$$

(g_j , s_j) → *fundamental frequencies* of the planetary system.

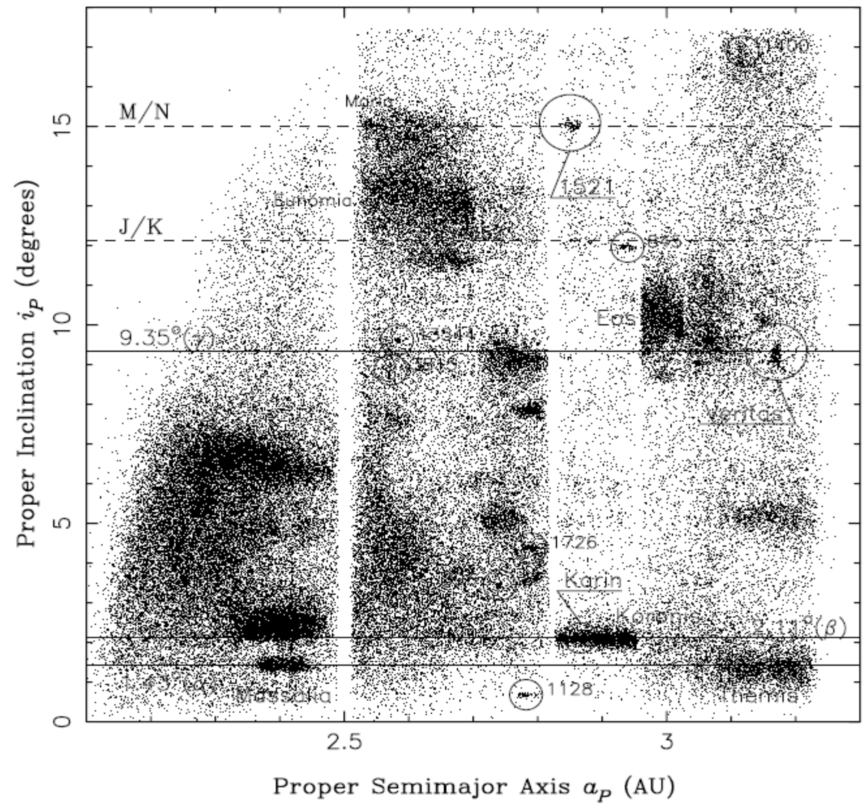
Now, the solution for a test-particle gives:

$$\boxed{\begin{aligned} a &= \quad \text{const} \\ e \exp(\iota\varpi) &= e_p \exp(i\varpi_p) + \textstyle\sum_j M_j \exp \iota(g_j t + \beta_j) \\ i \exp(\iota\Omega) &= i_p \exp(i\Omega_p) + \textstyle\sum_j N_j \exp \iota(s_j t + \delta_j) \end{aligned}}$$

i.e. the sum of *forced oscillations*, plus a *proper mode* → *linear proper elements*

They can be used to identify *asteroid families*, but they are not of the desired accuracy

→ we need a better approximation!



*Note that M_J, N_J contain *small divisors*...

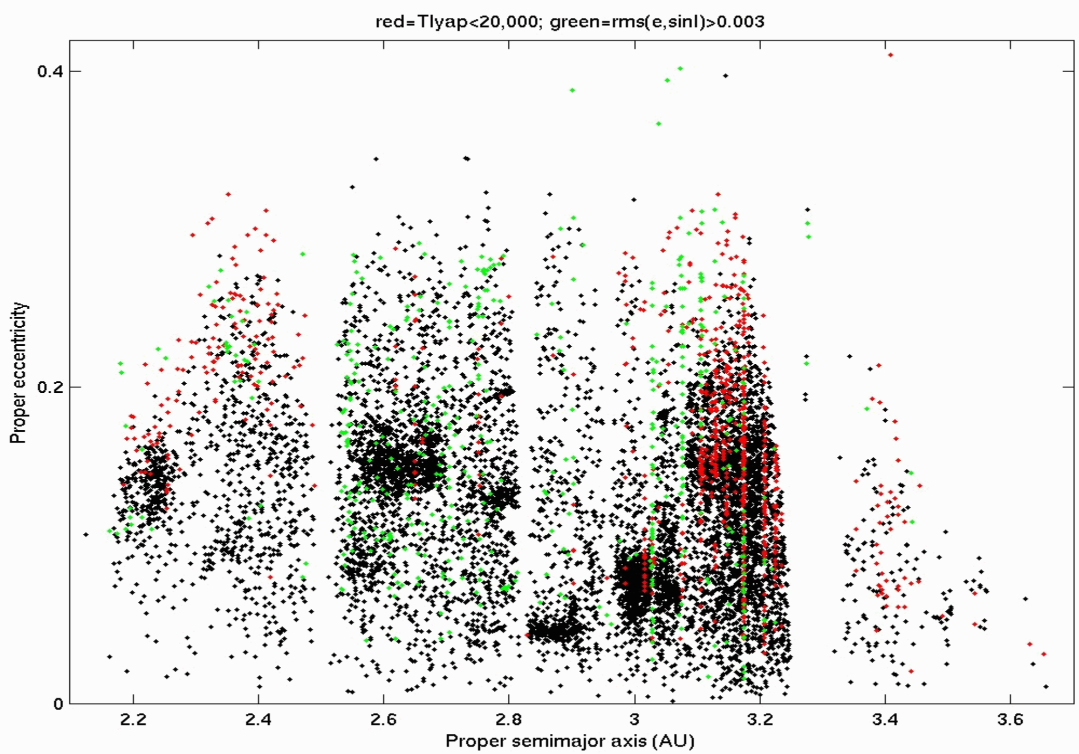
High-degree secular theories

First steps as before → derive a secular Hamiltonian *but keep higher-degree* terms (e.g. 4$^{\text{th}}$) in the expansion. Then, define *a new canonical transformation*

$$H' = H_0 + \epsilon H_1 + \epsilon\{H_0, \chi\} + \epsilon^2\{H_1, \chi\} + \frac{\epsilon^2}{2}\{\{H_0, \chi\}, \chi\} + \mathcal{O}(\epsilon^3)$$

to *eliminate all angles* and get an H_{sec} that depends *only* on the new momenta ⇨ the *new (more accurate) proper elements*

* Don't forget to check if some term eliminated to $O(\varepsilon)$ gives an important effect at $O(\varepsilon^2)$ → true for the 2:1 MMR (it's $O(m^2 e)$ strong)



- Can be done massively
- Also numerically (*synthetic pr.el.*)
- good for identifying *asteroid families*
- degraded accuracy near MMRs and SRs
- Other expansions needed e.g. for *high inclinations*

** Resonant proper elements can be defined (e.g. Trojans)*

Secular Satellite Theory

Remember: $$V=-\frac{\mu}{r}\sum_{n=0}^{\infty}\left(\frac{R_M}{r}\right)^n\sum_{m=0}^{n}P_{nm}(\sin\phi)\left[C_{nm}\cos m\lambda+S_{nm}\sin m\lambda\right]$$

$$\Rightarrow\quad \mathcal{H}=\mathcal{H}_0+\sum_{i=2}^{n}\mathcal{H}_{J_n}+\sum_{i=2}^{n}\sum_{m=1}^{n}\mathcal{H}_{C_{nm}}+\sum_{i=2}^{n}\sum_{m=1}^{n}\mathcal{H}_{S_{nm}}+\mathcal{H}_{n_M}$$

We *average** over the orbital period and look for the secular evolution of the orbit itself

$$\Rightarrow\quad \dot g=\partial\bar{\mathcal{H}}/\partial G,\qquad \dot G=-\partial\bar{\mathcal{H}}/\partial g$$
$$\dot h=\partial\bar{\mathcal{H}}/\partial H,\qquad \dot H=-\partial\bar{\mathcal{H}}/\partial h$$

If we add only J_3, $\overline{\mathcal{H}}_{J_3}=2J_3K\left(-4+5s^2\right)s\sin g$ and the equations give

$$\boxed{\begin{aligned}\dot g&=-\frac{3n_M J_2(1-5c^2)}{4(1-e^2)^2}\left(\frac{R}{a}\right)^2 N(e,I,g)\\ \dot e&=\frac{3J_3}{8}\frac{n_M}{(1-e^2)^2}\left(\frac{R}{a}\right)^3 s\left(1-5c^2\right)\cos g\end{aligned}}$$

and i = const, where

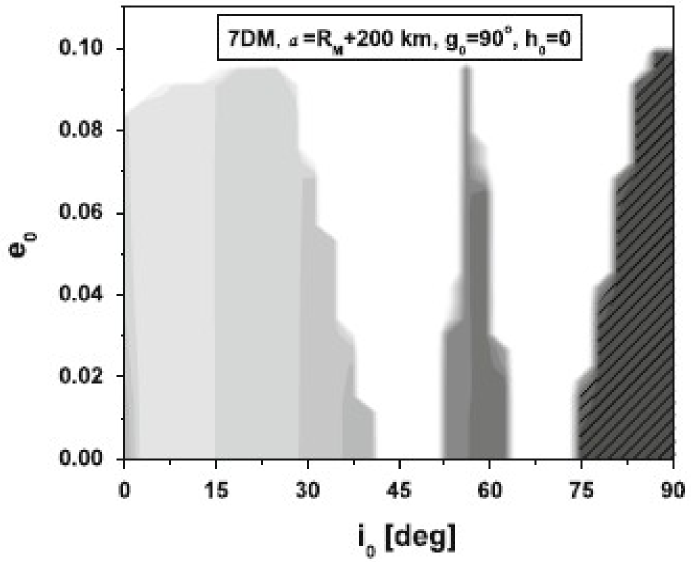
$$N(e,I,g)=1+\frac{J_3}{2J_2}\left(\frac{R}{a}\right)\left(\frac{1}{1-e^2}\right)\left(\frac{s^2-e^2c^2}{s}\right)\frac{\sin g}{e}$$

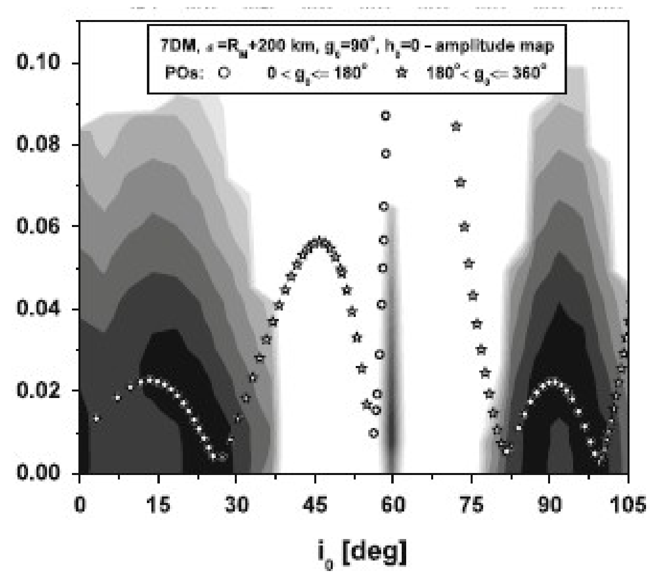
There are fixed points for $|g|=\pi/2$ and $e=e_{fr}$ at every inclination

$\Rightarrow$ *frozen orbits*

**higher-degree approximations can be obtained by averaging (over h) the Hamiltonian of the O(n) problem*

* In more complex gravity models, we can use *frequency analysis* on numerically integrated orbits to obtain a global view of the dynamics (maps of f_k, A_k)

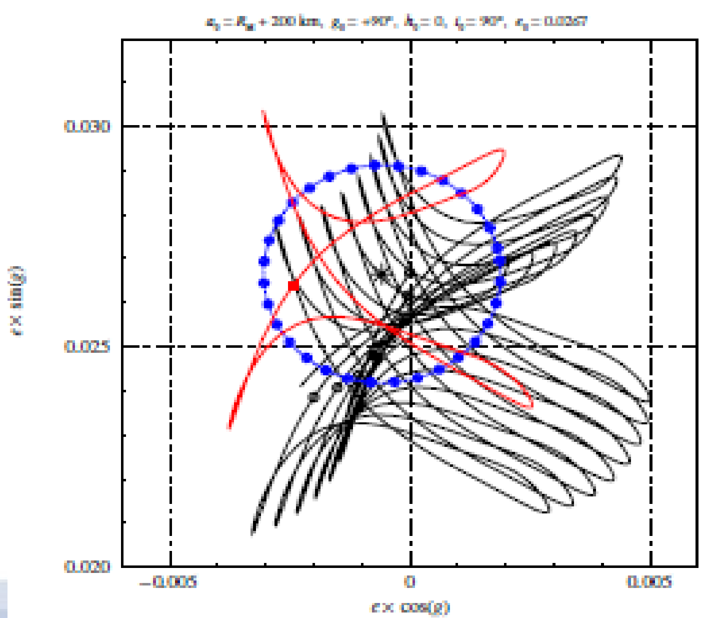




we can find e.g. a *minimal model* for our system (Moon)

$$\boxed{\mathcal{H}_{MM} = \mathcal{H}_0 + \sum_{i=2}^{3} \mathcal{H}_{J_n} + \mathcal{H}_{C_{2,2}} + \mathcal{H}_{C_{3,1}} + \mathcal{H}_{J_7} + \mathcal{H}_{n_M}}$$

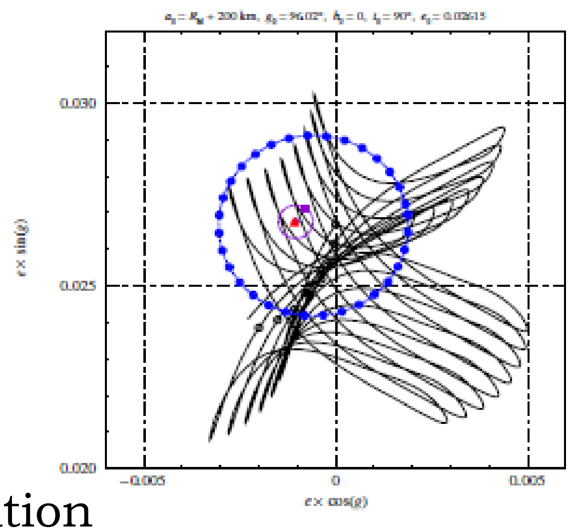
For orbits starting near the *FO*:

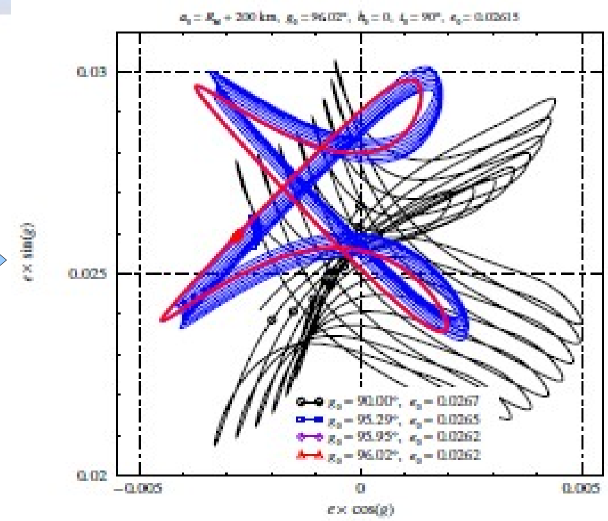


The long-periodic libration can lead to *collision* (for low *a*'s).

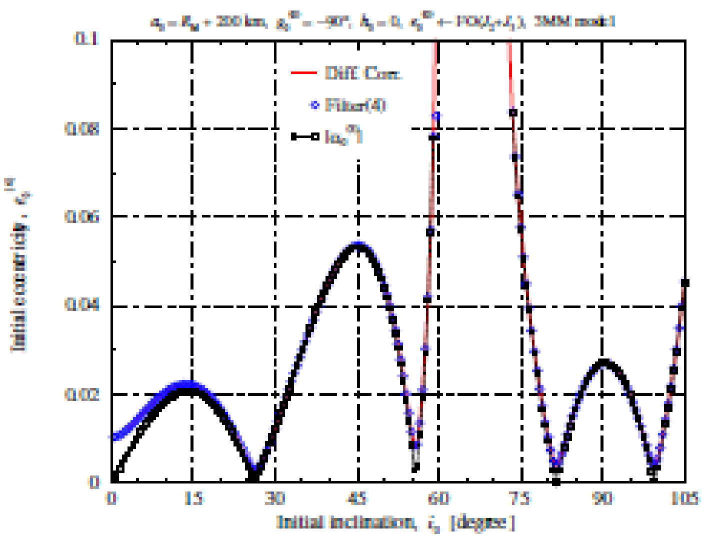
We can *filter out* this term from our decomposition and see if we get closer to the 'true' center of the motion

Successive iterations give *improved* i.c.'s:

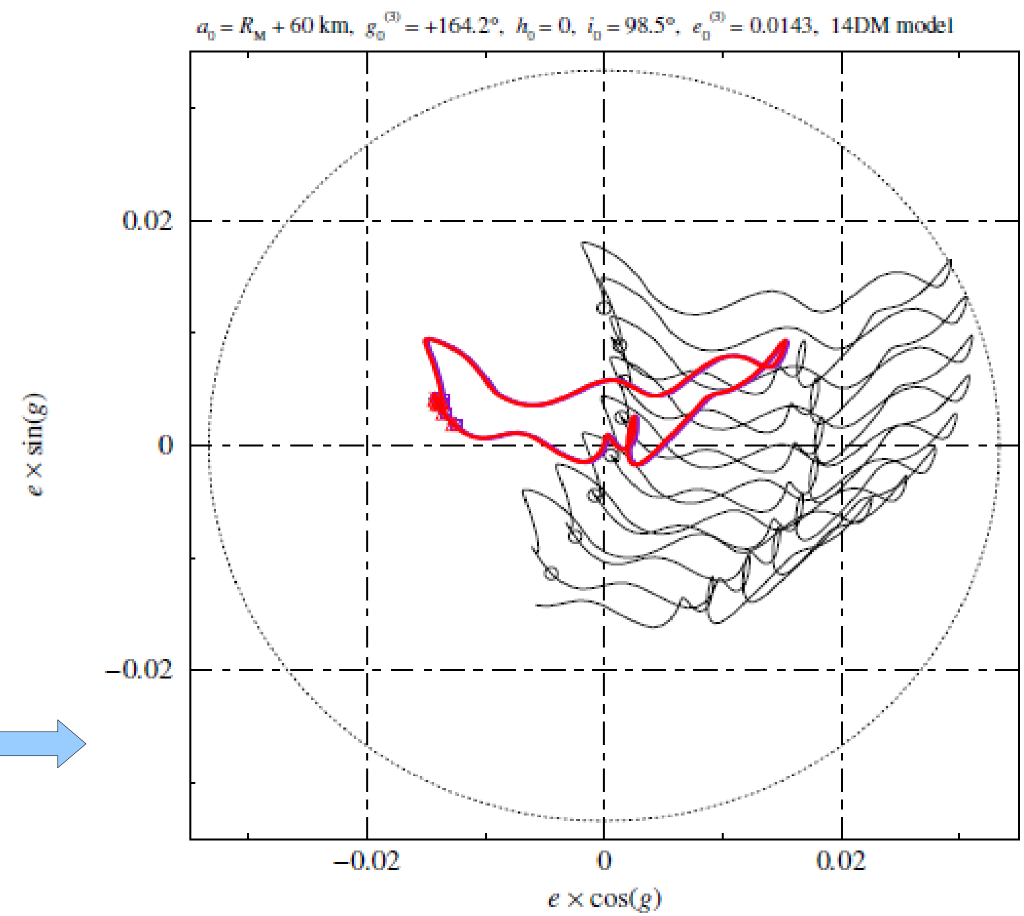




A very accurate approximation of *P.Os* at all inclinations can be found



... and an initial condition leading to collision can be *corrected*, to save the 'satellite'...



The MMR problem (2-bp and 3-bp)

Now, let's ask for $\chi(q',p')$ such that the new Hamiltonian has the following structure:

- we retain the lowest-degree *secular terms* as before (but in 2-D)
- we average all *short-period* terms (i.e. both λ,λ') , *except* a certain *resonant module* (k,q):

$$\boxed{\begin{aligned}\mathcal{H} = & - \tfrac{\mu_1^2}{2\Lambda^2} + n'\Lambda' - \mu\left[A_1(\Lambda)\,\Gamma + A_2(\Lambda)\,e'\sqrt{\Gamma}\cos(\gamma)\right] \\ & -\mu\textstyle\sum_{p=0}^{q} A_p(\Lambda, e')\,\Gamma^{p/2}\cos[k\lambda-(k+q)\lambda'-p\gamma]\end{aligned}}$$

There are *q+1* terms, satisfying the d'Alembert rules. This is frequently called the *resonant multiplet* of the *k:(k+q)* MMR. If we apply the same series of transformations as in $H_{\rm sec}$, we get

$$\mathcal{H} = -\frac{\mu_1^2}{2\Lambda^2} + n'\Lambda' - c_1\Phi - \mu\sum_{p=0}^{q} d_p\Phi^{p/2}\cos[k\lambda-(k+q)\lambda'-p\phi]$$

I can define the resonant angle $\psi = k\lambda - (k+q)\lambda'$ and its conjugate momentum $\Psi = \Lambda/k$ and expand the Keplerian part about $\Psi_{\rm res} = \sqrt{\mu_1 a_{\rm res}}/k$ to $O(2)$ in $J_\psi = \Psi - \Psi_{\rm res}$

The 2-D MMR Hamiltonian

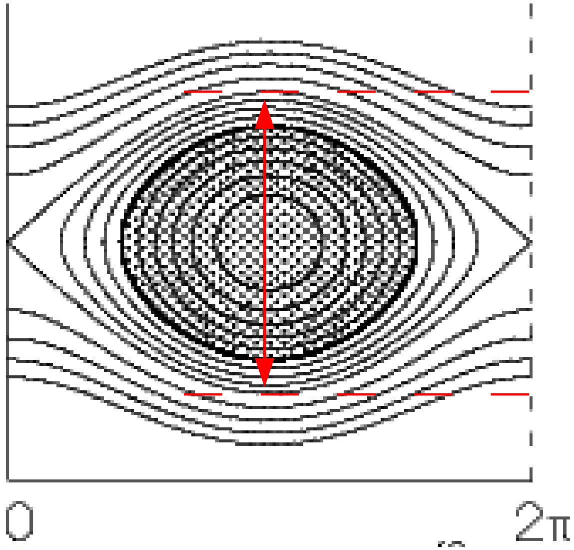
... $$\boxed{\mathcal{H} = \frac{1}{2}\beta J_\psi^2 - c_1\Phi - \mu\sum_{p=0}^{q} D_p\cos(\psi - p\phi)}$$ where $D_p = d_p\,\Phi^{p/2}$

** Bare with me on the following very simplistic approximations...*

If I could view each sub-resonance separately, and expand around a constant Φ value* $\mu D_{p,*} = d_p\,\Phi_*^{p/2}$ the *exact resonance* would be defined by:

$\dot{\psi} - p\dot{\phi} = 0 \Rightarrow J_{\psi,\mathrm{res}} \approx p\,c_1/\beta$ i.e. sub-resonances are $\delta J_\psi = c_1/\beta$ apart

and the width of the resonance is given by $\Delta J_\psi \sim \sqrt{\mu\,D_{p,*}/\beta}$



ΔJ

Chirikov's criterion suggests that for

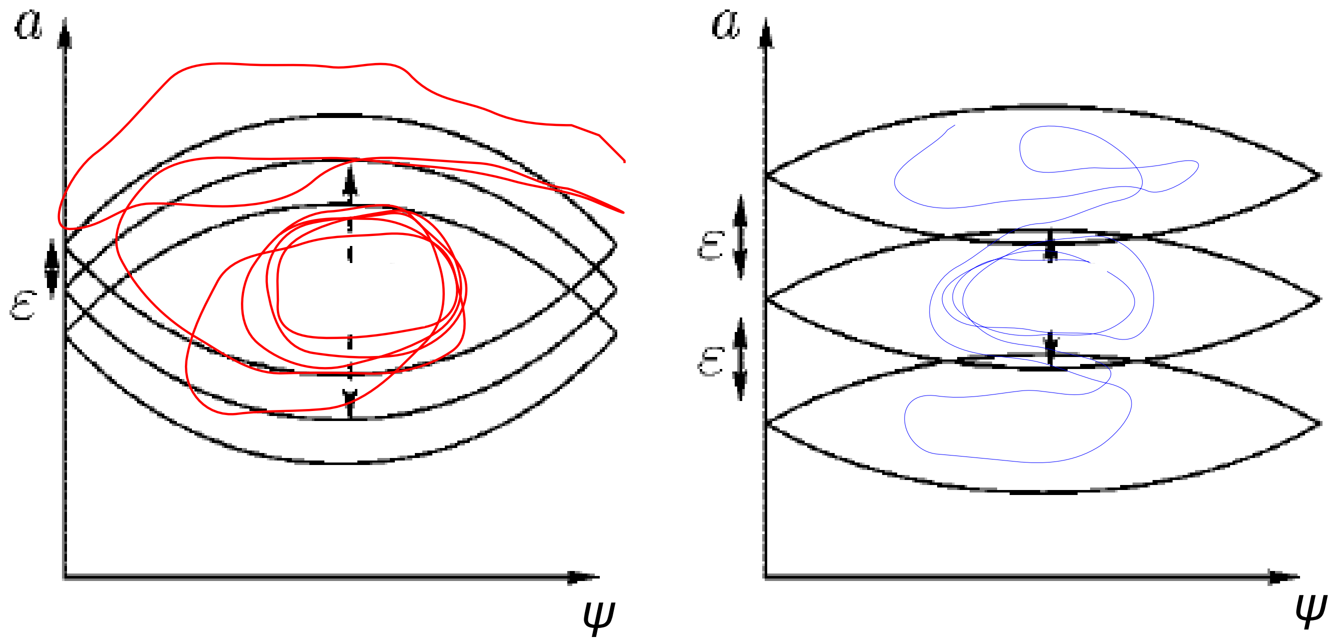
$$K \approx \left|\frac{\Delta J_{\psi,P} + \Delta J_{\psi,P+1}}{2\,\delta J_{\psi,P}}\right| > 1$$

we should expect *chaotic motion*

** the 1-res approx is a pendulum modulated by a harmonic oscillator...*

Chaotic diffusion

Depending on the size of each resonance, you have:



$D_p \sim 1 \;\rightarrow\; \Delta J \sim O(\mu^{1/2})$ $\qquad$ $D_p \sim \mu \;\rightarrow\; \Delta J \sim O(\mu)$

The 1st case can be approximated by a *slowly-modulated pendulum*

$$\mathcal{H} = \frac{1}{2}\beta J_\psi^2 - \mu \tilde{B}\cos(\psi - \tilde{Q}) \qquad \tilde{B} = \tilde{B}(\psi^0, \Psi^0)\ ,\ \ \tilde{Q} = \tilde{Q}(\psi^0, \Psi^0)$$

in which we *'freeze'* the *'slow'* d.o.f and for each set of frozen values (ψ^0, Ψ^0) we compute the solutions of the frozen pendulum

→ can give an approximation of the borders of the chaotic domain

* the $\mu^{2/7}$ - law

1st-order MMRs are located at $\alpha_{res} = \alpha/\alpha' = [j/(j+1)]^{2/3}$

and their distance in Λ is given by $\delta\Lambda = \Lambda_j - \Lambda_{j+1} = [j/(j+1)]^{1/3} - [(j+1)/(j+2)]^{1/3}$

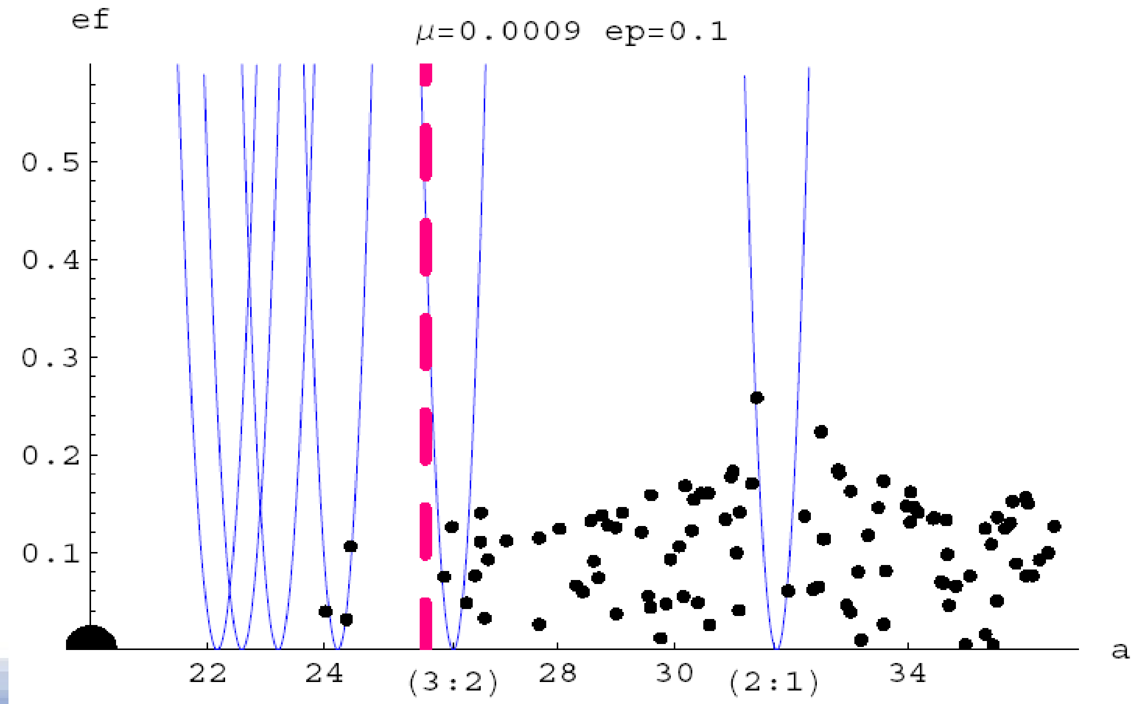
and each is described by a Hamiltonian of the form

$$H_c = \frac{1}{2}\beta J^2 - c_1 \Phi - \mu f_1 \sqrt{\Phi} \cos(\psi - \theta)$$

Wisdom applied Chirikov's criterion to find that a region of size

$$\Delta a \approx 1.5 a' \mu^{2/7} \approx 1 \text{ AU}$$

around the orbit of Jupiter should be empty (and it is...)



Three-body resonances (3b-MMRs)

Defined by: $\boxed{k_1\dot{\lambda}_1 + k_2\dot{\lambda}_2 + k_3\dot{\lambda}_3 \sim 0}$ (3 bodies involved, e.g. A-J-S)

The Hamiltonian is *very similar to the one found for MMRs* – they essentially differ in the formal size of the coefficients

- a bit more difficult to derive though...

Start from the asteroid's Hamiltonian, but with *two* perturbing planets

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon\mathcal{H}_1 = -\frac{1}{2\Lambda^2} + \sum_{j=1}^{N} n_j\Lambda_j + \varepsilon\mathcal{H}_1 \quad \text{i.e.} \quad \varepsilon\mathcal{H}_1 = \sum_{j=1}^{N} \varepsilon_j \mathcal{H}_1^{(j)}$$

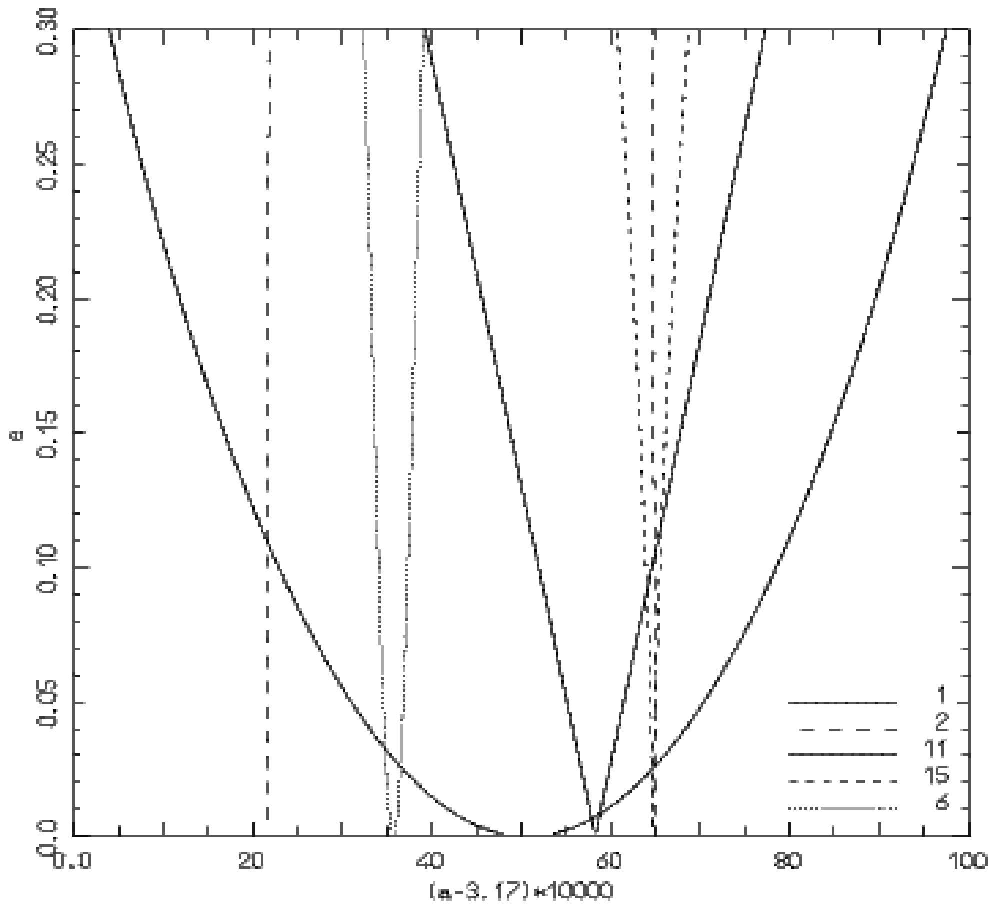
Perform the averaging over all λ's to $O(\varepsilon)$

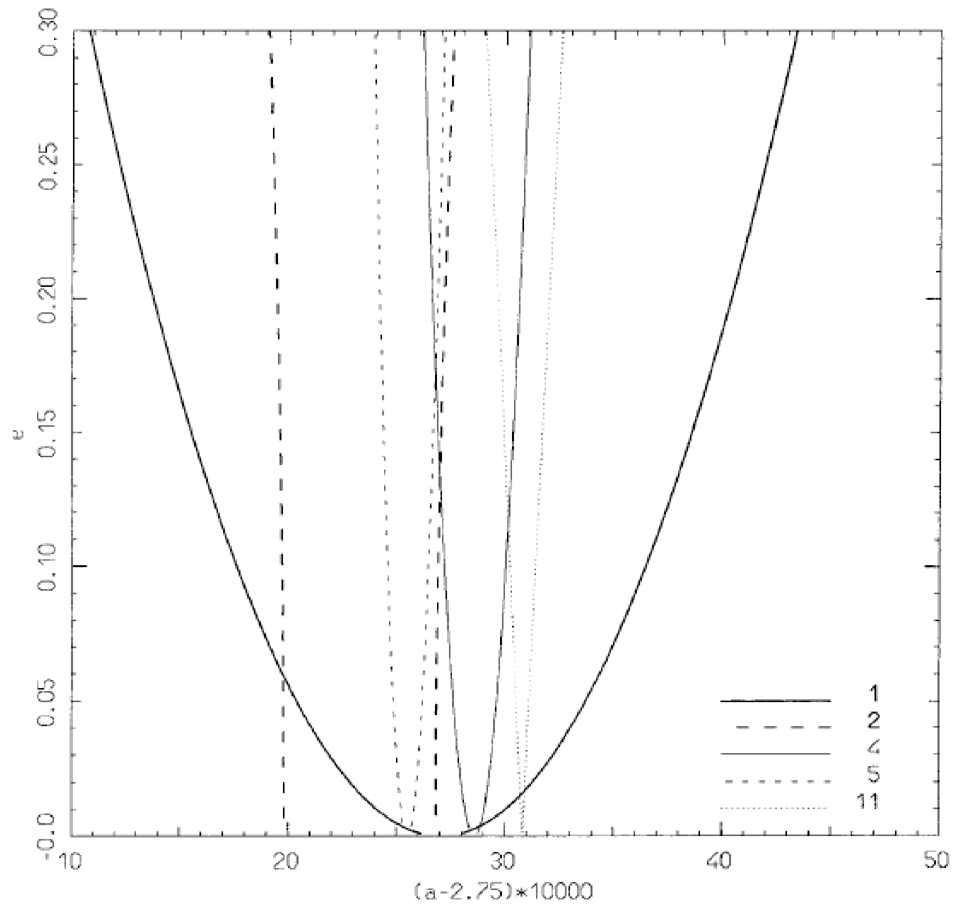
$$\{\mathcal{H}_0, \chi\} + \mathcal{H}_1 = \overline{\mathcal{H}}_1 \equiv \frac{1}{(2\pi)^2} \sum_{j=1}^{N} \int_0^{2\pi} \int_0^{2\pi} \mathcal{H}_1^{(j)} \mathrm{d}\lambda \mathrm{d}\lambda_j \quad \text{where} \quad \varepsilon\chi = \sum_{j=1}^{N} \varepsilon_j \chi^{(j)}$$

..but, now, *compute the $O(\varepsilon^2)$* terms:

$$\mathcal{H}^1 = \mathcal{H}_0 + \varepsilon\overline{\mathcal{H}}_1 + \frac{\varepsilon^2}{2}\Big(\{\mathcal{H}_1, \chi\} + \{\overline{\mathcal{H}}_1, \chi\}\Big) + \sum_{j=1}^{N} \sum_{k \neq j} \varepsilon_j \varepsilon_k \mathcal{H}_2^{(j,k)}$$

An important (and strange..) result



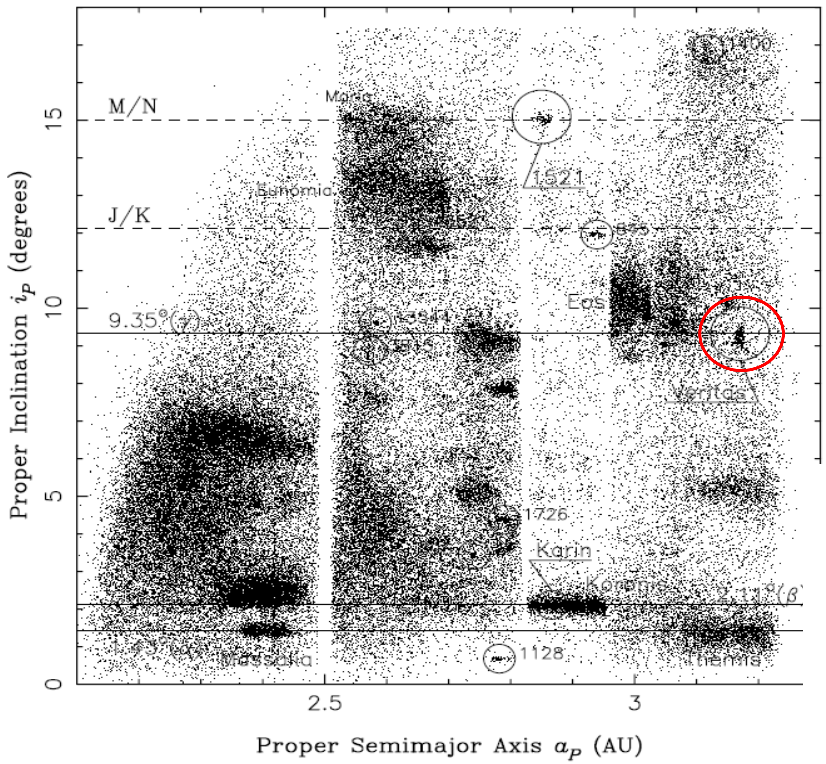


Low-order 3b-MMRs and high-order MMR have similar 'strength'

$D_p \sim \varepsilon \rightarrow \Delta J^2 / \tau_{sec} \sim O(\varepsilon^3)$

they all have a diffusion time-scale of ~ 1 Gy (and this is true!!!)

Veritas (… in vino)

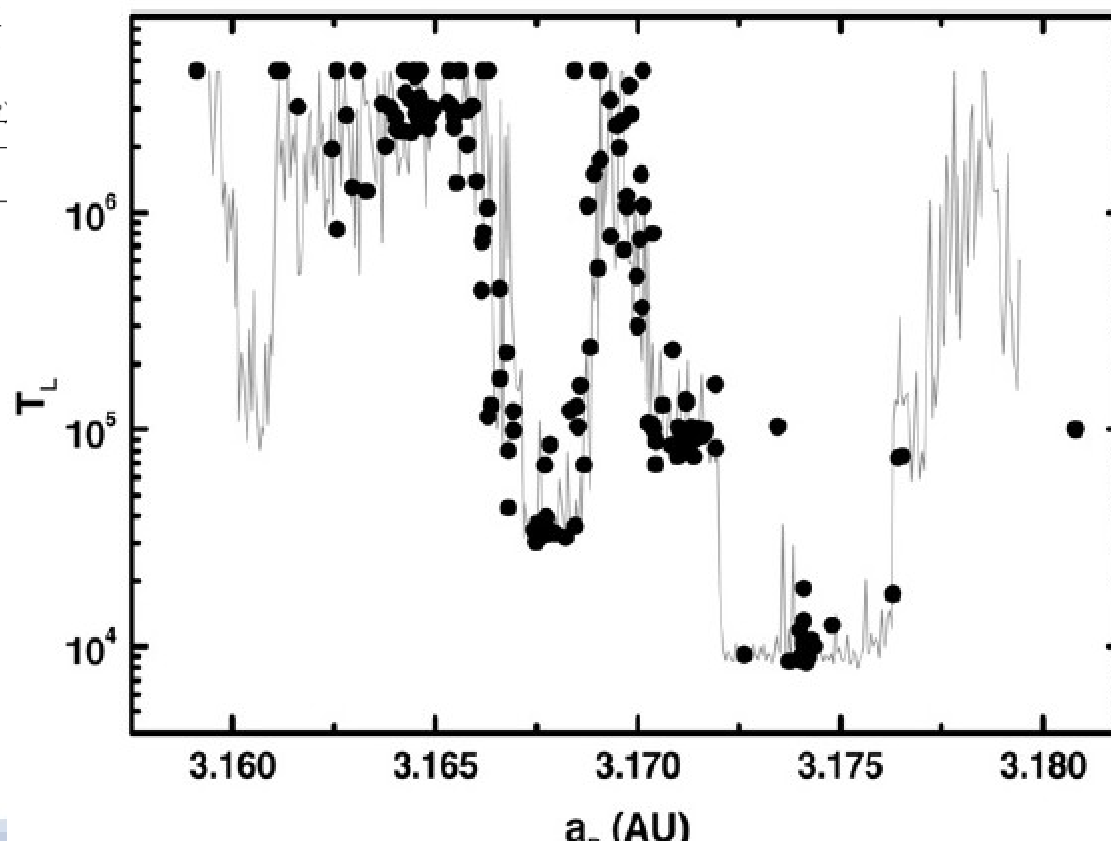


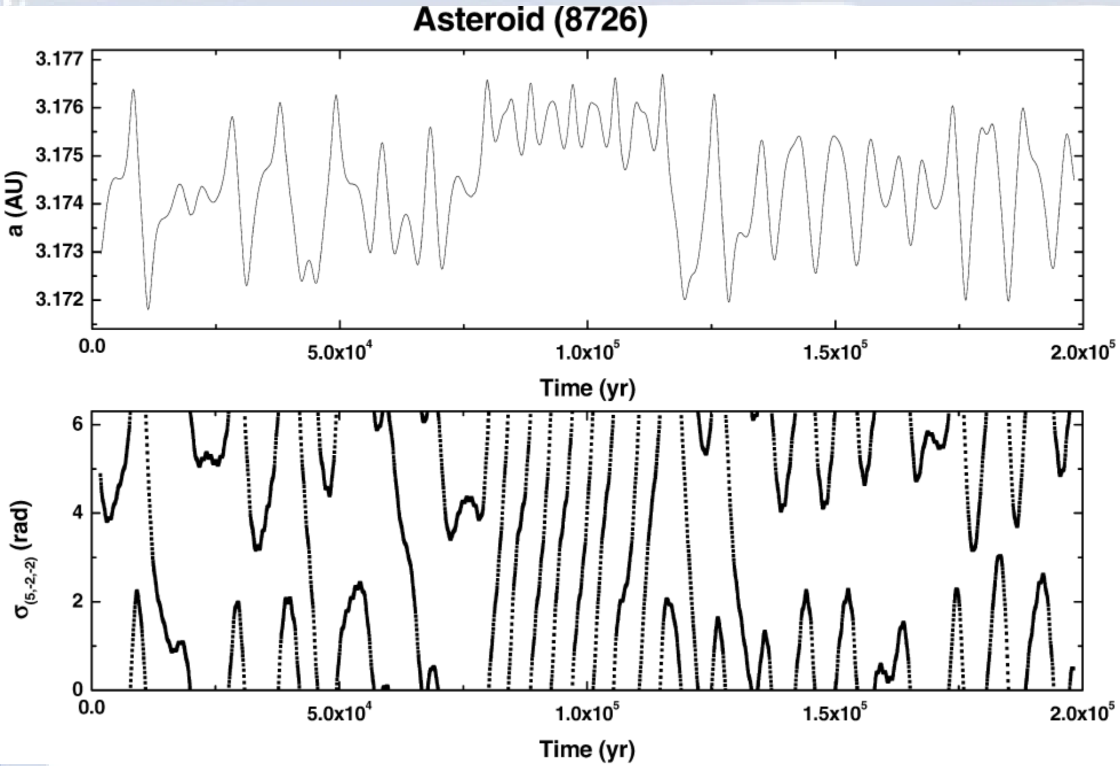
The asteroid family of *(490) Veritas* (a~3.17 AU) is cut through by several MMRs, most notably:

- the 3b-MMR (5, -2, -2) at 3.173 AU
- the 3b-MMR (3, 3, -2) at 3.168 AU

Lyapunov times are comparable

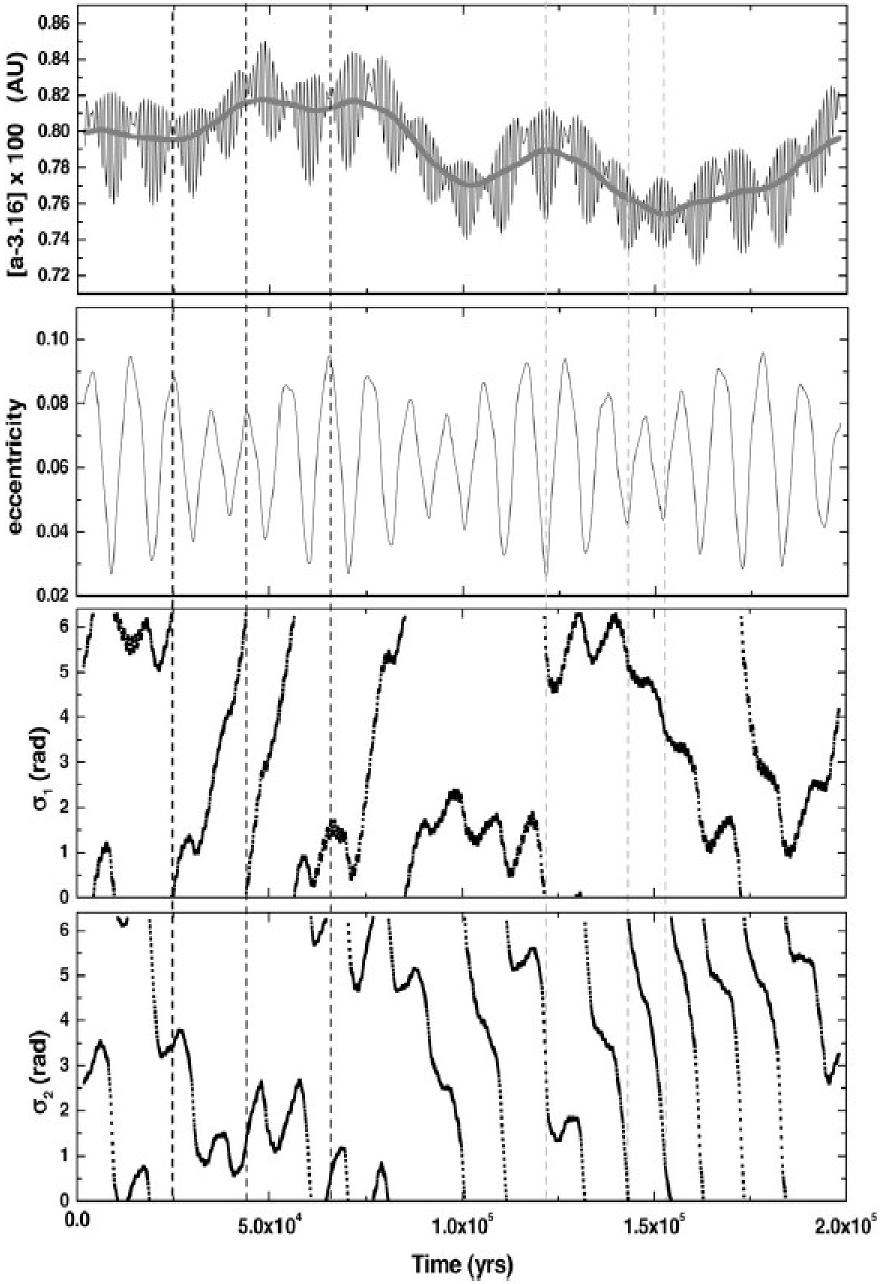
… but their long-term diffusion properties are not…

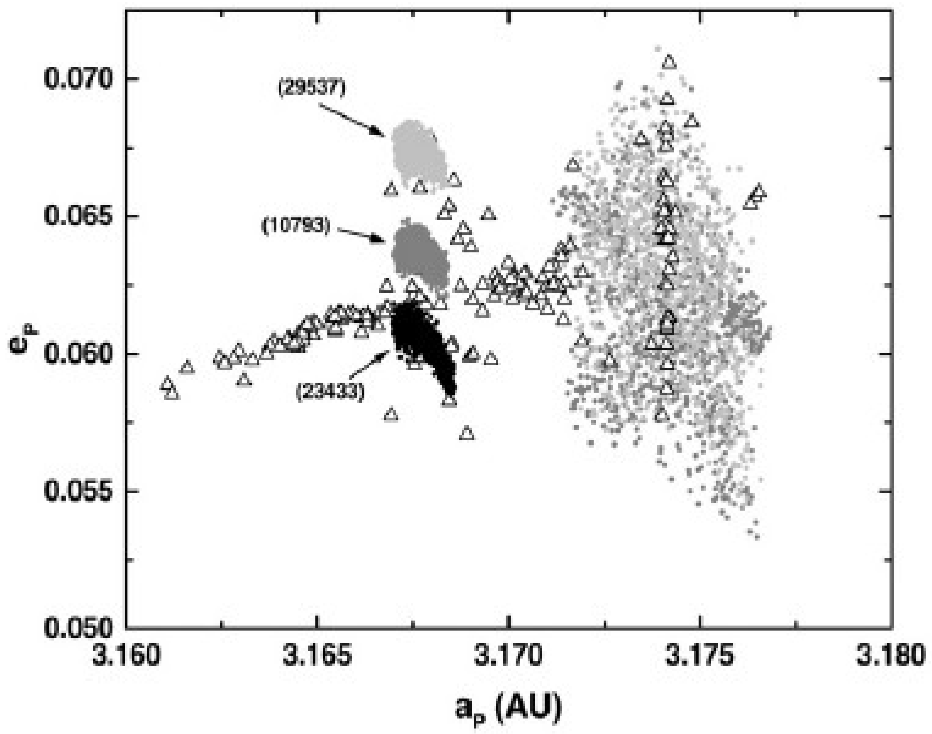


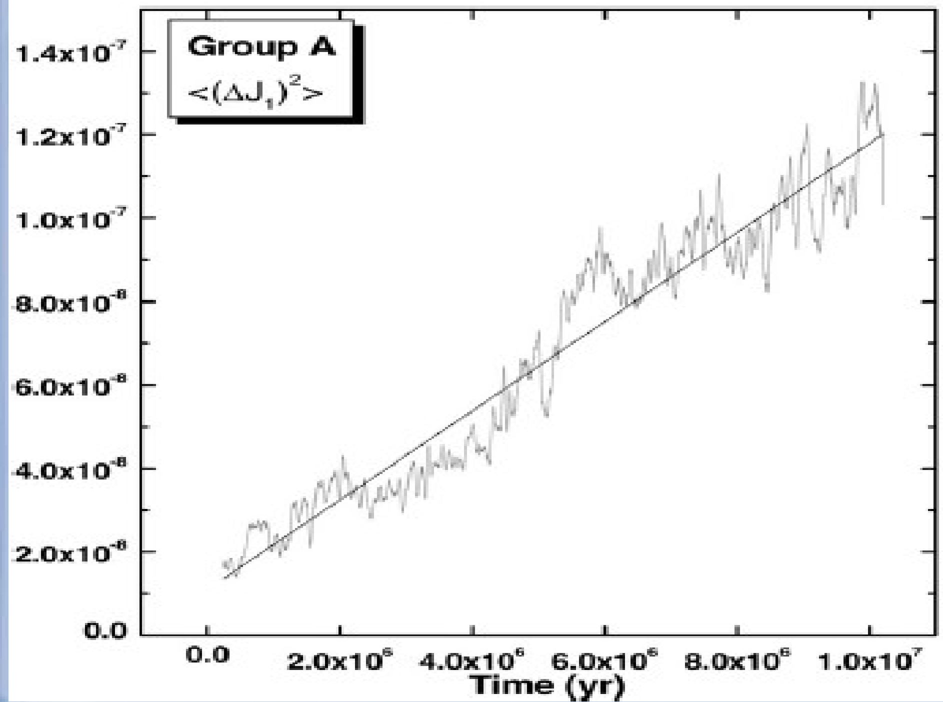


Typical (5,-2,-2) modulated pendulum dynamics

Typical (3,3,-2) dynamics (partial overlap)







Veritas long-term dynamics

Clear chaotic diffusion for the (5,-2,-2) group

→ a reasonable post-break-up configuration can extend to its current size within 8.7 +/- 1.7 My

→ *chaotic chronology* possible for relatively young families with sizable chaotic components...

- *D(J)* ~ 2-3 orders of magnitude smaller in the (3,3,-2) → looks like 1-res approximation...

→ *not all* MMRs lead to appreciable long-term diffusion *of chaotic orbits (??)*

Stable Chaos ...

The 1:1 *Resonance for Satellites*

This is a resonance between the orbital period of the satellite and the rotational period of the primary. The critical angle for the 1:1 case is:

$$\sigma = (\Omega + g + M) - \theta$$

Finding the terms that contain σ, we obtain the Hamiltonian. If we expand to $O(e^2)$, to get:

$$\mathcal{H} = -\dot{\theta}L' - \frac{\mu^2}{2L'^2} - \frac{C_{20}R^2\mu^4(-2+3\sin^3 i)}{4(1-e^2)^{3/2}L'^6} - \frac{3R^2\mu^4(1+\cos^2 i)}{4L'^6}$$

$$\times\left(1 - \frac{5e^2}{2} + \frac{13e^4}{16} + ...\right)[C_{22}\cos(2\sigma) + S_{22}\sin(2\sigma)]$$

$$-\frac{3C_{30}R^3\mu^5 e\sin i\,(-4+5\sin^2 i)\sin g}{8(1-e^2)^{5/2}L'^8}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned}\mathcal{H} &= -\dot{\theta}L' - \frac{\mu^2}{2L'^2} + C_{20}(f_{20} - G'g_{20}) + f_{22}(C_{22} + S_{22})\\ &+G'g_{22}(C_{22}\cos(2\sigma) + S_{22}\sin(2\sigma)) + C_{30}g_{30}\sqrt{-G'}\sin g\end{aligned}}$$

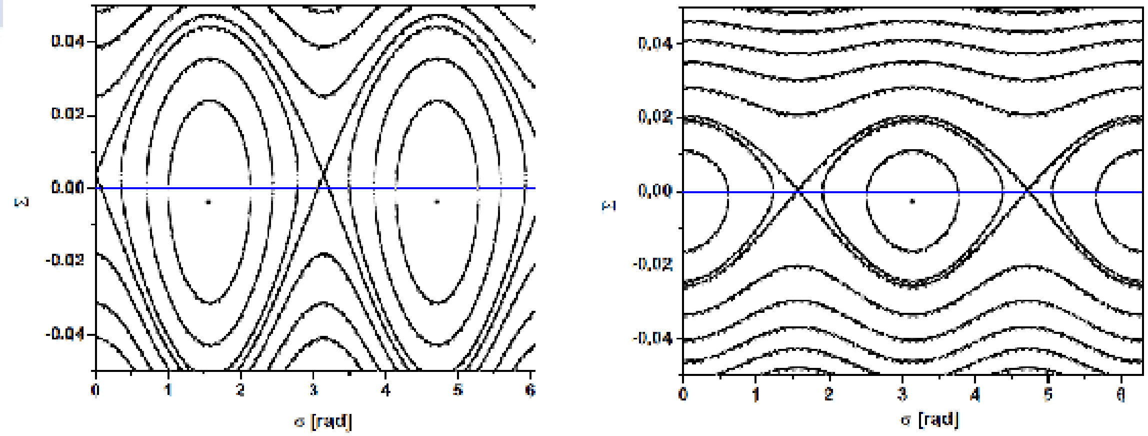
Following the same set of canonical transformations as in the rTBP, we find:

$$\mathcal{H} = c_1\Sigma^2 + c_2\Gamma\Sigma + c_3\Gamma + (c_4 + c_5\Sigma)\sqrt{-2\Gamma}$$

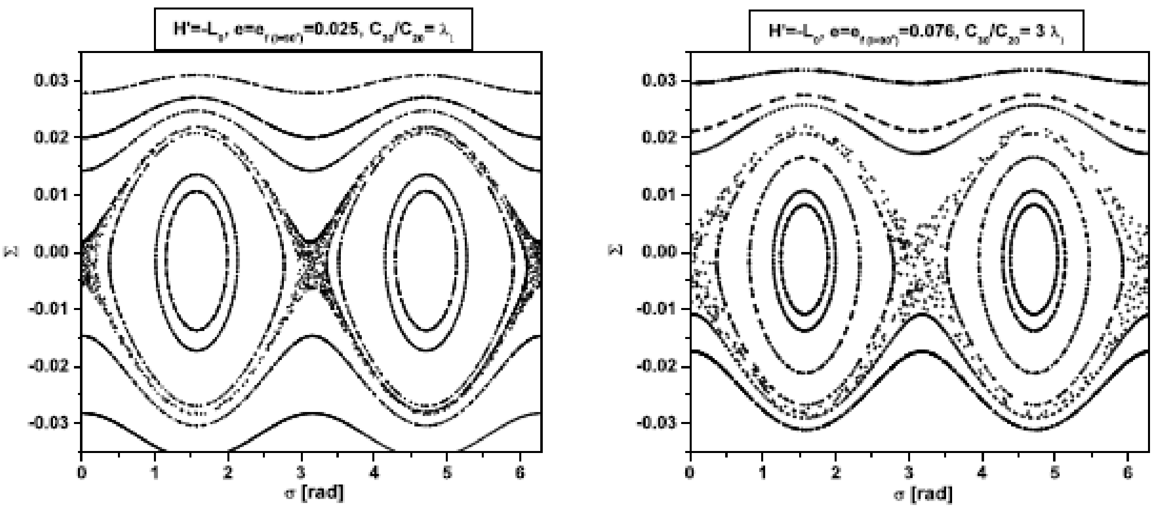
$$\times\{C_{22}[\sin(w-2\sigma) + \sin(w+2\sigma)] + S_{22}[\cos(w-2\sigma) - \cos(w+2\sigma)]\}$$

$$+[c_6 + c_7\Gamma + (c_8 + c_9\Gamma)\Sigma]\,[C_{22}\cos(2\sigma) + S_{22}\sin(2\sigma)]$$

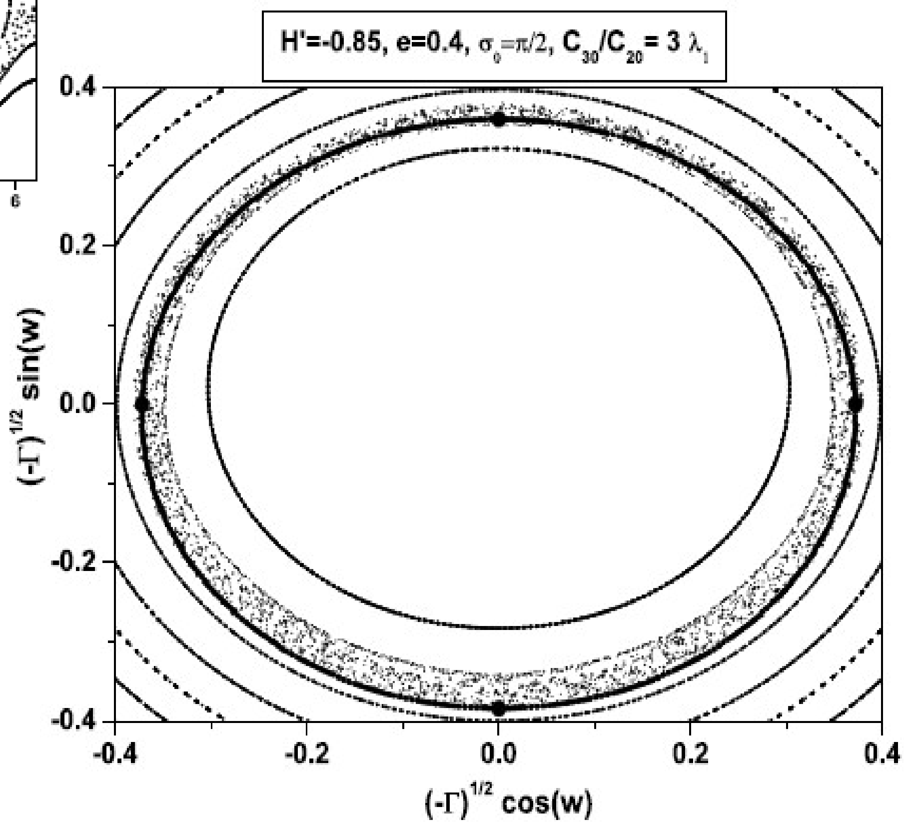
The **1-d.o.f.** system:



The **2 d.o.f.** system:



We can use the *frozen-pendulum approximation* to find approximately the limits of the chaotic domain *analytically*



The End

(hopefully I made it ...)