Newton's equations in spaces of constant curvature

Florin Diacu

Pacific Institute for the Mathematical Sciences
and
Department of Mathematics and Statistics
University of Victoria
CANADA

Tor Vergata, Roma, Italia
12 May 2014

Goal: to present some results from

- F. Diacu. On the singularities of the curved n-body problem, *Trans. Amer. Math. Soc.* **363**, 4 (2011), 2249-2264.
- F. Diacu and E. Pérez-Chavela. Homographic solutions of the curved 3-body problem, *J. Differential Equations* **250** (2011), 340-366.
- F. Diacu. Polygonal homographic orbits of the curved n-body problem, *Trans. Amer. Math. Soc.* **364**, 5 (2012), 2783-2802.
- F. Diacu. *Relative equilibria in the curved N-body problem*, Atlantis Studies in Dynamical Systems, vol. I, Atlantis Press, 2012.
- F. Diacu. Relative equilibria in the 3-dimensional curved n-body problem, *Memoirs Amer. Math. Soc.* **228**, 1071 (2013), ISBN: 978-0-8218-9136-0.
- F. Diacu and S. Kordlou. Rotopulsating orbits of the curved N-body problem *J. Differential Equations* **255** (2013), 2709-2750.

History of the problem

- 1830s Nikolai Lobachevsky and János Bolyai: 2-BP in $\mathbf{H}^3$
- 1852 Lejeune Dirichlet: 2-BP in $\mathbf{H}^3$
- 1860 Paul Joseph Serret: 2-BP in $\mathbf{S}^2$
- 1870 Ernst Schering: 2-BP in $\mathbf{H}^3$
- 1873 Rudolph Lipschitz: 2-BP in $\mathbf{S}^3$
- 1885 Wilhelm Killing: 2-BP in $\mathbf{H}^3$
- 1902 Heinrich Liebmann: 2-BP in $\mathbf{S}^2$ and $\mathbf{H}^2$ also proves an analogue of Bertrand's theorem
- 1940 Erwin Schrödinger: quantum 2-BP in $\mathbf{H}^3$
- 1945 Leopold Infeld and Alfred Schild: quantum 2-BP in $\mathbf{H}^3$
- 1990s Russian school of celestial mechanics
- 2005 José Cariñena, Manuel Rañada, Mariano Santander: 2-BP in $\mathbf{S}^2$ and $\mathbf{H}^2$

Setting

The space in which the motion of the bodies takes place is:

$$\mathbb{M}^3_\kappa = \{(w,x,y,z) | w^2 + x^2 + y^2 + \sigma z^2 = \kappa^{-1} (z > 0 \text{ if } \kappa < 0)\},$$

where σ is the *signum function*

$$\sigma = \begin{cases} +1, & \text{for} \quad \kappa > 0 \\ -1, & \text{for} \quad \kappa < 0 \end{cases}$$

Notice that

$$\mathbb{M}^3_1 = \mathbb{S}^3 \ \text{ and } \ \mathbb{M}^3_{-1} = \mathbb{H}^3$$

Notations

Consider $m_1, \ldots, m_n > 0$ in $\mathbb{R}^4$ for $\kappa > 0$ and $\mathbb{M}^{3,1}$ (Minkowski space) for $\kappa < 0$, with positions given by

$$\mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = \overline{1, n}$$

$\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ is the configuration of the system

$\nabla_{\mathbf{q}_i} := (\partial_{w_i}, \partial_{x_i}, \partial_{y_i}, \sigma\partial_{z_i}),\ \ \nabla := (\nabla_{\mathbf{q}_1}, \ldots, \nabla_{\mathbf{q}_n})$ is the gradient

For $\mathbf{a} := (a_w, a_x, a_y, a_z), \mathbf{b} := (b_w, b_x, b_y, b_z)$,

$$\mathbf{a} \cdot \mathbf{b} := (a_w b_w + a_x b_x + a_y b_y + \sigma a_z b_z)$$

is the inner product

Potential

For $\kappa \neq 0$, the force function is

$$U_\kappa(\mathbf{q}) = \sum_{1\leq i<j\leq n} \frac{m_i m_j |\kappa|^{1/2} \kappa \mathbf{q}_i \cdot \mathbf{q}_j}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{1/2}}$$

$-U_\kappa$ is the potential (a homogeneous function of degree 0).

Euler's formula for homogeneous functions:

$$\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, \quad i = \overline{1, n}.$$

Equations of motion

Using variational methods (constrained Lagrangian dynamics), we obtain the equations of motion:

$$m_i\ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i}U_\kappa(\mathbf{q}) - m_i\kappa(\dot{\mathbf{q}}_i\cdot\dot{\mathbf{q}}_i)\mathbf{q}_i,$$
$$\mathbf{q}_i\cdot\mathbf{q}_i = \kappa^{-1},\ \mathbf{q}_i\cdot\dot{\mathbf{q}}_i = 0,\ \kappa\neq 0,\ \ i=\overline{1,n}$$

$$\nabla_{\mathbf{q}_i}U_\kappa(\mathbf{q}) = \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{m_im_j|\kappa|^{3/2}(\kappa\mathbf{q}_j\cdot\mathbf{q}_j)[(\kappa\mathbf{q}_i\cdot\mathbf{q}_i)\mathbf{q}_j-(\kappa\mathbf{q}_i\cdot\mathbf{q}_j)\mathbf{q}_i]}{[\sigma(\kappa\mathbf{q}_i\cdot\mathbf{q}_i)(\kappa\mathbf{q}_j\cdot\mathbf{q}_j)-\sigma(\kappa\mathbf{q}_i\cdot\mathbf{q}_j)^2]^{3/2}},$$

$i=\overline{1,n}$

Elimination of κ

Coordinate and time-rescaling transformations

$$\mathbf{q}_i = |\kappa|^{-1/2}\mathbf{r}_i,\ i = \overline{1,n} \ \text{ and } \ \tau = |\kappa|^{3/4}t$$

lead to the equations of motion

$$\mathbf{r}_i'' = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{r}_j - \sigma(\mathbf{r}_i \cdot \mathbf{r}_j)\mathbf{r}_i]}{[\sigma - \sigma(\mathbf{r}_i \cdot \mathbf{r}_j)^2]^{3/2}} - \sigma(\mathbf{r}_i' \cdot \mathbf{r}_i')\mathbf{r}_i, \ \ i = \overline{1,n},$$

where

$$' = \frac{d}{d\tau}, \ \ \mathbf{r}_i \cdot \mathbf{r}_i = |\kappa|\mathbf{q}_i \cdot \mathbf{q}_i = |\kappa|\kappa^{-1} = \sigma$$

The positive case and the negative case

Equations of motion in $\mathbb{S}^3$:

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[1-(\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = 1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

Equations of motion in $\mathbb{H}^3$:

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{q}_j + (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[(\mathbf{q}_i \cdot \mathbf{q}_j)^2 - 1]^{3/2}} + (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = -1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

Hamiltonian form

$\mathbf{p} := (\mathbf{p}_1, \dots, \mathbf{p}_n)$, $\mathbf{p}_i := m_i \dot{\mathbf{q}}_i$, $i = \overline{1, n}$, momenta

$T(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^{n} m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i)(\sigma \mathbf{q}_i \cdot \mathbf{q}_i)$, kinetic energy

$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q})$, Hamiltonian function

$$\begin{cases} \dot{\mathbf{q}}_i = \nabla_{\mathbf{p}_i} H(\mathbf{q}, \mathbf{p}) = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = -\nabla_{\mathbf{q}_i} H(\mathbf{q}, \mathbf{p}) = \nabla_{\mathbf{q}_i} U(\mathbf{q}) - \sigma m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) \mathbf{q}_i, \\ \mathbf{q}_i \cdot \mathbf{q}_i = \sigma, \quad \mathbf{q}_i \cdot \mathbf{p}_i = 0, \quad i = \overline{1, n} \end{cases}$$

The wedge product

Consider the basis

$$\mathbf{e}_w = (1,0,0,0),\ \mathbf{e}_x = (0,1,0,0),\ \mathbf{e}_y = (0,0,1,0),\ \mathbf{e}_z = (0,0,0,1)$$

The wedge product of
$\mathbf{u} = (u_w, u_x, u_y, u_z), \mathbf{v} = (v_w, v_x, v_y, v_z) \in \mathbb{R}^4$ is defined as

$$\begin{aligned}\mathbf{u} \wedge \mathbf{v} := (u_w v_x - u_x v_w) e_w \wedge e_x + (u_w v_y - u_y v_w) e_w \wedge e_y + \\ (u_w v_z - u_z v_w) e_w \wedge e_z + (u_x v_y - u_y v_x) e_x \wedge e_y + \\ (u_x v_z - u_z v_x) e_x \wedge e_z + (u_y v_z - u_z v_y) e_y \wedge e_z,\end{aligned}$$

where $\mathbf{e}_w \wedge \mathbf{e}_x, \mathbf{e}_w \wedge \mathbf{e}_y, \mathbf{e}_w \wedge \mathbf{e}_z, \mathbf{e}_x \wedge \mathbf{e}_y, \mathbf{e}_x \wedge \mathbf{e}_z, \mathbf{e}_y \wedge \mathbf{e}_z$ represent the bivectors that form a canonical basis of the exterior Grassmann algebra over $\mathbb{R}^4$

Integrals of the total angular momentum

$$\sum_{i=1}^{n} m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i = \mathbf{c},$$

where $\mathbf{c} =$
$c_{wx}\mathbf{e}_w \wedge \mathbf{e}_x + c_{wy}\mathbf{e}_w \wedge \mathbf{e}_y + c_{wz}\mathbf{e}_w \wedge \mathbf{e}_z + c_{xy}\mathbf{e}_x \wedge \mathbf{e}_y + c_{xz}\mathbf{e}_x \wedge \mathbf{e}_z + c_{yz}\mathbf{e}_y \wedge \mathbf{e}_z,$
with the coefficients $c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz} \in \mathbb{R}$
– on components, 6 integrals:

$$\sum_{i=1}^{n} m_i(w_i\dot{x}_i - \dot{w}_i x_i) = c_{wx}, \quad \sum_{i=1}^{n} m_i(w_i\dot{y}_i - \dot{w}_i y_i) = c_{wy},$$

$$\sum_{i=1}^{n} m_i(w_i\dot{z}_i - \dot{w}_i z_i) = c_{wz}, \quad \sum_{i=1}^{n} m_i(x_i\dot{y}_i - \dot{x}_i y_i) = c_{xy},$$

$$\sum_{i=1}^{n} m_i(x_i\dot{z}_i - \dot{x}_i z_i) = c_{xz}, \quad \sum_{i=1}^{n} m_i(y_i\dot{z}_i - \dot{y}_i z_i) = c_{yz}$$

Isometries in $\mathbb{S}^3$

In some suitable basis, rotations can be written as

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \theta, \phi \in [0, 2\pi)$$

– simple rotations (elliptic): lead to new solutions
– double rotations (elliptic-elliptic): lead to new solutions

Isometries in $\mathbb{H}^3$

In some suitable basis, rotations can be written as

$$B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\phi & \sinh\phi \\ 0 & 0 & \sinh\phi & \cosh\phi \end{pmatrix}, \theta \in [0, 2\pi), \phi \in \mathbb{R},$$

– simple rotations (elliptic): lead to new solutions
– simple rotations (hyperbolic): lead to new solutions
– double rotations (elliptic-hyperbolic): lead to new solutions

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\xi & \xi \\ 0 & \xi & 1-\xi^2/2 & \xi^2/2 \\ 0 & \xi & -\xi^2/2 & 1+\xi^2/2 \end{pmatrix}, \xi \in \mathbb{R}.$$

– simple rotations (parabolic): lead to no solutions

Relative equilibria (RE) in $\mathbb{S}^3$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = \overline{1, n},$$

$$[\text{positive elliptic}]: \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \ (\text{constant}) \\ z_i(t) = z_i \ (\text{constant}), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2, \ r_i^2 + y_i^2 + z_i^2 = 1, \ i = \overline{1, n}$

$$[\text{positive elliptic}-\text{elliptic}]: \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \rho_i \cos(\beta t + b_i) \\ z_i(t) = \rho_i \sin(\beta t + b_i), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2, \ y_i^2 + z_i^2 = \rho_i^2, \ r_i^2 + \rho_i^2 = 1, \ i = \overline{1, n}$

Relative equilibria (RE) in $\mathbb{H}^3$

$$[\text{negative elliptic}]: \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \ (\text{constant}) \\ z_i(t) = z_i \ (\text{constant}), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2, \ r_i^2 + y_i^2 - z_i^2 = -1, \ i = \overline{1, n}$

$$[\text{negative hyperbolic}]: \quad \begin{cases} w_i(t) = w_i \ (\text{constant}) \\ x_i(t) = x_i \ (\text{constant}) \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases}$$

with $y_i^2 - z_i^2 = -\eta_i^2, \ w_i^2 + x_i^2 - \eta_i^2 = -1, \ i = \overline{1, n}$

$$[\text{negative elliptic}-\text{hyperbolic}]: \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2, \ y_i^2 - z_i^2 = -\eta_i^2$, so $r_i^2 - \eta_i^2 = -1, \ i = \overline{1, n}$

Fixed points (FP) in $\mathbb{S}^3$

– equilateral triangle on a great circle of a great sphere (equal masses, 3BP)
– any scalene acute triangle on a great circle of a great sphere (non-equal masses, 3BP)
– regular tetrahedron in a great sphere (equal masses, 4BP)
– two equilateral triangles, each on complementary great circles (equal masses, 6 BP):

$$
\begin{array}{llll}
w_1 = 1, & x_1 = 0, & y_1 = 0, & z_1 = 0, \\
w_2 = -1/2, & x_2 = \sqrt{3}/2, & y_2 = 0, & z_2 = 0, \\
w_3 = -1/2, & x_3 = -\sqrt{3}/2, & y_3 = 0, & z_3 = 0, \\
w_4 = 0, & x_4 = 0, & y_4 = 1, & z_4 = 0, \\
w_5 = 0, & x_5 = 0, & y_5 = -1/2, & z_5 = \sqrt{3}/2, \\
w_6 = 0, & x_6 = 0, & y_6 = -1/2, & z_6 = -\sqrt{3}/2,
\end{array}
$$

– two, not necessarily congruent, scalene acute triangles, each on one of two complementary great circles (non-equal masses, 6 BP)

Complementary circles in $\mathbb{S}^3$

Definition 1

Two great circles, C_1 and C_2, of two different great spheres of $\mathbb{S}^3$ are called *complementary* if there is a coordinate system $wxyz$ such that

$$C_1 = \mathbf{S}^1_{wx} = \{(0,0,y,z) | \, y^2 + z^2 = 1\},$$
$$C_2 = \mathbf{S}^1_{yz} = \{(w,x,0,0) | \, w^2 + x^2 = 1\}.$$

Complementary circles form a Hopf link in a Hopf fibration,

$$h \colon \mathbb{S}^3 \to \mathbb{S}^2, \ h(w,x,y,z) = (w^2+x^2-y^2-z^2, 2(wz+xy), 2(xz-wy)),$$

which takes circles of $\mathbb{S}^3$ to points of $\mathbb{S}^2$. Using the stereographic projection, it can be shown that the circles C_1 and C_2 are linked.

Distance between complementary circles

Since, in $\mathbb{S}^3$, the distance between two points, $\mathbf{a}$ and $\mathbf{b}$, is

$$d(\mathbf{a}, \mathbf{b}) = \cos^{-1}(\mathbf{a} \cdot \mathbf{b}),$$

it follows that if $\mathbf{a} \in C_1$ and $\mathbf{b} \in C_2$, then

$$d(\mathbf{a}, \mathbf{b}) = \pi/2 = \text{constant}$$

Therefore if the body m_1 is on C_1 and the body m_2 is on C_2, the magnitude of the attraction between them is the same, no matter where each of them lies on the respective circle

Clifford tori in $\mathbb{S}^3$

A remarkable family of surfaces in $\mathbb{R}^4$ are the Clifford tori

$$\mathbf{T}^2_{r\rho} = \{(r\cos\theta, r\sin\theta, \rho\cos\phi, \rho\sin\phi) \mid r^2+\rho^2 = 1, 0 \le \theta, \phi < 2\pi\},$$

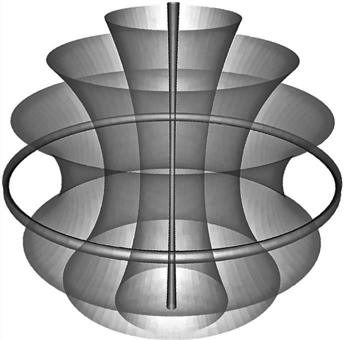
which lie in $\mathbb{S}^3$. Indeed, the Euclidean distance from the origin of the coordinate system to any point of a Clifford torus is

$$(r^2\cos^2\theta + r^2\sin^2\theta + \rho^2\cos^2\phi + \rho^2\sin^2\phi)^{1/2} = (r^2+\rho^2)^{1/2} = 1$$

Unlike the standard torus, the Clifford torus is a flat surface, which divides $\mathbb{S}^3$ into two solid tori, for which it forms the boundary

Heegaard splitting of $\mathbb{S}^3$

The Clifford torus with $r = \rho = 1/\sqrt{2}$ provides the standard genus 1 splitting of $\mathbb{S}^3$, a case in which the two solid tori are congruent.



A 3D projection of a 4D foliation of $\mathbb{S}^3$ into Clifford tori

Qualitative behaviour of RE in $\mathbb{S}^3$

Theorem 2

Assume that, in the curved n-body problem in $\mathbb{S}^3$, $n \geq 2$, with bodies of masses $m_1, \dots, m_n > 0$, positive elliptic and positive elliptic-elliptic relative equilibria exist. Then the corresponding solution $\mathbf{q}$ may have one of the following properties:

(i) it is a (simply rotating) positive elliptic RE, with the body of mass m_i moving on a (not necessarily geodesic) circle $\mathcal{C}_i, i = \overline{1, n}$, of a 2-sphere in $\mathbb{S}^3$; in the hyperplanes wxy and wxz, the circles $\mathcal{C}_i$ are parallel with the plane wx; another possibility is that some bodies rotate on a great circle of a great sphere, while the other bodies stay fixed on a complementary great circle of another great sphere.

(ii) it is a (doubly rotating) positive elliptic-elliptic RE, with some bodies rotating on a great circle of a great sphere and the other bodies rotating on a complementary great circle of another great sphere; another possibility is that each body m_i is moving on the Clifford torus $\mathbf{T}^2_{r_i \rho_i}, i = \overline{1, n}$.

Lagrangian RE as in (i)

$$
\begin{aligned}
w_1(t) &= r\cos\omega t, & x_1(t) &= r\sin\omega t,\\
y_1(t) &= y \ \text{(constant)}, & z_1(t) &= z \ \text{(constant)},\\
w_2(t) &= r\cos(\omega t + 2\pi/3), & x_2(t) &= r\sin(\omega t + 2\pi/3),\\
y_2(t) &= y \ \text{(constant)}, & z_2(t) &= z \ \text{(constant)},\\
w_3(t) &= r\cos(\omega t + 4\pi/3), & x_3(t) &= r\sin(\omega t + 4\pi/3),\\
y_3(t) &= y \ \text{(constant)}, & z_3(t) &= z \ \text{(constant)}.
\end{aligned}
$$

Given $m := m_1 = m_2 = m_3 > 0,\ r \in (0,1)$, and y, z with $r^2 + y^2 + z^2 = 1$, we can always find two frequencies,

$$
\alpha^+ = \frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}} \ \text{ and } \ \alpha^- = -\frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}};
$$

$$
c_{wx} = 3m\omega \neq 0 \ \text{ and } \ c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0.
$$

Stability of Lagrangian RE in $\mathbb{S}^2$

Regina Martínez and Carles Simó: On $\mathbb{S}^2$, the Lagrangian RE with masses $m_1 = m_2 = m_3 = 1$ are linearly stable for $r \in (r_1, r_2) \cup (r_3, 1)$, where $r = \sqrt{1 - z^2}$,

$$r_1 = 0.55778526844099498188467226566148375,$$

$$r_2 = 0.68145469725865414807206661241888645,$$

$$r_3 = 0.92893280143637470996280353121615412,$$

truncated to 35 decimal digits.

Example of RE as in (ii) on Clifford tori

Place the bodies $m_1 = m_2 = m_3 = m_4$ at the vertices of a regular tetrahedron. Then m_1 and m_2 move on the Clifford torus with $r = 0$ and $\rho = 1$, which is the only Clifford torus in the class of a given foliation of $\mathbb{S}^3$ that is also a great circle of $\mathbb{S}^3$. The bodies of mass m_3 and m_4 move on the Clifford torus with $r = \frac{\sqrt{6}}{3}$ and $\rho = \frac{\sqrt{3}}{3}$:

$$w_1 = 0,\ x_1 = 0,\ y_1 = \cos(\alpha t + \pi/2),\ z_1 = \sin(\alpha t + \pi/2),$$

$$w_2 = 0,\ x_2 = 0,\ y_2 = \cos(\alpha t + b_2),\ z_2 = \sin(\alpha t + b_2),$$

with $\sin b_2 = -\frac{1}{3}$ and $\cos b_2 = \frac{2\sqrt{2}}{3}$,

Example of RE as in (ii) on Clifford tori

$$w_3 = \frac{\sqrt{6}}{3}\cos(\alpha t + 3\pi/2),\ x_3 = \frac{\sqrt{6}}{3}\sin(\alpha t + 3\pi/2),$$

$$y_3 = \frac{\sqrt{3}}{3}\cos(\alpha t + b_3),\ z_3 = \frac{\sqrt{3}}{3}\sin(\alpha t + b_3),$$

with $\cos b_3 = -\frac{\sqrt{6}}{3}$ and $\sin b_3 = -\frac{\sqrt{3}}{3}$, and

$$w_4 = \frac{\sqrt{6}}{3}\cos(\alpha t + \pi/2),\ x_4 = \frac{\sqrt{6}}{3}\sin(\alpha t + \pi/2),$$

$$y_4 = \frac{\sqrt{3}}{3}\cos(\alpha t + b_4),\ z_4 = \frac{\sqrt{3}}{3}\sin(\alpha t + b_4),$$

with $\cos b_4 = -\frac{\sqrt{6}}{3}$ and $\sin b_4 = -\frac{\sqrt{3}}{3}$. Notice that $b_3 = b_4$.

RE generated from FP configurations in $\mathbb{S}^3$

Theorem 3

Consider the bodies of masses $m_1, \ldots, m_n > 0, n \geq 2$*, in* $\mathbb{S}^3$*. Then an RE generated from a fixed point configuration may have one of the following properties:*
(i) it is a (simply rotating) positive elliptic RE for which all bodies rotate on the same great circle of a great sphere of $\mathbb{S}^3$*;*
(ii) it is a (simply rotating) positive elliptic RE for which some bodies rotate on a great circle of a great sphere, while the other bodies are fixed on a complementary great circle of a different great sphere;
(iii) it is a (doubly rotating) positive elliptic-elliptic RE for which some bodies rotate with frequency $\alpha \neq 0$ *on a great circle of a great sphere, while the other bodies rotate with frequency* $\beta \neq 0$ *on a complementary great circle of a different sphere; the frequencies may be different in size, i.e.* $|\alpha| \neq |\beta|$*;*
(iv) it is a (doubly rotating) positive elliptic-elliptic RE with frequencies $\alpha, \beta \neq 0$ *equal in size, i.e.* $|\alpha| = |\beta|$*.*

Example of RE as in (ii)

This is a solution of the 6-body problem with two equilateral triangles, one inscribed in a great circle of a great sphere and the other inscribed in a complementary great circle of another great sphere. The first triangle rotates uniformly, while the second triangle is fixed:

$$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 =: m,$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i \in \{1, 2, 3, 4, 5, 6\},$$

$$\begin{array}{llll}
w_1 = \cos\alpha t, & x_1 = \sin\alpha t, & y_1 = 0, & z_1 = 0, \\
w_2 = \cos(\alpha t + a), & x_2 = \sin(\alpha t + a), & y_2 = 0, & z_2 = 0, \\
w_3 = \cos(\alpha t + b), & x_3 = \sin(\alpha t + b), & y_3 = 0, & z_3 = 0, \\
w_4 = 0, & x_4 = 0, & y_4 = 1, & z_4 = 0, \\
w_5 = 0, & x_5 = 0, & y_5 = -\dfrac{1}{2}, & z_5 = \dfrac{\sqrt{3}}{2}, \\
w_6 = 0, & x_6 = 0, & y_6 = -\dfrac{1}{2}, & z_6 = -\dfrac{\sqrt{3}}{2},
\end{array}$$

where $a = 2\pi/3$ and $b = 4\pi/3$.

Example of RE as in (iii)

In general, the orbit described below is quasiperiodic:

$$
\begin{aligned}
w_1 &= \cos \alpha t, & x_1 &= \sin \alpha t,\\
y_1 &= 0, & z_1 &= 0,\\
w_2 &= \cos(\alpha t + 2\pi/3), & x_2 &= \sin(\alpha t + 2\pi/3),\\
y_2 &= 0, & z_2 &= 0,\\
w_3 &= \cos(\alpha t + 4\pi/3), & x_3 &= \sin(\alpha t + 4\pi/3),\\
y_3 &= 0, & z_3 &= 0,\\
w_4 &= 0, & x_4 &= 0,\\
y_4 &= \cos \beta t, & z_4 &= \sin \beta t,\\
w_5 &= 0, & x_5 &= 0,\\
y_5 &= \cos(\beta t + 2\pi/3), & z_5 &= \sin(\beta t + 2\pi/3),\\
w_6 &= 0, & x_6 &= 0,\\
y_6 &= \cos(\beta t + 4\pi/3), & z_6 &= \sin(\beta t + 4\pi/3).
\end{aligned}
$$

$$c_{wx} = 3m\alpha \neq 0, \;\; c_{yz} = 3m\beta \neq 0, \;\; c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$$

Qualitative behaviour of RE in $\mathbb{H}^3$

Theorem 4

In the curved n-body problem in $\mathbb{H}^3$, $n \geq 2$, with bodies of masses $m_1, \ldots, m_n > 0$, every RE may have one of the following properties:
(i) it is a (simply rotating) negative elliptic RE, with the body of mass m_i moving on a circle $\mathcal{C}^i,\ i = \overline{1, n}$, of a hyperbolic 2-sphere in $\mathbb{H}^3$; in the hyperplanes wxy and wxz, the planes of the circles $\mathcal{C}^i$ are parallel with the plane wx;
(ii) it is a (simply rotating) negative hyperbolic relative equilibrium, with the body of mass m_i moving on some (not necessarily geodesic) hyperbola $\mathcal{H}_i$ of a hyperbolic 2-sphere in $\mathbb{H}^3,\ i = \overline{1, n}$; in the hyperplanes wyz and xyz, the planes of the hyperbolas $\mathcal{C}^i$ are parallel with the plane yz;
(iii) it is a (doubly rotating) negative elliptic-hyperbolic relative equilibrium, with the body of mass m_i moving on the hyperbolic cylinder

$$\mathbf{C}^2_{r_i\rho_i} = \{(r_i\cos\theta, r_i\sin\theta, \eta_i\sinh\iota, \eta_i\cosh\iota) \mid r_i^2 - \eta_i^2 = -1,\ \theta \in [0, 2\pi), \iota \in \mathbb{R}\},$$

$i = \overline{1, n}.$

Eulerian RE as in (ii)

The motion described below takes place on a hyperbolic 2-sphere, and is not periodic:

$$
\begin{array}{llll}
w_1 = 0, & x_1 = 0, & y_1 = \sinh \beta t, & z_1 = \cosh \beta t, \\
w_2 = 0, & x_2 = x \ (\text{constant}), & y_2 = \eta \sinh \beta t, & z_2 = \eta \cosh \beta t, \\
w_3 = 0, & x_3 = -x \ (\text{constant}), & y_3 = \eta \sinh \beta t, & z_3 = \eta \cosh \beta t,
\end{array}
$$

Given $m := m_1 = m_2 = m_3 > 0, x > 0, \eta > 0$ with $x^2 - \eta^2 = -1$, there exist two non-zero frequencies,

$$
\beta^+ = \frac{1}{2\eta}\sqrt{\frac{1+4\eta^2}{\eta(\eta^2-1)^{3/2}}} \ \text{and} \ \beta^- = -\frac{1}{2\eta}\sqrt{\frac{1+4\eta^2}{\eta(\eta^2-1)^{3/2}}};
$$

$$
c_{wx} = c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0, \ c_{yz} = m\beta(1-2\eta^2)
$$

Example of RE as in (iii)

The motion described below takes place on a hyperbolic cylinder, and is not periodic:

$$
\begin{array}{llll}
w_1 = 0, & x_1 = 0, & y_1 = \sinh \beta t, & z_1 = \cosh \beta t, \\
w_2 = r \cos \alpha t, & x_2 = r \sin \alpha t, & y_2 = \eta \sinh \beta t, & z_2 = \eta \cosh \beta t, \\
w_3 = -r \cos \alpha t, & x_3 = -r \sin \alpha t, & y_3 = \eta \sinh \beta t, & z_3 = \eta \cosh \beta t.
\end{array}
$$

$$c_{wx} = 2m\alpha r^2, c_{yz} = -1 - 2\beta\eta^2, c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$$

Extension of the equations to $\kappa = 0$

$$m_i\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j \left[\mathbf{q}_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right)\mathbf{q}_i\right]}{r_{ij}^3\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - \kappa m_i(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i, \quad i = \overline{1, n},$$

where $m_1, m_2, \ldots, m_n > 0$ represent the masses, the vectors $\mathbf{r}_i$ are given by

$$\mathbf{q}_i = \mathbf{r}_i + (0, 0, 0, (\sigma\kappa)^{1/2}),\ \mathbf{r}_i = (x_i, y_i, z_i, \omega_i),\ i = \overline{1, n},$$

and

$$r_{ij} := \begin{cases} [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 + (\omega_i - \omega_j)^2]^{1/2}, & \kappa > 0 \\ [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2}, & \kappa = 0 \\ [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (\omega_i - \omega_j)^2]^{1/2}, & \kappa < 0. \end{cases}$$

The explicit equations

$$\begin{cases} \ddot{x}_i = \sum_{j=1, j\neq i}^{n} \dfrac{m_j\left[x_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right)x_i\right]}{r_{ij}^3\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)x_i \\ \ddot{y}_i = \sum_{j=1, j\neq i}^{n} \dfrac{m_j\left[y_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right)y_i\right]}{r_{ij}^3\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)y_i \\ \ddot{z}_i = \sum_{j=1, j\neq i}^{n} \dfrac{m_j\left[z_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right)z_i\right]}{r_{ij}^3\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)z_i \\ \ddot{\omega}_i = \sum_{j=1, j\neq i}^{n} \dfrac{m_j\left[\omega_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right)\omega_i + \frac{\sigma(\sigma\kappa)^{\frac{1}{2}} r_{ij}^2}{2}\right]}{r_{ij}^3\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)[\kappa\omega_i + \sigma(\sigma\kappa)^{\frac{1}{2}}], \end{cases}$$

$i = \overline{1, n}$.

Constraints

$$\kappa(x_i^2 + y_i^2 + z_i^2 + \sigma\omega_i^2) + 2(\sigma\kappa)^{1/2}\omega_i = 0,$$
$$\kappa(x_i\dot{x}_i + y_i\dot{y}_i + z_i\dot{z}_i + \sigma\omega_i\dot{\omega}_i) + (\sigma\kappa)^{1/2}\dot{\omega}_i = 0, \quad i = \overline{1, n}.$$

Newtonian equations

For $\kappa = 0$ we recover the Newtonian equations:

$$m_i\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad i = \overline{1,n},$$

with $\mathbf{r}_i = (x_i, y_i, z_i, 0), i = \overline{1,n}$

Bifurcation of the first integrals

– Integral of energy:

for all $\kappa \in \mathbb{R}$: 1 integral (no bifurcation)

– Integrals of the centre of mass:

$\kappa = 0$: 3 integrals

$\kappa \neq 0$: 0 integrals

– Integrals of the linear momentum:

$\kappa = 0$: 3 integrals

$\kappa \neq 0$: 0 integrals

– Integrals of the total angular momentum:

$\kappa = 0$: 3 integrals

$\kappa \neq 0$: 6 integrals

Thank you very much!